PENRITH HIGH SCHDOL

2015
HSC TRIAL EXAMINATION

## Mathematics Extension 1

## General Instructions:

- Reading time -5 minutes
- Working time -2 hours
- Write using black or blue pen Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In questions $11-14$, show relevant mathematical reasoning and/or calculations
- Answer each question on a new sheet of paper


## Total marks-70

## SECTION I <br> Pages 3-5

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section


## SECTION II <br> Pages 6-9

60 marks

- Attempt Questions 11-14
- Allow about 1 hours 45 minutes for this section


## Section I

## 10 marks

Attempt Questions 1-10
Allow about 15 minutes for this section

Use the provided multiple-choice answer sheet for Questions 1-10
1 For $x>1, e^{x}-\ln x$ is:
(A) $=0$
(B) $>0$
(C) $<0$
(D) $=e$

2 Consider the function $f(x)$ given in the graph below:


Which domain of the function $f(x)$ above is valid for the inverse function to exist?:
(A) $\quad x>0$
(B) $-a<x<a$
(C) $0<x<a$
(D) $x<0$

3 What is the acute angle between the lines $2 x-y-7=0$ and $3 x-5 y-2=0$ ?
(A) $\quad 4^{\circ} 24^{\prime}$
(B) $32^{\circ} 28^{\prime}$
(C) $57^{\circ} 32^{\prime}$
(D) $85^{\circ} 36^{\prime}$

4 A particle is moving in a straight line with velocity $v \mathrm{~m} / \mathrm{s}$ and acceleration $a \mathrm{~m} / \mathrm{s}^{2}$. Initially the particle started moving to the left of a fixed point $O$. The particle is noticed to be slowing down during the course of the motion from 0 to $A$. It turns around at $A$, keeps speeding up for the rest of the course of motion, passing $O$ and $B$ and continues. The particle never comes back. Take left to be the negative direction.


During the course of the particle's motion from $O$ to $A$, which statement of the following is correct?
(A) $\quad v>0$ and $a>0$
(B) $\quad v>0$ and $a<0$
(C) $\quad v<0$ and $a>0$
(D) $\quad v<0$ and $a<0$

5 A particle is moving in a straight line with velocity, $v=\frac{1}{1+x} \mathrm{~m} / \mathrm{s}$ where $x$ is the displacement of the particle from a fixed point $O$. If the particle was observed to have reached the position $x=-2 m$ at a certain moment of time, then this particle:
(A) will definitely reach the position $x=1 m$
(B) may reach the position $x=1 \mathrm{~m}$
(C) will never reach the position $x=1 \mathrm{~m}$
(D) will come to rest before reaching $x=1 m$

6 If the rate of change of a function $y=f(x)$ at any point is proportional to the value of the function at that point then the function $y=f(x)$ is a:
(A) Polynomial function
(B) Trigonometric function
(C) Exponential function
(D) Quadratic function

7 Let $\alpha$ and $\beta$ be any two acute angles such that $\alpha<\beta$. Which of the following statements is correct ?
(A) $\sin \alpha<\sin \beta$
(B) $\cos \alpha<\cos \beta$
(C) $\operatorname{cosec} \alpha<\operatorname{cosec} \beta$
(D) $\cot \alpha<\cot \beta$

8 Which of the following is a primitive function of $\sin ^{2} x+x^{2}$ ?
(A) $x-\frac{1}{2} \sin 2 x+\frac{x^{3}}{3}+c$
(B) $\frac{1}{2} x-\frac{1}{4} \sin 2 x+\frac{x^{3}}{3}+c$
(C) $\quad x-\frac{1}{2} \sin 2 x+2 x+c$
(D) $\frac{1}{2} x-\frac{1}{4} \sin 2 x+2 x+c$

