Name: ______________________________

Teacher's Name: __________________

Mr Keanan-Brown
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PYMBLE LADIES' COLLEGE

2000 TRIAL H.S.C. EXAMINATION

MATHEMATICS

3/4 UNIT

Time Allowed: 2 hours
plus 5 minutes reading time

INSTRUCTIONS TO CANDIDATES:

1. All questions must be attempted.
2. All necessary working must be shown.
3. Start each question on a new page.
4. Put your name and your teachers' name on every sheet of paper.
5. Marks may be deducted for careless or untidy work.
6. Only approved calculators may be used.
7. DO NOT staple different questions together.
8. Hand this question paper in with your answers.
9. All rough working paper must be attached to the back of the last question.
10. All questions are of equal value.

There are seven (7) questions in this paper.

Question 1

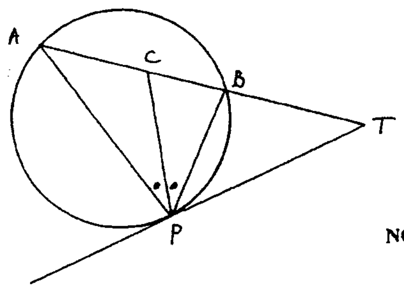
		Marks
(a)	Find $\frac{d}{dx}(\sec 2x)$	1
(b)	If $\log_m a = 0\cdot7$, $\log_m b = 0\cdot3$, $\log_m c = 0\cdot2$, find the value of $\log_m \frac{\sqrt{a}}{b^2c^3}$	2
(c)	Find the exact value of $\int_e^{e^2} \frac{dx}{x\ln x}$ (you may use the substitution $u = \ln x$ if you wish).	3
(d)	Evaluate $\lim_{x\to 0} \frac{\sin 2x}{5x}$	1
(e)	Find $\int \sin x \cos x \, dx$	2
(f)	(i) Sketch on the same diagram, $y = \frac{1}{x}$ and $y = \sqrt{x}$	3
	(ii) Hence, or otherwise, solve $\frac{1}{x} \geq \sqrt{x}$	

Question 2 **(Start a new page)** **Marks**

(a) Use the substitution $u = \sqrt{x}$ to evaluate $\int_0^3 \frac{dx}{\sqrt{x}(x+1)}$ 4

(b) Evaluate $\int_0^{\frac{1}{6}} \frac{3dx}{\sqrt{1-9x^2}}$ 3

(c) 5



NOT TO SCALE

A chord AB of a circle is produced to a point T. From T, a tangent is drawn, touching the circle at P. C is a point on AB such that CP bisects $\angle APB$.

(i) Copy the diagram onto your writing paper

(ii) Prove that $TP = TC$, giving reasons.

(iii) If $AT = 9$ and $TB = 4$, find TP and hence AC.

Units are in centimetres

Question 3 **(Start a new page)** **Marks**

(a) Evaluate $\int_0^{\frac{\pi}{6}} \sin^2 2x\, dx$ 4

(b) The area of a circle is A cm^2 and the circumference is C cm at time t seconds. 4

If the area is increasing at a rate of 4 cm^2/s, find the rate at which the circumference is increasing when the radius is 2 cm.

(c) (i) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$ where α is in radians. 4

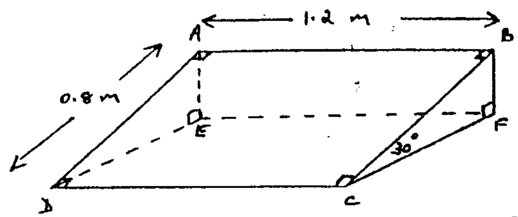
(ii) Hence, or otherwise, find the general solution of the equation $\sqrt{3}\cos\theta - \sin\theta = 1$

Question 4 **(Start a new page)** **Marks**

(a) Sketch $y = \tan^{-1} x$ 3

What is the maximum value that the gradient of the inverse tangent curve can have? Give reasons for your answer.

(b) 3



NOT TO SCALE

An architect's desk has a sloping work surface which measures 1.2 metres by 0.8 metres, as shown. The sloping work surface $ABCD$ makes an angle of 30° with the horizontal $EFCD$.

Find (i) the length of BF

(ii) the length of AC, correct to 2 decimal places

(iii) the angle that the diagonal AC makes with the horizontal, giving your answer to the nearest degree.

(c) The tangent at $P(6t, 3t^2)$ on the parabola $x^2 = 12y$ cuts the x, y axes at A, B respectively. 0 is the origin and C is the point such that $0ACB$ is a rectangle. 6

Find (i) the equation of the tangent at P

(ii) the coordinates of A, B and C

(iii) the locus of C as P moves on the parabola.

