Penrith Selective High School

2012

Higher School Certificate
Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Circle the correct answer to Questions 1-10 on the multiple choice answer sheet provided at the back of this paper
- Show all necessary working in Questions 11-14

Total Marks – 70

Section I Pages 2-4

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II Pages 5-9

60 marks

- Attempt Questions 11-14
- Allow about 1 hour 45 minutes for this section

Student Number: ______________________________

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2012 Higher School Certificate Examination.

SECTION 1: Circle the correct answer on the multiple choice answer sheet provided

Q1. $y = f(x)$ is a linear function with slope $\frac{1}{3}$, find the slope of $y = f^{-1}(x)$.

(A) 3

(B) $\frac{1}{3}$

(C) -3

(D) $-\frac{1}{3}$

Q2. The interval joining the points $A(-3, 2)$ and $B(-9, y)$ is divided externally in the ratio $5 : 3$ by the point $P(x, -13)$. What are the values of x and y?

(A) $x = -27,\ \ y = 22$

(B) $x = 27,\ \ y = 4$

(C) $x = 6,\ \ y = 12$

(D) $x = -18,\ \ y = -4$

Q3. The value of $\lim_{x \to 0} \frac{\sin\frac{\pi x}{5}\cos\frac{\pi x}{5}}{4x} =$

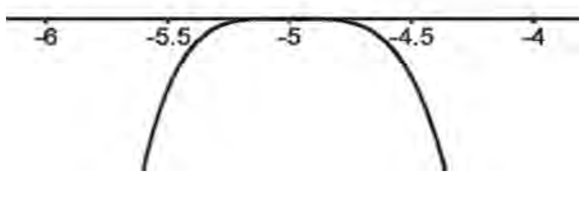
(A) $\frac{5\pi}{4}$

(B) $\frac{20}{\pi}$

(C) $\frac{4}{5\pi}$

(D) $\frac{\pi}{20}$

Q4. Part of the graph of $y = P(x)$, where $P(x)$ is a polynomial of degree five, is shown below. This graph could be part of which of the following polynomials?



(A) $P(x) = (x + 5)^3(x - 5)^2$

(B) $P(x) = (x + 1)(x + 5)^4$

(C) $P(x) = (x - 5)^5$

(D) $P(x) = (x - 1)^4(x + 5)$

Q5. Using $u = x^2 + 1$, the value that is equal to $\int_0^1 3x(x^2+1)^5 dx$ is

(A) $\frac{1}{4}$ (B) $\frac{16}{3}$

(C) $\frac{63}{4}$ (D) 32

Q6. The number N of animals in a population at time t years is given by $N = 250 + Ae^{kt}$ for constants $A > 0$ and $k > 0$. Which of the following is the correct differential equation?

(A) $\frac{dN}{dt} = k(N - 250)$ (B) $\frac{dN}{dt} = k(N + 250)$

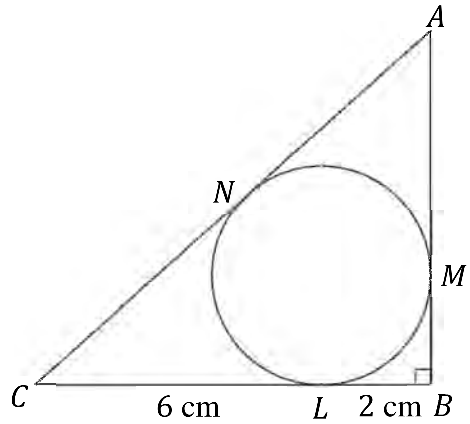
(C) $\frac{dN}{dt} = -k(N - 250)$ (D) $\frac{dN}{dt} = -k(N + 250)$

Q7. Which of the following represents $f'(x)$ if $(x) = \cos^{-1} 3x + x \cos^{-1} 3x$?

(A) $\cos^{-1} 3x - \frac{x+1}{\sqrt{9 - x^2}}$ (B) $\cos^{-1} 3x - \frac{3(x+1)}{\sqrt{1 - 9x^2}}$

(C) $\cos^{-1} 3x + \frac{x-1}{\sqrt{9 - x^2}}$ (D) $\cos^{-1} 3x + \frac{3(x-1)}{\sqrt{1 - 9x^2}}$

Q8. AC is a tangent to the circle at the point N, AB is a tangent to the circle at the point M and BC is a tangent to the circle at the point L. Find the exact length of AM if CL is 6 cm and BL is 2 cm.