9 Consider the binomial expansion $(1+x)^{n}=1+n_{C_{1}} x+n_{C_{2}} x^{2}+\cdots+n_{C_{n}} x^{n}$. Which of the following expressions is correct?
(A) $\quad{ }^{n} c_{1}+2^{n} c_{2}+\cdots+n^{n} c_{n}=n 2^{n-1}$
(B) $\quad{ }^{n} c_{1}+2^{n} c_{2}+\cdots+n^{n} c_{n}=n 2^{n+1}$
(C) $\quad{ }^{n} c_{1}+{ }^{n} c_{2}+\cdots+{ }^{n} c_{n}=2^{n-1}$
(D) $\quad{ }^{n} c_{1}+{ }^{n} c_{2}+\cdots+{ }^{n} c_{n}=2^{n+1}$

10 Using the substitution, $u=1+\sqrt{x}$, find the value of $\int_{1}^{4} \frac{1}{(1+\sqrt{x})^{2}} \frac{1}{\sqrt{x}} d x$ is:
(A) $\frac{6}{5}$
(B) $\frac{1}{3}$
(C) $\frac{2}{3}$
(D) $\frac{3}{2}$

## Section II

## 60 marks

## Attempt Questions 11-14

Allow about $\mathbf{1}$ hour and $\mathbf{4 5}$ minutes for this section
Begin each question on a new sheet of paper. Extra sheets of paper are available.
In questions 11-14, your responses should include relevant mathematical reasoning and/or calculations

Question 11 (15 marks) Start a new sheet of paper.
(a) $\quad(x+1)$ and $(x-2)$ are factors of $A(x)=x^{3}-4 x^{2}+x+6$. Find the third factor.
(b) Find the coordinates of the point $P$ which divides the interval $A B$ internally in the ratio 2:3
with $A(-3,7)$ and $B(15,-6)$
(c) Solve the inequality $\frac{1}{4 x-1}<2$, graphing your solution on a number line.
(d) Use the method of mathematical induction to prove that, for all positive integers $n$ :

$$
1^{2}+3^{2}+5^{2}+7^{2}+\cdots+(2 n-1)^{2}=\frac{n}{3}(2 n-1)(2 n+1)
$$

(e)


PT is the common tangent to the two circles which touch at T.
PA is the tangent to the smaller circle at Q , intersecting the larger circle at points $B$ and $A$ as shown.
i) State the property which would be used to explain why $P T^{2}=P A \times P B$
ii) If $P T=m, Q A=n$ and $Q B=r$, prove that $m=\frac{n r}{n-r}$

Question 12 (15 marks) Start a new sheet of paper.
(a) The equation $\sin x=1-2 x$ has a root near $x=0.3$. Use one application of Newton's methods to find a better approximation, giving your answer correct to 2 decimal places.
(b) Five couples sit at a round table. How many different seating arrangements are possible if:
i) there are no restrictions?
ii) each person sits next to their partner?
(c) In the expansion of $\left(4+2 x-3 x^{2}\right)\left(2-\frac{x}{5}\right)^{6}$, find the coefficient of $x^{5}$
(d) i) Write the binomial expansion for $(1+x)^{n}$
ii) Using part (i), show that $\int_{0}^{3}(1+x)^{n} d x=\sum_{k=0}^{n} \frac{1}{k+1}{ }^{n} C_{k} 3^{k+1}$
iii) Hence show that $\sum_{k=0}^{n} \frac{1}{k+1}{ }^{n} C_{k} 3^{k+1}=\frac{1}{n+1}\left[4^{n+1}-1\right]$