Question 5 **(Start a new page)** **Marks**

(a) The velocity v ms^{-1} of a particle moving in simple harmonic motion along the x axis is given by $v^2 = 6 + 4x - 2x^2$ 8

(i) Between which two points is the particle oscillating?

(ii) What is the amplitude of the motion?

(iii) Find the acceleration of the particle in terms of x.

(iv) Write down the period of the oscillation.

(v) What is the maximum speed of the particle?

(b) Prove, by mathematical induction, that

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}).....(1 - \frac{1}{n^2}) = \frac{n+1}{2n} \text{ for all } n \geq 2.$$ 4

Question 6 **(Start a new page)** **Marks**

(a) The roots of the equation $x^3 - 6x^2 + 3x + k = 0$ are consecutive terms of an arithmetic sequence. Find the value of k. **5**

(b) Consider the function $f(x) = \dfrac{x-4}{x-2}$ for $x > 2$ **7**

(i) Show that $f(x)$ is an increasing function for all values of x in its domain.

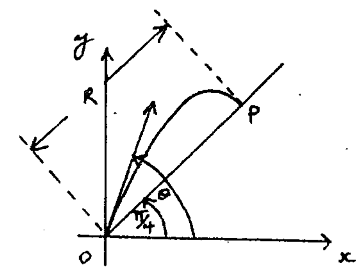
(ii) Explain briefly why the inverse function $f^{-1}(x)$ exists.

(iii) State the domain and range of $f^{-1}(x)$

(iv) Find the gradient of the tangent to $y = f^{-1}(x)$ at the point (0, 4) on it.

Question 7 **(Start a new page)** **Marks**

12



A cat can jump with a velocity of $5ms^{-1}$. It is standing at O, at the bottom of a slope inclined at $\dfrac{\pi}{4}$ to the horizontal.

The cat jumps at an angle of θ to the horizontal, where $\dfrac{\pi}{4} < \theta < \dfrac{\pi}{2}$.

The equations of motion of the cat are $\ddot{x} = 0, \ddot{y} = -10$

(i) Use calculus to show that the coordinates of the cat's position at time t seconds are given by $x = 5t\cos\theta$ and $y = -5t^2 + 5t\sin\theta$.

(ii) The cat lands at P, where the length of $OP = R$ metres. Explain why $x = y = \dfrac{R}{\sqrt{2}}$ at P.

(iii) Show that $R = 5\sqrt{2}(\cos\theta\sin\theta - \cos^2\theta)$

(iv) By differentiation, find the value of θ for the cat to achieve maximum distance R.

(v) The cat had seen a mouse sitting 1.8m up the slope from 0. If the cat attains maximum distance R, will it need to run up the slope or down the slope in its attempt to catch the mouse (assuming the mouse remains stationary)? Justify your answer.

[Note: No animal was harmed in the writing of this question - the mouse escaped].

END OF PAPER

$$\frac{d}{dx}(\sec 2x) = 2\sec 2x \tan 2x \qquad \boxed{1}$$

$$\log_m \frac{\sqrt{a}}{b^2c^3} = \frac{1}{2}\log_m a - 2\log_m b - 3\log_m c$$

$$= \frac{1}{2}\times 0.7 - 2\times 0.3 - 3\times 0.2 \qquad \boxed{2}$$

$$= -0.85$$

$$\int_e^{e^2} \frac{dx}{x\ln x} = \Big[\ln(\ln x)\Big]_e^{e^2}$$

$$= \ln(\ln e^2) - \ln(\ln e) \qquad \boxed{3}$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

Alt. Method:

$$\int_e^{e^2} \frac{dx}{x\ln x} = \int_1^2 \frac{du}{u}$$

$$= \Big[\ln u\Big]_1^2$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

If $u = \ln x$,

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

When $x = e$, $u = 1$

When $x = e^2$, $u = 2$

$$\lim_{x\to 0} \frac{\sin 2x}{5x} = \frac{2}{5} \lim_{2x\to 0} \frac{\sin 2x}{2x}$$

$$= \frac{2}{5}\times 1$$

$$= \frac{2}{5} \qquad \boxed{1}$$

C

$$= \frac{1}{2}\int \sin 2x \, dx \qquad \boxed{2}$$

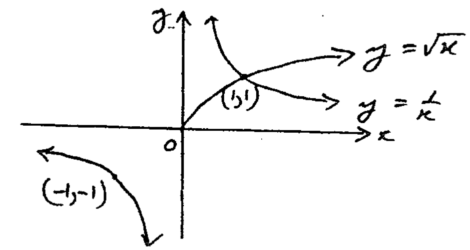
$$= -\frac{1}{4}\cos 2x + c$$

Alt. methods:

$$\int \sin x \cos x \, dx \qquad \text{or} \qquad \int \sin x \cos x \, dx$$

$$= \frac{1}{2}\sin^2 x + c_1 \qquad\qquad = -\frac{1}{2}\cos^2 x + c_2$$

(4) (i)