(A) 3 cm

(B) 4 cm

(C) 5 cm

(D) 6 cm

Q9. A football is kicked at an angle of φ to the horizontal. The position of the ball at time t seconds is given by $x = Vt\cos\varphi$ and $y = Vt\sin\varphi - \frac{1}{2}gt^2$ where g m/s^2 is the acceleration due to gravity and V m/s is the initial velocity of projection. What is the maximum height reached by the ball?

(A) $\dfrac{V\sin\varphi}{g}$

(B) $\dfrac{g\sin\varphi}{V}$

(C) $\dfrac{V^2\sin^2\varphi}{2g}$

(D) $\dfrac{g\sin^2\varphi}{2V^2}$

Q10. What is the term that is independent of x in the expansion of $\left(5x^2 - \frac{3}{x}\right)^{12}$?

(A) ${}^{12}C_4 \times 5^8 \times (-3)^4$

(B) ${}^{12}C_4 \times 5^4 \times (-3)^8$

(C) ${}^{12}C_8 \times 5^8 \times (-3)^4$

(D) ${}^{12}C_8 \times 5^4 \times (-3)^8$

SECTION 2

Question 1 (15 marks) Start on a new page **Marks**

a) Solve $\dfrac{2x+3}{x-5} \geq 1$ for all real x. **2**

b) For $f(x) = -4\sin^{-1}\left(\dfrac{2x}{3}\right)$

i) State the domain and range **2**

ii) Hence or otherwise, sketch $f(x)$, marking clearly any end points. **1**

c) Find the reminder when $P(x) = 5x^3 - 12x + 7$ is divided by $x + 3$. **1**

d) Find the area between the curve $y = \cos^2 2x$ and the x-axis from $x = 0$ and $x = \dfrac{\pi}{6}$ **2**

e) Use the substitution $u = e^x$ to find: $\displaystyle\int \frac{dx}{e^x + 9e^{-x}}$ **3**

f) i) Find the equation of the tangent to the curve $y = e^{\sin x} + x^2$ at the point where $x = \pi$. **2**

ii) Find the obtuse angle between the line $\dfrac{x}{2} + \dfrac{y}{5} = 1$ and the tangent found in part i). Give your answer to the nearest minute. **2**

Question 2 (15 marks) Start on a new page **Marks**

a) A particle moves in a straight line and its position at a time t is given by $x = a\cos(9t + \theta)$. The particle is initially at the origin moving with a velocity of $15\ \text{m/s}$ in the negative direction.

i) Show that the particle is undergoing simple harmonic motion. **1**

ii) Find the period of the motion. **1**

iii) Find the value of the constants a and θ. **2**

iv) Find the position of the particle after 6 seconds, correct your answer to 2 decimal places. **1**

b) A curve is defined by the parametric equations $x = t - 5,\ y = t^2 - 25$.

i) Find $\dfrac{dy}{dx}$ in terms of t. **1**

ii) Find the equation of the tangent to the curve at the point where $t = -6$. **1**

iii) Express the equation of this curve in the simplest Cartesian form. **1**

c) Taking $x = 3$ as the first approximation for the root of $\tan x = 1 - \log_e x$, use Newton's method to find a second approximation and correct your answer to 4 significant figures. **2**

d) Two of the roots of the cubic polynomial $P(x) = 4x^3 - bx^2 - 64x - 16$ are reciprocals of each other, and two of the roots of $P(x)$ are of opposite signs of each other.

i) Find the value of b. **2**

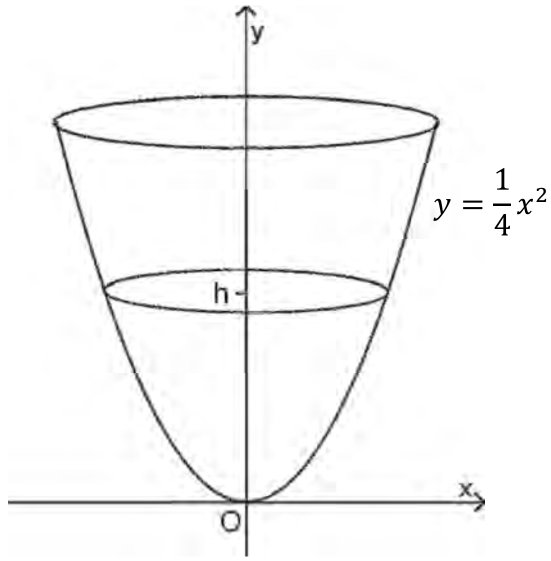
ii) Factorise $P(x)$ completely. **1**

e) Find the greatest coefficient in the expansion of $(4a + 9)^{17}$. Express your answer in index form. **2**

Question 3 (15 marks) Start on a new page **Marks**

a) A large industrial container is in the shape of a paraboloid, which is formed by rotating the parabola $y = \frac{1}{4}x^2$ around the y-axis.