Question 13 (15 marks) Start a new sheet of paper.
(a) Use the substitution $u=\frac{x}{\sqrt{1-x^{2}}}$ to show that $\frac{d}{d x}\left[\tan ^{-1}\left(\frac{x}{\sqrt{1-x^{2}}}\right)\right]=\frac{1}{\sqrt{1-x^{2}}}$
(You can use the result that $\int \frac{1}{a^{2}+x^{2}} d x=\frac{1}{a} \tan ^{-1} \frac{x}{a}+c$. You do not need to prove this).
(b) Give the exact value for $\int_{-\sqrt{3}}^{3} \frac{d x}{3+x^{2}}$
(c) The distinct points $P, Q$ have parameters $t=t_{1}$ and $t=t_{2}$ respectively on the parabola $x=2 t, y=t^{2}$. The equations of the tangents to the parabola at $P$ and $Q$ respectively are given by:
$y-t_{1} x+t_{1}{ }^{2}=0$ and $y-t_{2} x+t_{2}{ }^{2}=0$ (You do not need to prove these)
i) Show that the equation of the chord $P Q$ is $2 y-\left(t_{1}+t_{2}\right) x+2 t_{1} t_{2}=0$
ii) Show that $M$, the point of intersection of the tangents to the parabola at P and Q , has coordinates $\left(t_{1}+t_{2}, t_{1} t_{2}\right)$.
iii) $\quad \propto)$ Prove that for any value of $t_{1}$, except $t_{1}=0$, there are exactly two values of $t_{2}$ for which $M$ lies on the parabola $x^{2}=-4 y$.
$\beta$ ) Find these two values of $t_{2}$ in terms of $t_{1}$.

Question 14 (15 marks) Start a new sheet of paper.
(a) A particle $P$ is moving in simple harmonic motion on the $x$ axis, according to the law $x=4 \sin 3 t$ where $x$ is the displacement of $P$ in centimetres from $O$ at time $t$ seconds.
i) State the period and amplitude of the motion.
ii) Find the first time when the particle is 2 cm to the positive side of the origin and it's velocity at this time.
iii) Find the greatest speed and greatest acceleration of $P$
(b)


A projectile is fired from $O$, with speed $V m s^{-1}$, at an angle of elevation of $\theta$ to the horizontal.
After $t$ seconds, its horizontal and vertical displacements from $O$ (as shown) are $x$ metres and $y$ metres, repectively.
i) Prove that $x=V t \cos \theta$ and $y=-\frac{1}{2} g t^{2}+V t \sin \theta$

3
ii) Show that the time taken to reach $P$ is given by $t=\frac{2 V \sin \theta}{g}$
iii) The projectile falls to $Q$, where its angle of depression from $O$ is $\theta$. 3 Prove that, in its flight from $O$ to $Q, P$ is the half-way point in terms of time.

## End of paper

## STANDARD INTEGRALS

$$
\begin{aligned}
& \int x^{n} d x=\frac{1}{n+1} x^{n+1}, n \neq-1 ; x \neq 0, \text { if } n<0 \\
& \int \frac{1}{x} d x=\ln x, x>0 \\
& \int e^{a x} d x=\frac{1}{a} e^{a x}, a \neq 0 \\
& \int \cos a x d x=\frac{1}{a} \sin a x, a \neq 0 \\
& \int \sin a x d x=-\frac{1}{a} \cos a x, a \neq 0 \\
& \int \sec ^{2} a x d x=\frac{1}{a} \tan ^{2} a x, a \neq 0 \\
& \int \sec a x \tan a x d x=\frac{1}{a} \sec a x, a \neq 0 \\
& \int \frac{1}{a^{2}+x^{2}} d x=\frac{1}{a} \tan ^{-1} \frac{x}{a}, a \neq 0 \\
& \int \frac{1}{\sqrt{a^{2}-x^{2}}} d x=\sin ^{-1} \frac{x}{a}, a \neq 0,-a<x<a \\
& \int \frac{1}{\sqrt{x^{2}-a^{2}}} d x=\ln \left(x+\sqrt{x^{2}-a^{2}}\right), x>a>0 \\
& \int \frac{1}{\sqrt{x^{2}+a^{2}}} d x=\ln \left(x+\sqrt{x^{2}+a^{2}}\right) \\
&
\end{aligned}
$$

NOTE: $\ln x=\log _{e} x, x>0$
$\qquad$

## Section I - Multiple Choice

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.
Sample: $2+4=$
(A) 2
(B) 6
(C) 8
(D) 9
$\mathrm{A} \bigcirc$
B
$\mathrm{C} \bigcirc$
D

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.
A
B
$\mathrm{C} \bigcirc$
D $\bigcirc$