$\boxed{3}$

(ii) $\frac{1}{x} \geq \sqrt{x}$ when $0 < x \leq 1$

$$= \int_0^{\sqrt{3}} \frac{2\,du}{u^2+1}$$ ✓

$$= 2\left[\tan^{-1} u\right]_0^{\sqrt{3}}$$ ✓

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\therefore 2\,du = \frac{dx}{\sqrt{x}}$$

When $x=0$, $u=0$
When $x=3$, $u=\sqrt{3}$ ✓

$$= 2\left[\tan^{-1}\sqrt{3} - \tan^{-1} 0\right]$$

$\boxed{4}$

$$= \frac{2\pi}{3}$$ ✓

b) $$\int_0^{\frac{1}{6}} \frac{3\,dx}{\sqrt{1-9x^2}}$$

$$= \int_0^{\frac{1}{6}} \frac{3\,dx}{\sqrt{9\left(\frac{1}{9}-x^2\right)}}$$

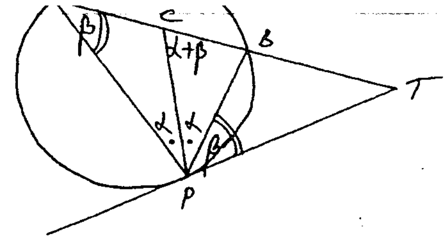
$$= \int_0^{\frac{1}{6}} \frac{dx}{\sqrt{\frac{1}{9}-x^2}}$$ ✓

$\boxed{3}$

$$= \left[\sin^{-1} 3x\right]_0^{1/6}$$ ✓

$$= \sin^{-1}\tfrac{1}{2} - \sin^{-1} 0$$

$$= \frac{\pi}{6}$$ ✓



(ii) Let $\angle APC = \angle CPB = \alpha$ and let $\angle BPT = \beta$

Now, $\angle BAP = \angle BPT = \beta$ (angle between tangent & chord equals angle in alternate segment)

$\therefore \angle BCP = \angle CPA + \angle CAP$ (exterior angle of ΔCAP) ✓

$= \alpha + \beta$

Also, $\angle CPT = \angle CPB + \angle BPT$

$= \alpha + \beta$

$\therefore \angle BCP = \angle CPT$ (both $(\alpha+\beta)$) ✓

$\therefore \Delta TCP$ is isosceles (base angles equal)

$\therefore TP = TC$ (equal sides of isos. Δ)

(iii) $PT^2 = AT \cdot TB$ $\boxed{5}$

$= 9 \cdot 4$

$= 36$

$\therefore PT = 6$ ✓

$\therefore TC = 6$ ($TP = TC$, from part (ii))

Now, $AC = AT - TC$

$= 9 - 6$

$\therefore AC = 3$ ✓

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}(1-\cos 4x)\,dx$$ ✓

$$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{6}}$$ ✓

$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3}\right]$$ ✓ $\boxed{4}$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2}\right]$$

$$= \frac{\pi}{12} - \frac{\sqrt{3}}{16}$$ ✓

(b) $A = \pi r^2$, $C = 2\pi r$, $\frac{dA}{dt} = 4$

$\therefore \frac{dA}{dr} = 2\pi r$

Now, $\frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt}$

$$\therefore \frac{dr}{dt} = \frac{\frac{dA}{dt}}{\frac{dA}{dr}}$$

$$= \frac{4}{2\pi r}$$ ✓✓

$\boxed{4}$

Also, $C = 2\pi r$

$\therefore \frac{dC}{dr} = 2\pi$

Now, $\frac{dC}{dt} = \frac{dC}{dr}\cdot\frac{dr}{dt}$

$$= 2\pi\cdot\frac{4}{2\pi\cdot 2}$$ ✓ when $r = 2$

$$= 2$$ ✓

$\therefore$ Circumf. is increasing at 2 . m/s.

$$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

$$\therefore \left.\begin{aligned} R\cos\alpha &= \sqrt{3} \\ R\sin\alpha &= 1 \end{aligned}\right\}$$

$R^2\sin^2\alpha + R^2\cos^2\alpha = 1^2 + \sqrt{3}^2$, $\quad \frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{3}}$

