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PYMBLE LADIES' COLLEGE

YEAR 12

TRIAL HIGHER SCHOOL CERTIFICATE - 1995

3/4 UNIT MATHEMATICS

Time Allowed: 2 Hours

Plus 5 Minutes reading time

DIRECTIONS TO CANDIDATES:

* Attempt all questions.
* All questions are of equal value.
* All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
* Each question is to be started on a new page and labelled with your name and your teacher's name.
* Standard integrals are attached.
* Approved calculators may be used.

PLC 1995

Question 1 — **Marks**

(a) If $f(x) = x \sin^{-1} x$ evaluate $f'(\frac{\sqrt{3}}{2})$ 3

(b) (i) Two dice are tossed and the two numbers are added. State the probability that this total is 10 or greater. 3

(ii) Two dice are tossed eight times. Find the probability that a total of 10 or greater occurs:

(α) at least once

(β) exactly once

(give answers correct to 3 significant figures)

(c) (i) Use the factor theorem to factorise $p^3 + 2p + 12$ 6

(ii) $P(2p, p^2)$ is a point on the parabola $x^2 = 4y$

(α) show that the equation of the normal at P is $x + py = p^3 + 2p$

(β) Find the co-ordinates of P if this normal cuts the x axis at $(-12, 0)$

Question 2 (Start a new page)

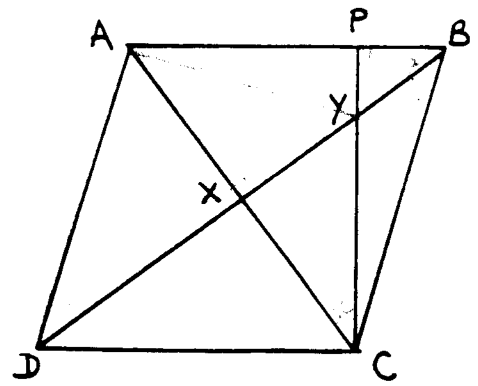
(a) Find $\int \frac{x\,dx}{1 + 2x}$ using the substitution $u = 1 + 2x$ 4

(b) (i) Evaluate $\int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$ 8

(ii) Use the substitution $x = \cos 2\theta$ to evaluate $\int_{\frac{1}{2}}^{1} \sqrt{\frac{1 - x}{1 + x}}\, dx$

Question 3 (Start a new page) **Marks**

(a) 5



ABCD is a rhombus whose diagonals intersect at X. The perpendicular CP from C to AB cuts BD at Y.

Prove that:

(i) points B, P, X, C are concyclic

(ii) points A, Y, C, D are concyclic

(b) (i) Sketch $y = 4\tan^{-1}x$ 7

(ii) By drawing a suitable line, l, on this graph, show that the equation

$$4\tan^{-1}x + x - 4 = 0$$

has a root close to $x = 1$.
Clearly state the equation of l.

(iii) Use Newton's method once to find a better approximation to this root. (Give your answer correct to 3 decimal places).

Question 4 (Start a new page)

(a) A population of bacteria is treated with a new drug. B, the number of bacteria present after t minutes is given by 6

$$B = 200 + 500\, e^{kt}$$

If initially the population decreases at the rate of 10 bacteria/minute

(i) evaluate k, and

(ii) find the time it takes for the population to halve.

(iii) Sketch the graph of B against t.

(b) A function f(x) is defined as 6
$f(x) = x^3 + 6x^2 + 12x - 14$ for $-1 \le x \le 4$

(i) Show that f(x) is an increasing function.

(ii) Explain why the inverse function $y = f^{-1}(x)$ exists and state its domain and range.

(iii) Find the gradient of $y = f^{-1}(x)$ at the point (5, 1) on it.

Question 5 (Start a new page) **Marks**

(a) Given $x(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^{r+1}$ 3

show by differentiation that

$$\binom{n}{o} + 2\binom{n}{1} + 3\binom{n}{2} + \ldots + (n+1)\binom{n}{n} = (n+2)\,2^{n-1}$$

(b) A particle is oscillating in simple harmonic motion on a straight line. Its displacement, x cm, from the origin 0 at time t seconds 9

is given by $x = 3 + 2\cos(\tfrac{1}{2}t + \frac{\pi}{3})$

(i) Find its initial position.

(ii) State the frequency and the amplitude of the oscillation.

(iii) Find the first two times that the particle moves through its centre of oscillation.

(iv) If its velocity at time t seconds is v cm/sec, express v^2 as a function of x.