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.
A
あ
correct
B
$\mathrm{C} \bigcirc$
$\mathrm{D} \bigcirc$

Start Here
$\mathrm{D} O$
2. $\mathrm{A} \bigcirc$

B $\bigcirc$
$\mathrm{c} \bigcirc$
D
3. $\mathrm{A} \bigcirc$

B $\bigcirc$
$\mathrm{C} \bigcirc$
D $\bigcirc$
4. $\mathrm{A} \bigcirc$

B $\bigcirc$
$\mathrm{C} \bigcirc$
D $\bigcirc$
5. $\mathrm{A} \bigcirc$

B $\bigcirc$
$\mathrm{C} \bigcirc$
D $\bigcirc$
6. $\mathrm{A} \bigcirc$

B $\bigcirc$
$\mathrm{C} \bigcirc$
D $\bigcirc$
7. $\mathrm{A} \bigcirc$

B $\bigcirc$
$\mathrm{C} \bigcirc$
D $\bigcirc$
8. A $\bigcirc$

B $\bigcirc$
$\mathrm{c} \bigcirc$
D $\bigcirc$
9. $\mathrm{A} \bigcirc$

B $\bigcirc$
$\mathrm{c} \bigcirc$
D $\bigcirc$
10. A $\bigcirc$

B $\bigcirc$D $\bigcirc$
a)

$$
\begin{aligned}
P(3) & =27-4 \times 9+3+6 \\
& =0
\end{aligned}
$$

$\therefore(x-3)$ is thied factor.
b)

$$
\begin{aligned}
& A(-3,7) \quad B(15,-6) \quad M: N \\
& x=\frac{3(-3)+2(15)}{5}=\frac{21}{5} \\
& y=\frac{3(7)+2(-6)}{5}=\frac{9}{5} \\
& P\left(\frac{21}{5}, \frac{9}{5}\right)
\end{aligned}
$$

c) $\frac{1}{4 x-1}<2$

$$
\begin{aligned}
& \quad \begin{array}{l}
4 x-1<2(4 x-1)^{2} \\
2(4 x-1)^{2}-(4 x-1)>0 \\
(4 x-1)[2(4 x-1)-1]>0 \\
(4 x-1)(8 x-3)>0
\end{array} \\
& x<\frac{1}{4} \\
& \text { or } x>\frac{3}{8} \quad \frac{1}{4}
\end{aligned}
$$

Mult Choice

1) $B$
2) $(3) B 4)(5) C$
b) $C$
3) $A$
4) $B$ 9/A 10) $B$
a) Test for $n=1$.

$$
\begin{aligned}
\text { L.H.S } & =[2(1)-1]^{2} \\
& =1 \\
\text { R.HS } & =\frac{1}{3}(1)(3) \\
& =1
\end{aligned}
$$

-True for $n=1$
Assume true for $n=k$

$$
\begin{aligned}
& \text { true for } n=k \\
& 1^{2}+3^{2}+5^{2}+\cdots+(2 k-1)^{2}=\frac{k}{3}(2 k-1)(2 k+1)
\end{aligned}
$$

Test for $n=k+1$
that is we are required to prove

$$
\begin{aligned}
\text { LW. } & =\frac{k}{3}(2 k-1)(2 k+1)+(2 k+1)^{2} \quad \text { by indiction assumption } \\
& =(2 k+1)\left[\frac{k}{3}(2 k-1)+(2 k+1)\right] \\
& =(2 k+1)\left[\frac{2 k^{2}-k}{3}+(2 k+1)\right] \\
& =(2 k+1)\left[\frac{2 k^{2}-k+6 k+3}{3}\right] \\
& =(2 k+1)\left(\frac{1}{3}\right)\left(2 k^{2}+5 k+3\right) \\
& =\frac{1}{3}(2 k+1)(k+1)(2 k+3) \\
& =\frac{k+1}{3}(2 k+1)(2 k+3) \\
& =\text { R.H.S } \quad \therefore \text { proved by mathematical induction }
\end{aligned}
$$

well clone, some students didn't realise $S_{k+1}=S_{k}+T_{k+1}$ and tried to prove

$$
\begin{aligned}
& \text { re are required to prove } \\
& 1^{1^{2}+3^{2}+5^{2}+\cdots+(2 k-1)^{2}}+(2 k+1)^{2}=\frac{k+1}{3}(2 k+1)(2 k+3) \\
& S_{k+1}=T_{k+1}
\end{aligned}
$$