Liquid is poured into the container at the rate of 6 m^3 per minute.



i) Prove that the volume V of liquid in the container when the depth of liquid is h, is given by $V = 2\pi h^2$. **1**

ii) At what rate is the height of the liquid rising when the depth is 5 m? Leave your answer in exact form. **2**

iii) If the container is 12 m high, how long will it take to fill the container? Leave your answer to two decimal places. **1**

b) i) Use mathematical induction to prove **3**

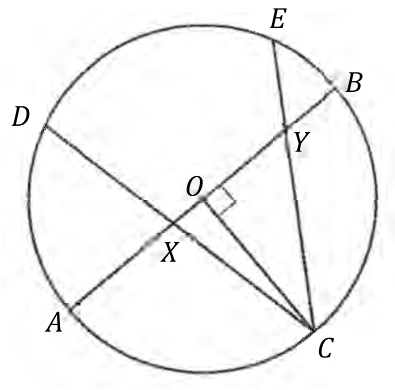
$$4(1^3 + 2^3 + 3^3 + \cdots + n^3) = n^2(n+1)^2$$

for all positive integers n.

ii) Hence find the value of $\lim_{n\to\infty}\left(\frac{1^3 + 2^3 + 3^3 + \cdots + n^3}{n^4}\right)$ **1**

Question 3 continued

c) In the diagram below, AB is the diameter of a circle with centre O. The radius OC is drawn perpendicular to AB. The chords CD and CE intersect the diameter at the points X and Y respectively.



Copy the diagram before attempting this question.

i) Prove that $\angle CBA = \angle CAB = 45°$. **1**

ii) Prove that $\angle CBD = \angle CXB$. **2**

iii) Prove that $XYED$ is a cyclic quadrilateral. **1**

d) Assume that the rate at which a body cools in air is proportional to the difference between its temperature T and the constant temperature P of the surrounding air. **3**

The body of the murder victim is discovered at 1am when its temperature is 33.5°C. Two hours later its temperature has fallen to 28°C. The body is cooling according to the equation $T = P + Ae^{kt}$, where t is the time in minutes and k and A are constants. If the room temperature remains constant at 22°C and assuming normal body temperature is 37°C, calculate the time the victim passed away to the nearest minute.

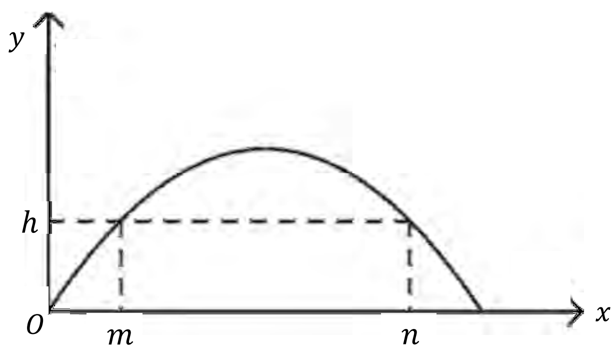
Question 4 (15 marks) Start on a new page **Marks**

a) i) When considering motion in a straight line, prove that **2**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$$

ii) An object moving in a straight line has an acceleration given by $\ddot{x} = x^2(4 - x^{-3})$ where x is the displacement in metres. When the object is 1 metre to the right of the origin, it has a speed of 3 m/s. Find its speed to 2 decimal places when it is 5 metres to the right of the origin. **2**

b) An object is projected with velocity V ms^{-1} from a point O at an angle of elevation α. Axes x and y are taken horizontally and vertically through O. The object just clears two vertical chimneys of height h meters at horizontal distance of m metres and n metres from O. The acceleration due to gravity is taken as 10 ms^{-2} and air resistance is ignored.