$R^2(\sin^2\alpha + \cos^2\alpha) = 1 + 3$

$R^2 = 4$ $\quad \tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore R = 2$

$\therefore \alpha = \pi/6$

$\therefore \sqrt{3}\cos\theta - \sin\theta = 2\cos(\theta + \pi/6)$ ✓✓

(ii) $\sqrt{3}\cos\theta - \sin\theta = 1$ $\boxed{4}$

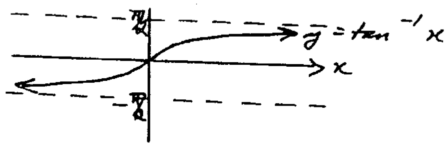
$\therefore 2\cos(\theta + \pi/6) = 1$

$\cos(\theta + \pi/6) = \frac{1}{2}$

$\theta + \pi/6 = 2n\pi \pm \pi/3$ ✓

$$\left.\begin{aligned} \therefore\ \theta &= 2n\pi + \pi/6 \\ \underline{\underline{\text{or}}}\ \ \theta &= 2n\pi - \pi/2 \end{aligned}\right\}$$ ✓

(4)



✓

$y = \tan^{-1} x$ [3]

$\frac{dy}{dx} = \frac{1}{1+x^2}$

Now, $1+x^2 \geq 1$ for all real values of x ✓

$\therefore \frac{1}{1+x^2} \leq 1$

$\therefore$ Max. value of gradient $= 1$ ✓

(b) (i) In $\triangle BCF$, $\sin 30^\circ = \frac{BF}{BC}$

$\therefore BF = BC \sin 30^\circ$

$= 0.8 \times \frac{1}{2}$ (as $BC = AD$)

$\therefore$ length of $BF = 0.4$ m ✓

(ii) In $\triangle ABC$, $AC^2 = (1.2)^2 + (0.8)^2$

$\therefore AC = \sqrt{2.08} = 1.4422\ldots\ldots$

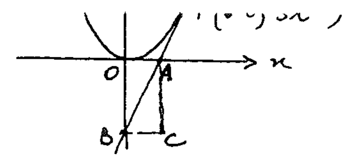
$\therefore$ length of $AC = 1.44$ m (correct to 2 d.p.) ✓

(iii) Required angle is $\angle ACE$ [3]

$\sin \angle ACE = \frac{AE}{AC}$

$= \frac{0.4}{1.44}$ (as $AE = BF$)

$\therefore \angle ACE = 16^\circ$ (to nearest degree) ✓



(i) $y = \frac{1}{12} x^2$

$\frac{dy}{dx} = \frac{1}{6} x$

$\therefore$ Grad. of tangent at $P = \frac{1}{6} \cdot 6t = t$

$\therefore$ Eqn. of tangent at P is

$y - 3t^2 = t(x - 6t)$

$y - 3t^2 = tx - 6t^2$

$\therefore tx - y - 3t^2 = 0$ ✓✓

(ii) Cuts x axis when $y = 0$ [6]

$\therefore A$ is $(3t, 0)$ ✓

Cuts y axis when $x = 0$

$\therefore B$ is $(0, -3t^2)$ ✓

As $OACB$ is a rectangle,

C is $(3t, -3t^2)$ ✓

(iii) $x = 3t$, $y = -3t^2$

$\therefore t = \frac{x}{3}$

$y = -3\left(\frac{x}{3}\right)^2$

$y = -\frac{x^2}{3}$ ✓

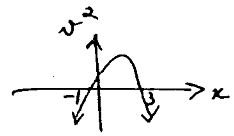
$\therefore x^2 = -3y$ is locus of C.

(i) For motion to exist, $v^2 \geq 0$

$$6 + 4x - 2x^2 \geq 0$$

$$2(3-x)(1+x) \geq 0$$

$$\therefore -1 \leq x \leq 3$$



i.e. Particle is oscillating between $x=-1$ and $x=3$.