Question 6 (Start a new page)

(a) A (t, e^t) and B $(-t, e^{-t})$ are two points on $y = e^x$, $(t > 0)$. The tangents at A and B intersect at an angle of 45° 6

(i) Show that $e^t - e^{-t} = 2$

(ii) Solve this equation for t

(b) When x cm from the origin, the acceleration of a particle moving on a straight line is given by 6

$$\frac{d^2x}{dt^2} = \frac{-4}{(x+1)^3}$$

It has an initial velocity of 2 cm/second at x = 0

(i) If its velocity is V cm/second, show that $V = \dfrac{2}{x+1}$

(ii) Find the time taken for the particle to reach x = 4

Question 7 (Start a new page) **Marks**

(a) Solve $\sin \alpha - 3 \cos \alpha = 3$ for $0° \le \alpha \le 360°$ 4

(b) 8

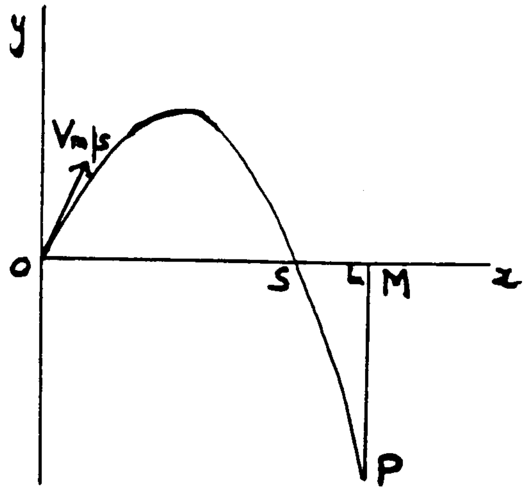


Diagram not to scale

A projectile is fired from a point O with initial speed of V m/s at an angle of elevation θ. If x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds later then $x = Vt \cos\theta$ and $y = Vt \sin\theta - \frac{1}{2}gt^2$ where g m/s^2 is the acceleration due to gravity.

The projectile falls to a point P below the level of O such that PM = OM.

(i) Prove that the time taken to reach P is $2V \dfrac{(\sin\theta + \cos\theta)}{g}$ seconds

(ii) Show that the distance OM is

$$\frac{V^2}{g}(\sin 2\theta + \cos 2\theta + 1) \textit{ metres}$$

(iii) If the horizontal range of the projectile level with O (i.e. OS) is r metres and OM = $\dfrac{4r}{3}$, $(r > 0)$

Prove that $\sin 2\theta - 3 \cos 2\theta = 3$

(iv) Hence by using part (a), find the value of θ.

(v) If the magnitude of the velocity of the projectile at P is KV m/s find the exact value of K

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

3 unit 1995 Solutions

1 (a) $f(x) = x \sin^{-1} x$

(3) $f'(x) = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$ 1

$f'\left(\frac{\sqrt3}{2}\right) = \sin^{-1}\frac{\sqrt3}{2} + \frac{\sqrt3/2}{\sqrt{1-3/4}}$

$= \pi/3 + \sqrt3$. 2

(b) (i) $\frac{1}{6}$ 1

(3)

(ii) (α) P(at least once) $= 1 - \left(\frac{5}{6}\right)^8$

$= 0.767$

(β) P(exactly once) $= {}^8C_1 \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^7$

$= 0.372$

(6) (c) (i) $f(p) = p^3 + 2p + 12$

$f(-2) = -8 - 4 + 12 = 0$

$\therefore (p+2)$ is a factor

$f(p) = (p+2)(p^2 - 2p + 6)$

(ii) $x = 2p \qquad y = p^2$

$\frac{dx}{dp} = 2 \qquad \frac{dy}{dp} = 2p$

$\therefore \frac{dy}{dx} = p$

grad. of normal $\therefore -\frac{1}{p}$

(x)

Eqn of normal: $y - p^2 = -\frac{1}{p}(x - 2p)$

$$py - p^3 = -x + 2p$$

1 $$x + py = p^3 + 2p$$

$(-12, 0)$ lies on normal

$$-12 = p^3 + 2p$$

$$p^3 + 2p + 12 = 0$$

2 $\therefore p = -2$ is only soln

1 $$P(-4, 4)$$

2. a) $\displaystyle\int \frac{x\,dx}{1+2x}$ $\qquad u = 1+$ ... $\quad du = 2d$...