i) Show that the expression of the particle after time t seconds for the horizontal displacement is $x = Vt\cos\alpha$ and the vertical displacement is $y = Vt\sin\alpha - \frac{1}{2}gt^2$. **2**

ii) Show that $V^2 = \dfrac{5m^2(1+\tan^2\alpha)}{m\tan\alpha - h}$ **2**

iii) Show that $\tan\alpha = \dfrac{h(m+n)}{mn}$ **3**

c) It is given that $(1+x)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k}x^k$. Show that

i) $\sum_{k=0}^{2n}\binom{2n}{k} = 4^n$ **1**

ii) $\sum_{k=0}^{2n}\binom{2n}{k}\dfrac{1}{k+1} = \dfrac{4^{n+1}-2}{4n+2}$ **3**

-End of Paper-

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Student Number:____

Multiple Choice Answer Sheet

1.	A	B	C	D
2.	A	B	C	D
3.	A	B	C	D
4.	A	B	C	D
5.	A	B	C	D
6.	A	B	C	D
7.	A	B	C	D
8.	A	B	C	D
9.	A	B	C	D
10.	A	B	C	D

2012 HSC Ext 1 Trial Exam Solution

Section 1

Q1. A

$y = mx + b$

$y = \frac{1}{3}x + b$

inverse function

$x = \frac{1}{3}y + b$

$y = 3x - b$

$\therefore m = 3$ for $f^{-1}(x)$

Q2. D

$A(-3, 2) \quad B(-9, y)$

ratio $-5 : 3$ (externally)

$P(x, -13)$

$x = \frac{-5 \times -9 + 3 \times -3}{-5 + 3}$

$x = -18$

$-13 = \frac{-5 \times y + 3 \times 2}{-5 + 3}$

$26 = -5y + 6$

$y = -4$

Q3. D

$\lim_{x \to 0} \frac{\sin \frac{\pi x}{5} \cos \frac{\pi x}{5}}{4x}$

$= \lim_{x \to 0} \frac{\frac{1}{2} \sin \frac{2\pi x}{5}}{4x}$

$= \frac{1}{2} \times \frac{2\pi}{20}$

$= \frac{\pi}{20}$

Q4. B

Q5. C

$u = x^2 + 1 \quad du = 2xdx$

$x = 1 \quad u = 2$

$x = 0 \quad u = 1$

$\int_0^1 3x(x^2 + 1)^5 dx$

$= \frac{3}{2} \int_0^1 (x^2 + 1)^5 \times 2xdx$

$= \frac{3}{2} \int_1^2 u^5 du$

$= \frac{3}{2} \left[\frac{u^6}{6}\right]_1^2$

$= \frac{63}{4}$

Q6. A

$N = 250 + Ae^{kt}$

$\frac{dN}{dt} = kAe^{kt}$

$\frac{dN}{dt} = k(N - 250)$

Q7. B

$f(x) = \cos^{-1} 3x + x \cos^{-1} 3x$

$f'(x) = \frac{-3}{\sqrt{1 - 9x^2}} + \frac{-3x}{\sqrt{1 - 9x^2}} + \cos^{-1} 3x$

$f'(x) = \cos^{-1} 3x - \frac{3(x + 1)}{\sqrt{1 - 9x^2}}$

Q8. B

$CL = CN = 6$ cm

$LB = BM = 2$ cm

$AN = AM = x$ cm

$(x + 2)^2 + 8^2 = (x + 6)^2$

$x^2 + 4x + 4 + 64 = x^2 + 12x + 36$

$8x = 32$

$x = 4$

Q9. C

Maximum height when $\dot{y} = 0$

$y = Vt \sin \varphi - \frac{1}{2}gt^2$

$\dot{y} = V \sin \varphi - gt$

$V \sin \varphi - gt = 0$

$t = \frac{V \sin \varphi}{g}$

Maximum height

$y = V \times \frac{V \sin \varphi}{g} \times \sin \varphi - \frac{1}{2} g \left(\frac{V \sin \varphi}{g}\right)^2$

$y = \frac{V^2 \sin^2 \varphi}{2g}$

Q10. D

$\left(5x^2 - \frac{3}{x}\right)^{12}$

$T_{r+1} = {}^{12}C_r (5x^2)^{12-r} \times \left(-\frac{3}{x}\right)^r$

$T_{r+1} = {}^{12}C_r \times 5^{12-r} \times (-3)^r \times x^{24-2r} \times x^{-r}$

$T_{r+1} = {}^{12}C_r \times 5^{12-r} \times (-3)^r \times x^{24-3r}$

Term independent of x, so $24 - 3r = 0$

$r = 8$

$T_9 = {}^{12}C_8 \times 5^4 \times (-3)^8$

Section 2

Q1.

a)

$\frac{2x+3}{x-5} \geq 1$

$(2x+3)(x-5) \geq (x-5)^2 \qquad x \neq 5$

$2x^2 - 7x - 15 \geq x^2 - 10x + 25$

$x^2 + 3x - 40 \geq 0$

$(x+8)(x-5) \geq 0$

$x \leq -8, \;\; x > 5$

b) i)