(i) tangent-secent theorem.
(ii) $P T=m \quad Q A=n \quad Q B=r \quad P T^{2}=P A \times P B$.
$P T=P Q$ (equal tangents from an external $\begin{aligned} & \text { poniti) }\end{aligned}$

$$
\begin{array}{rlrl}
\therefore P Q & =m \\
P A & =P Q+P A & P B & =P Q-Q B \\
& =m+n & & =m-r
\end{array}
$$

Only bree answer was required.
students needed to stater reason why $P T=P Q$

$$
\begin{aligned}
&=m+n \\
& \therefore m^{2}=(m+n)(m-r) \Rightarrow
\end{aligned} \frac{m x^{2}=n^{2}+m n-r m-r n}{m(n-r)}=r n \quad m=\frac{n r}{n-r}
$$



a) Let $u=\frac{x}{\sqrt{1-x^{2}}}$ Let $y=\tan ^{-1}\left(\frac{x}{\sqrt{1-x^{2}}}\right)$

$$
\begin{aligned}
& \begin{array}{l}
\frac{d y}{d x}
\end{array}=\frac{d y}{d u} \times \frac{d u}{d x} \\
& \frac{d y}{d u}=\frac{d}{d u}(\tan -1 u)=\frac{1}{1+u^{2}} \\
& \frac{d y}{d u}=\frac{1}{1+\frac{x^{2}}{1-x^{2}}}=\frac{1}{\frac{1-x^{2}+x^{2}}{1-x^{2}}}=1-x^{2} \\
& \frac{y}{\frac{d u}{d x}}=x^{-x} \times \frac{1}{2}\left(1-x^{2}\right)^{-\frac{1}{2}} \times-2 x+1\left(1-x^{2}\right)^{-\frac{1}{2}} \\
&=\frac{x^{2}}{\left(\sqrt{1-x^{2}}\right)^{3}}+\frac{1}{\sqrt{1-x^{2}}}=\frac{x^{2}+1-x^{2}}{\left(\sqrt{1-x^{2}}\right)^{3}} \\
& \frac{d y}{d x}=\left(1-x^{2}\right) \times \frac{1}{\left(\sqrt{1-x^{2}}\right)^{3}}=\sqrt{1-x^{2}}
\end{aligned}
$$ did not use the guan substitution which was required.

Note: If you use Product rule $v=x, v=\left(1-x^{2}\right)^{-\frac{1}{2}}$ For quotiontrule $v=x, \quad v=\left(1-x^{2}\right)^{\frac{1}{2}}$ students were mixing these up.
b)

$$
\begin{aligned}
& \int_{-\sqrt{3}}^{3} \frac{1}{3+x^{2}} d x=\frac{1}{\sqrt{3}}\left[\tan ^{-1} \frac{x}{\sqrt{3}}\right]_{-\sqrt{3}}^{3} \\
= & \frac{1}{\sqrt{3}}\left(\tan ^{-1} \frac{3}{\sqrt{3}}-\tan ^{-1} \frac{-\sqrt{3}}{3}\right) \\
= & \frac{1}{\sqrt{3}}\left(\tan ^{-1} \sqrt{3}-\tan ^{-1}(-1)\right) \\
= & \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) \\
= & \frac{7 \pi}{12 \sqrt{3}}\left(\operatorname{tor} \frac{7 \pi \sqrt{3}}{36}\right)
\end{aligned}
$$

- This question was well answered
- Some students did not knows the range of
$\tan ^{-1}$ to be $-\frac{\pi}{2}$ to $\frac{\pi}{2}$