(4)

$$= \int \frac{u-1}{4u}\,du \qquad 1$$

$$= \frac{1}{4}\int\left(1-\frac{1}{u}\right)du \qquad 1$$

$$= \frac{1}{4}\left(u - \ln u\right) + c \qquad 1$$

$$= \frac{1}{4}(1+2x) - \frac{1}{4}\ln(1+2x) + c$$

(b) (i) (4) $\displaystyle\int_0^{\pi/6} \sin^2\theta\,d\theta$

$$= \frac{1}{2}\int_0^{\pi/6}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/6}$$

$$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

(ii) (4) $\displaystyle\int_{1/2}^{1}\sqrt{\frac{1-x}{1+x}}\,dx$

$x = \cos$ ...
$dx = -2\sin 2\theta$ ...
$x=1 \quad \theta = 0$
$x = \frac{1}{2} \quad \theta =$ ... $\qquad 1$

$$= \int_{\pi/6}^{0}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\cdot -2\sin 2\theta\,d\theta \qquad 1$$

$$= \int_{\pi/6}^{0}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\; -2\sin 2\theta\,d\theta$$

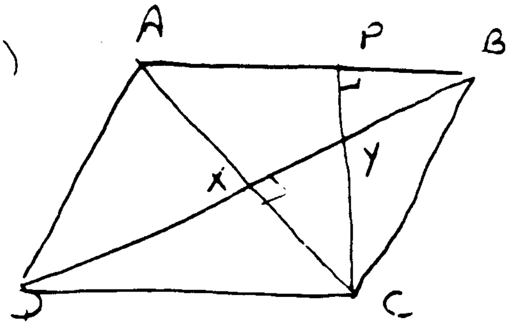
$$= 4\int_0^{\pi/6}\frac{\sin\theta}{\cos\theta}\times\sin\theta\cos\theta\,d\theta \qquad 1$$

$$= 4\int_0^{\pi/6}\sin^2\theta\,d\theta \qquad 1$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2} \qquad 1$$

3 (a)

5

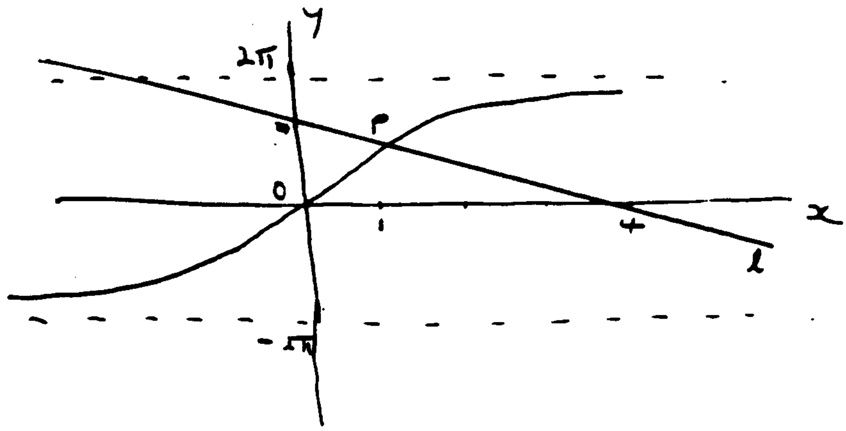


Proof (i) $\angle BXC = 90°$ (diags of rhombus are $\perp$)
$\angle BPC = 90°$ (given)
$\therefore$ pts B, P, X, C are concyclic (equal $\angle^s$ on BC & on same side of

(ii) $AB = AD$ (sides of rhombus)
$\therefore \angle ABD = \angle ADB$ (base $\angle^s$ of isosceles ΔABD)
$\angle PBX = \angle PCX$ (angles in same segment of circle BPXC)
$\therefore \angle ADY = \angle ACY$

$\therefore$ pts A, Y, C, D are concyclic (equal $\angle^s$ on AY & on same side

b)



1 $y = 4\tan^{-1}x$ + asympt

1 line l + scale

l is line $y = 4 - x$ 1

P, point of intersection, has x value close to

$$f(x) = 4\tan^{-1}x + x - 4$$

$$f'(x) = \frac{4}{1+x^2} + 1 \qquad 1$$

$$f(1) = 4\tan^{-1}1 - 3$$

$$= \pi - 3 \qquad 1$$

$$f'(1) = 3 \qquad 1$$

$$x_1 = 1$$

$$x_2 = 1 - \frac{\pi - 3}{3}$$

$$= 0.953. \ (3dp) \qquad 1$$

4. a) $B = 200 + 500e^{kt}$

6

$t=0 \quad \frac{dB}{dt} = -10$.

$\frac{dB}{dt} = 500ke^{kt}$ 1

$\therefore -10 = 500k$

$k = -\frac{1}{50}$. 1

b) Initially $B = 700$.