$f(x) = -4\sin^{-1}\left(\frac{2x}{3}\right)$

Domain:

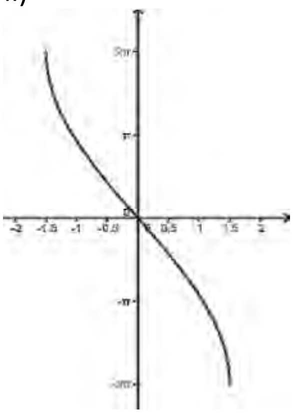
$-1 \leq \frac{2x}{3} \leq 1$

$-\frac{3}{2} \leq x \leq \frac{3}{2}$

Range:

$-2\pi \leq y \leq 2\pi$

ii)



c)

$P(x) = 5x^3 - 12x + 7$ is divided by $x + 3$

$R(x) = P(-3) = 5 \times (-3)^3 - 12 \times (-3) + 7$

$R(x) = -92$

d)

$A = \int_0^{\frac{\pi}{6}} \cos^2 2x \, dx$

$A = \frac{1}{2}\int_0^{\frac{\pi}{6}} (\cos 4x + 1) dx$

$V = \frac{1}{2}\left[\frac{1}{4}\sin 4x + x\right]_0^{\frac{\pi}{6}}$

$V = \frac{1}{2}\left[\frac{1}{4} \times \frac{\sqrt{3}}{2} + \frac{\pi}{6}\right]$

$V = \frac{\sqrt{3}}{16} + \frac{\pi}{12} \;\; \text{units}^3$

e)

$u = e^x$

$du = e^x dx$

$dx = \frac{1}{u} du$

$\int \frac{dx}{e^x + 9e^{-x}}$

$= \int \frac{1}{u + \frac{9}{u}} \times \frac{1}{u} du$

$= \int \frac{u}{u^2 + 9} \times \frac{1}{u} du$

$= \int \frac{du}{u^2 + 9}$

$= \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$

$= \frac{1}{3}\tan^{-1}\left(\frac{e^x}{3}\right) + C$

f) i)

$y = e^{\sin x} + x^2$

$\frac{dy}{dx} = \cos x \, e^{\sin x} + 2x$

at $x = \pi, y = 1 + \pi^2$

$\frac{dy}{dx} = \cos \pi \, e^{\sin \pi} + 2\pi$

$\frac{dy}{dx} = 2\pi - 1$

Equation of tangent is

$(y - 1 - \pi^2) = (2\pi - 1)(x - \pi)$

$y = (2\pi - 1)x - 2\pi^2 + \pi + \pi^2 + 1$

$y = (2\pi - 1)x - \pi^2 + \pi + 1$

ii)

$\frac{x}{2} + \frac{y}{5} = 1$

$5x + 2y = 10$

$y = -\frac{5}{2}x + 10$

$m_1 = -\frac{5}{2} \qquad m_2 = 2\pi - 1$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\theta = \left|\frac{-\frac{5}{2} - (2\pi - 1)}{1 + \left(-\frac{5}{2}\right) \times (2\pi - 1)}\right|$

$\theta = 32°31'$ (nearest minute)

obtuse angle $= 147°29'$ (nearest minute)

Q2.

a) i)

$x = a\cos(9t+\theta)$

$\dot{x} = -9a\sin(9t+\theta)$

$\ddot{x} = -81a\cos(9t+\theta)$

$\ddot{x} = -81x$

As the particle's motion can be described in the form $\ddot{x} = -n^2x$, where $n = 9$, it is undergoing simple harmonic motion.

ii)

The period of the motion is $\frac{2\pi}{n} = \frac{2\pi}{9}$

iii)

$x = a\cos(9t+\theta)$

When $t = 0, x = 0$

$0 = a\cos\theta$

$\theta = \frac{\pi}{2}$

$\therefore x = a\cos\left(9t+\frac{\pi}{2}\right)$

$\dot{x} = -9a\sin\left(9t+\frac{\pi}{2}\right)$

When $t = 0, v = -15$

$-15 = -9a\sin\left(\frac{\pi}{2}\right)$

$a = \frac{5}{3}\text{ m}$

iv)

$x = \frac{5}{3}\cos\left(9t+\frac{\pi}{2}\right)$

When $t = 6$

$x = \frac{5}{3}\cos\left(54+\frac{\pi}{2}\right)$

$x = 0.93$ m (2 d.p.)