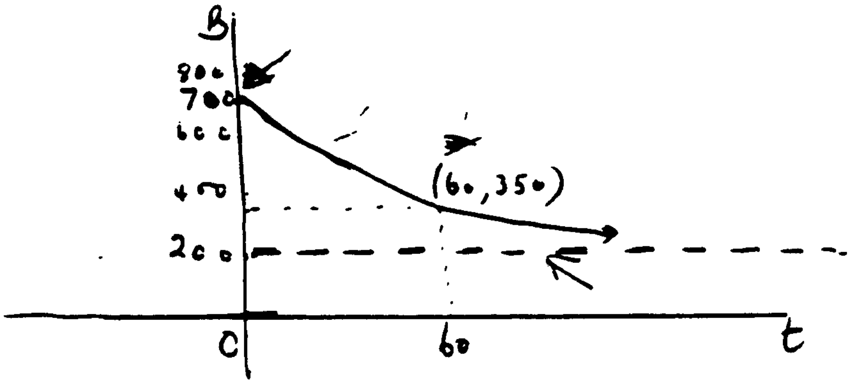
When $B = 350$.

$350 = 200 + 500e^{-\frac{t}{50}}$ 1

$e^{-\frac{t}{50}} = \frac{150}{500}$

$t = 60.2$ 1

It takes 60.2 minutes.



2

$t \geq 0$
scale
asymptote

(b) $f(x) = x^3 + 6x^2 + 12x - 14 \quad , \quad -1 \le x \le 4$

6

$f'(x) = 3x^2 + 12x + 12$
$= 3(x+2)^2$ 1

Since $(x+2)^2 > 0$ for $-1 \le x \le 4$ ...
$f'(x)$ is positive
$\therefore f(x)$ is an increasing fn 1

(ii) Since $f(x)$ is an increasing fn
it is a 1-1 fn
$\therefore$ it has an inverse.

$f^{-1}(x) \quad D: -21 \le x \le 194$ 1
$R: -1 \le y \le 4$ 1

(iii) $f'(1) = 27$

Grad of inverse fn at $x = 5$ is $\frac{1}{27}$

5(a) (3) $$x(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^{r+1}$$

≤4

$$= \binom{n}{0}x + \binom{n}{1}x^2 + \binom{n}{2}x^3 + \dots + \binom{n}{n}x^{n+1}$$

Differentiate

1 $$(1+x)^n + nx(1+x)^{n-1} = \binom{n}{0} + 2\binom{n}{1}x + 3x^2\binom{n}{2} + \dots (n+1)$$

Let $x = 1$

1 $$2^n + n.2^{n-1} = \binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} + \dots + (n+1)\binom{n}{n}$$

$$2^{n-1}(2+n) = \binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} + \dots + (n+1)$$

(b) (9) $$x = 3 + 2\cos\left(\tfrac{1}{2}t + \tfrac{\pi}{3}\right)$$

(i) Initially $t = 0$

$$x = 3 + 2\cos\tfrac{\pi}{3}$$
$$= 4$$

1 1

1 Particle is 4 cm to right of O

1 (ii) Frequency $= \frac{1}{4\pi}$ osc/sec

1 Amplitude $= 2$ cm.

(iii) Centre of oscillation $x = 3$

$$3 + 2\cos\left(\tfrac{1}{2}t + \tfrac{\pi}{3}\right) = 3$$

$$2\cos\left(\tfrac{1}{2}t + \tfrac{\pi}{3}\right) = 0.$$

$$t = \tfrac{\pi}{3} \text{ and } \tfrac{7\pi}{3}$$

1 (iv) $$v = -\sin\left(\tfrac{1}{2}t + \tfrac{\pi}{3}\right)$$

$$v^2 = \sin^2\left(\tfrac{1}{2}t + \tfrac{\pi}{3}\right)$$

1 $$= 1 - \cos^2\left(\tfrac{1}{2}t + \tfrac{\pi}{3}\right)$$

1 $$= 1 - \frac{(x-3)^2}{4}$$

6 (a) $y = e^x$

(b) $\frac{dy}{dx} = e^x$

at A, grad $= e^t = m_1$

at B, grad $= e^{-t} = m_2$

$\tan \theta = 1$ $\qquad \theta$ is angle between tang

$\tan \theta = \frac{m_1 - m_1}{1 + m_1 m_2}$

$1 = \frac{e^t - e^{-t}}{1 + 1}$

2 $\quad \therefore 2 = e^t - e^{-t}$

$e^t - \frac{1}{e^t} = 2.$

1 $\quad e^{2t} - 2e^t - 1 = 0.$

1 $\quad e^t = \frac{2 \pm \sqrt{4+4}}{2}$

$= 1 \pm \sqrt{2}.$

1 $\quad e^t = 1 + \sqrt{2}$ $\quad$ since $e^t > 0.$

1 $\quad \therefore t = \ln(1+\sqrt{2})$