The particle is about 0.93 m to the right of origin.

b) i)

$x = t-5, \qquad y = t^2-25$

$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = 2t$

ii)

$t = -6$

$\frac{dy}{dx} = 2\times -6 = -12$

$x = -11, \qquad y = 11$

$y - 11 = -12(x+11)$

$12x + y + 121 = 0$

iii)

$t = x+5$

$y = (x+5)^2 - 25$

$y = x^2 + 10x + 25 - 25$

$y = x^2 + 10x$

c)

Let $f(x) = \tan x + \log_e x - 1$

$f'(x) = \sec^2 x + \frac{1}{x}$

Let $x_1 = 3$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_2 = 3 - \frac{f(3)}{f'(3)}$

$x_2 = 3 - \frac{\tan 3 + \log_e 3 - 1}{\sec^2 3 + \frac{1}{3}}$

$x_2 = 3.032$ (4 sig figs)

d) i)

$P(x) = 4x^3 - bx^2 - 64x - 16$

$P(x) = 4\left(x^3 - \frac{b}{4}x^2 - 16x - 4\right)$

Let the roots be $\alpha, \frac{1}{\alpha}$ and β

$\alpha\times\frac{1}{\alpha}\times\beta = 4$

$\beta = 4$

$\alpha = -4, \quad \frac{1}{\alpha} = -\frac{1}{4}$

$\alpha + \beta + \frac{1}{\alpha} = \frac{b}{4}$

$b = -1$

ii)

$P(x) = 4(x-4)(x+4)\left(x+\frac{1}{4}\right)$

$P(x) = (x-4)(x+4)(4x+1)$

e)

d) $(4a+9)^{17}$

$$T_{k+1} = \binom{17}{k}(4a)^{17-k}(9)^k$$
$$T_{k+1} = \binom{17}{k}4^{17-k}\cdot 9^k\cdot a^{17-k}$$
$$T_k = \binom{17}{k-1}(4a)^{17-(k-1)}(9)^{k-1}$$
$$T_k = \binom{17}{k-1}4^{18-k}\cdot 7^{k-1}\cdot a^{18-k}$$

Compare coefficients

$$\frac{t_{k+1}}{t_k} = \frac{\binom{17}{k}4^{17-k}\cdot 9^k}{\binom{17}{k-1}4^{18-k}\cdot 9^{k-1}}$$
$$\frac{t_{k+1}}{t_k} = \frac{\binom{17}{k}\cdot 9}{\binom{17}{k-1}\cdot 4}$$
$$\frac{t_{k+1}}{t_k} = \frac{9}{4}\times\frac{\frac{17!}{(17-k)!\,k!}}{\frac{17!}{\big(17-(k-1)\big)!\,(k-1)!}}$$
$$\frac{t_{k+1}}{t_k} = \frac{9}{4}\times\frac{(18-k)!\,(k-1)!}{(17-k)!\,k!}$$
$$\frac{t_{k+1}}{t_k} = \frac{9}{4}\times\frac{18-k}{k}$$
$$\frac{t_{k+1}}{t_k} = \frac{162-9k}{4k}$$

For the greatest coefficient,

$$\frac{t_{k+1}}{t_k} > 1$$
$$\frac{162-9k}{4k} > 1$$
$$162-9k > 4k$$
$$-13k > -162$$
$$k < 12\frac{6}{13}$$

So the term with the greatest coefficient occurs when $k = 12$.

$$T_{13} = \binom{17}{12}4^{17-12}\cdot 9^{12}\cdot a^{17-12}$$
$$= \binom{17}{12}4^5\cdot 9^{12}\cdot a^5$$

So the greatest coefficient is $\binom{17}{12}4^5\cdot 9^{12}$

Q3.

a) i)

$$V = \pi\int_0^h x^2\,dy$$
$$V = \pi\int_0^h 4y\,dy$$
$$V = \pi[2y^2]_0^h$$
$$V = 2\pi h^2$$

ii)

$$\frac{dV}{dt} = \frac{dV}{dh}\times\frac{dh}{dt}$$
$$\frac{dV}{dt} = 4\pi h\times\frac{dh}{dt}$$
$$\frac{dV}{dt} = 6,\quad h = 5$$
$$6 = 4\pi\times 5\times\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{3}{10\pi}\text{ m/min}$$

iii)

$$\frac{dh}{dt} = \frac{3}{2\pi h}$$
$$t = \int_0^{12}\frac{2\pi h}{3}\,dh$$
$$t = \left[\frac{\pi h^2}{3}\right]_0^{12}$$
$$t = \frac{144\pi}{3}\text{ min} \approx 150.80\text{ min}$$

b) i)

$4(1^3+2^3+3^3+\cdots+n^3) = n^2(n+1)^2$

Step 1: for $n = 1$

$LHS = 4\times 1^3 = 4$

$RHS = 1^2\times(1+1)^2 = 4$

$LHS = RHS$

∴ Statement is true for $n = 1$

Step 2: Assume statement is true for $n = k$

$4(1^3+2^3+3^3+\cdots+k^3) = k^2(k+1)^2$

Step 3: Prove statement is true for $n = k+1$

i.e. $4(1^3+2^3+3^3+\cdots+k^3+(k+1)^3)$
$= (k+1)^2(k+2)^2$

$LHS = 4(1^3+2^3+3^3+\cdots+k^3+(k+1)^3)$

$LHS = 4(1^3+2^3+3^3+\cdots+k^3)+4(k+1)^3$

$LHS = k^2(k+1)^2+4(k+1)^3$ (from step 2)

$LHS = (k+1)^2\big(k^2+4(k+1)\big)$

$LHS = (k+1)^2(k^2+4k+4)$

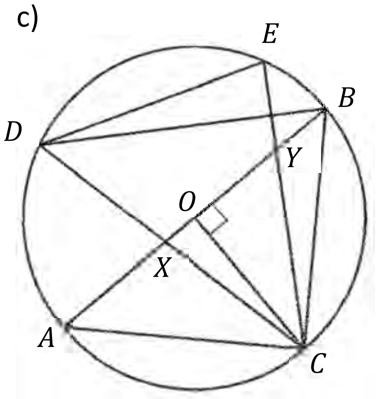
$LHS = (k+1)^2(k+2)^2 = RHS$

∴ Statement is true for all positive integers n by mathematical induction.

ii)

$$\lim_{n\to\infty}\left(\frac{1^3+2^3+3^3+\cdots+n^3}{n^2}\right)$$
$$=\lim_{n\to\infty}\left(\frac{n^2(n+1)^2}{4n^4}\right)$$
$$=\lim_{n\to\infty}\left(\frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}\right)$$
$$=\frac{1}{4}$$

c)



i)
Join AC
$\angle CBA = \frac{1}{2}\angle AOC = 45°$ (angle at the centre is twice the angle at the circumference)
$\angle CAB = \frac{1}{2}\angle AOC = 45°$ (angle at the centre is twice the angle at the circumference)
$\therefore \angle CBA = \angle CAB = 45°$

ii)
Join DB
$\angle DBA = \angle DCA$ (angles at the circumference standing on the same arc AD)
Let $\angle DBA = \angle DCA = \alpha$
$\angle DBC = \angle DBA + \angle ABC = \alpha + 45°$ (adjacent angles)
$\angle BXC = \angle BAC + \angle DCA = \alpha + 45°$ (exterior angle of ΔAXC equals to sum of the opposite interior angles)
$\therefore \angle DBC = \angle BXC$

iii)
Join DE
$\angle DBC = \angle DEC = \alpha + 45°$ (angles at the circumference standing on the same arc CD)
$\therefore \angle DEC = \angle BXC = \alpha + 45°$
$\therefore XYED$ is a cyclic quadrilateral as the exterior angle equals to opposite interior angle)

d)
$T = P + Ae^{kt}$
Let 1am be time zero
$P = 22°\text{C},\ \ t = 0,\ \ T = 33.5°\text{C}$
$33.5 = 22 + Ae^0$
$A = 11.5$

2 hours after 1am, $t = 120, T = 28°\text{C}$
$28 = 22 + 11.5e^{120k}$
$$\frac{6}{11.5} = e^{120k}$$
$$k = \frac{\ln\left(\frac{6}{11.5}\right)}{120}$$

Body temperature was originally 37°C
$37 = 22 + 11.5e^{kt}$
$15 = 11.5e^{kt}$
$$kt = \ln\left(\frac{15}{11.5}\right)$$
$$t = \frac{\ln\left(\frac{15}{11.5}\right)}{k}$$
$t = -49.00859092\ldots$
$t \approx -49$
$\therefore$ The victim passed away 49 min before 1am, so the time would be 12: 11am in the morning.

Q4.

a) i)

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2} \times 2 \times v \times \frac{dv}{dx}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dx}{dt} \times \frac{dv}{dx}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dv}{dt}$$
$$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d^2x}{dt^2}$$

ii)

$$\ddot{x} = x^2(4 - x^{-3})$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x^2(4 - x^{-3})$$
$$\frac{1}{2}v^2 = \int \left(4x^2 - \frac{1}{x}\right) dx$$
$$\frac{1}{2}v^2 = \frac{4x^3}{3} - \ln x + C$$
$$v^2 = \frac{8x^3}{3} - 2\ln x + C$$

When $x = 1, v = 3$

$$9 = \frac{8}{3} + C$$
$$C = 6\frac{1}{3}$$
$$\therefore v^2 = \frac{8x^3}{3} - 2\ln x + 6\frac{1}{3}$$

$x = 5$

$$v^2 = \frac{8}{3} \times 5^3 - 2\ln 5 + 6\frac{1}{3}$$

$v = \pm 18.34$ m/s (2 d.p.)

The speed is 18.34 m/s when the particle is 5 metres to the right of the origin.

b) i)

Horizontal	Vertically
$\ddot{x} = 0$	$\ddot{y} = -g$
$\dot{x} = V\cos\alpha$	$\dot{y} = V\sin\alpha - gt$
$x = Vt\cos\alpha$	$y = Vt\sin\alpha - \frac{1}{2}gt^2$

ii)

at $x = m, y = h, g = 10$

$$m = Vt\cos\alpha$$
$$t = \frac{m}{V\cos\alpha}$$
$$h = V \times \frac{m}{V\cos\alpha} \times \sin\alpha - \frac{1}{2} \times 10 \times \left(\frac{m}{V\cos\alpha}\right)^2$$
$$h = m\tan\alpha - \frac{5m^2}{V^2\cos^2\alpha}$$
$$h = m\tan\alpha - \frac{5m^2(\tan^2\alpha + 1)}{V^2}$$
$$5m^2(\tan^2\alpha + 1) = V^2(m\tan\alpha - h)$$
$$V^2 = \frac{5m^2(\tan^2\alpha + 1)}{m\tan\alpha - h}$$

iii)

$$V^2 = \frac{5n^2(\tan^2\alpha + 1)}{n\tan\alpha - h}$$
$$\frac{5m^2(\tan^2\alpha + 1)}{m\tan\alpha - h} = \frac{5n^2(\tan^2\alpha + 1)}{n\tan\alpha - h}$$
$$\frac{m^2}{m\tan\alpha - h} = \frac{n^2}{n\tan\alpha - h}$$
$$m^2(n\tan\alpha - h) = n^2(m\tan\alpha - h)$$
$$m^2n\tan\alpha - m^2h = n^2m\tan\alpha - n^2h$$
$$m^2n\tan\alpha - n^2m\tan\alpha = m^2h - n^2h$$
$$(m^2n - n^2m)\tan\alpha = m^2h - n^2h$$
$$\tan\alpha = \frac{m^2h - n^2h}{m^2n - n^2m}$$
$$\tan\alpha = \frac{(m^2 - n^2)h}{mn(n - m)}$$
$$\tan\alpha = \frac{(m+n)(m-n)h}{mn(n-m)}$$
$$\tan\alpha = \frac{h(m+n)}{mn}$$

c) i)

$$(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k$$

Let $x = 1$

$$2^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} 1^k$$

$$4^n = \sum_{k=0}^{2n} \binom{2n}{k}$$

ii)

Integrate both sides

$$\frac{(1+x)^{2n+1}}{2n+1} = \sum_{k=0}^{2n} \binom{2n}{k} \frac{x^{k+1}}{k+1} + C$$

Let $x = 0$

$$\frac{1}{2n+1} = C$$

$$\frac{(1+x)^{2n+1}}{2n+1} = \sum_{k=0}^{2n} \binom{2n}{k} \frac{x^{k+1}}{k+1} + \frac{1}{2n+1}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \frac{x^{k+1}}{k+1} = \frac{(1+x)^{2n+1}}{2n+1} - \frac{1}{2n+1}$$

Let $x = 1$

$$\sum_{k=0}^{2n} \binom{2n}{k} \frac{1}{k+1} = \frac{2^{2n+1}}{2n+1} - \frac{1}{2n+1}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \frac{1}{k+1} = \frac{2^{2n+1} - 1}{2n+1}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \frac{1}{k+1} = \frac{2 \times 2^{2n} - 1}{2n+1} \times \frac{2}{2}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \frac{1}{k+1} = \frac{4 \times 4^n - 2}{4n+2}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \frac{1}{k+1} = \frac{4^{n+1} - 2}{4n+2}$$