(ii) Amplitude of motion = 2 metres

(iii) Acceleration $= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$$= \frac{d}{dx}\left(3+2x-x^2\right)$$

$$= 2-2x$$

$$= -2(x-1)$$

(iv) Period $= \frac{2\pi}{n}$

$$= \frac{2\pi}{\sqrt{2}}$$

$$= \sqrt{2}\,\pi \text{ seconds}$$

$\boxed{8}$

(v) Max. speed occurs as particle passes through centre of motion $x=1$

$$v^2 = 6+4-2 = 8$$

$$v = \pm\sqrt{8}$$

$\therefore$ Max. speed $= 2\sqrt{2}$ ms^{-1}

Alt. method :

$$v^2 = 6+4x-2x^2$$

$$= -2\left[x^2-2x-3\right]$$

$$= -2\left[(x^2-2x+1)-3-1\right]$$

$$= -2\left[(x-1)^2-4\right]$$

$$= -2(x-1)^2+8$$

$\therefore v^2_{max} = 8$

$\therefore$ Max. speed $= 2\sqrt{2}$ ms^{-1}

When $n=2$, LHS $= 1-\frac{1}{2^2}$ $= 1-\frac{1}{4}$ $= \frac{3}{4}$, RHS $= \frac{2+1}{2.2}$ $= \frac{3}{4}$

$\therefore$ True for $n=2$

Assume true for $n=k$

i.e. Assume that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$

When $n=k+1$,

$$\underbrace{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{k^2}\right)}\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \left(\frac{k+1}{2k}\right)\cdot\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$= \left(\frac{k+1}{2k}\right)\cdot\left(\frac{k^2+2k}{(k+1)^2}\right)$$

$\boxed{4}$

$$= \left(\frac{\cancel{k+1}}{2\cancel{k}}\right)\cdot\left(\frac{\cancel{k}(k+2)}{(k+1)^{\cancel{2}}}\right)$$

$$= \frac{k+2}{2(k+1)}$$

$\therefore$ Statement is true for $n=k+1$ if true for $n=k$.

As true for $n=2$, it is true for $n=2+1=3$

As true for $n=3$, it is true for $n=3+1=4$

etc.

$\therefore$ True for all $n \geq 2$.

sequence, let roots be $\alpha - d, \alpha, \alpha + d$.

Sum of roots $= -\frac{b}{a}$

$(\alpha - d) + \alpha + (\alpha + d) = 6$

$3\alpha = 6$

$\therefore \alpha = 2$

$\boxed{5}$

As $\alpha = 2$ is one of the roots, it must satisfy equation

$\therefore 2^3 - 6.2^2 + 3.2 + k = 0$

$8 - 24 + 6 + k = 0$

$\therefore k = 10$

(b) $f(x) = \frac{x-4}{x-2}$ for $x > 2$

(i) $f'(x) = \frac{(x-2).1 - (x-4).1}{(x-2)^2}$

$= \frac{2}{(x-2)^2}$

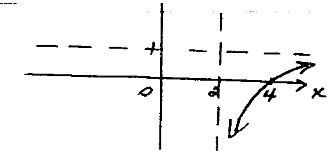
Now, $(x-2)^2 \geq 0$ for all real x

$\therefore f'(x) > 0$ for all x in domain $x > 2$

$\therefore f(x)$ is an increasing function

(ii) As $f(x)$ is an increasing function it is a one-one function

$\therefore$ The inverse function $f^{-1}(x)$ exists.



Domain is $x > 2$

Range is $y < 1$

$\therefore$ For $f^{-1}(x)$, Domain is $x < 1$

Range is $y > 2$

$\boxed{7}$

(iv) $f'(x) = \frac{2}{(x-2)^2}$ (from part (i))

$f'(4) = \frac{2}{(4-2)^2} = \frac{1}{2}$

$\therefore$ Grad. of tangent to $y = f^{-1}(x)$ at the point $(0,4) = 2$.

Alt. Method:

Inverse is $x = \frac{y-4}{y-2}$

$xy - 2x = y - 4$

$xy - y = 2x - 4$

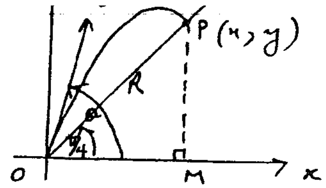
$\therefore y = \frac{2(x-2)}{x-1}$

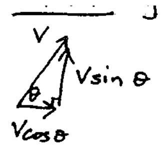
$\frac{dy}{dx} = \frac{(x-1).2 - (2x-4).1}{(x-1)^2}$

$= \frac{2}{(x-1)^2}$

$= 2$ when $x = 0$

$\therefore$ Gra of tangent to inverse fn. $= 2$.





i) **Horizontal motion:**

$\ddot{x} = 0$

Integrating w.r.t. t,

$\dot{x} = c_1$

When $t = 0$, $\dot{x} = V\cos\theta$

$\therefore c_1 = 5\cos\theta$

$\therefore \dot{x} = 5\cos\theta$

Integrating w.r.t. t,

$x = 5t\cos\theta + c_2$

When $t = 0$, $x = 0$

$\therefore c_2 = 0$

$\therefore x = 5t\cos\theta$

Vertical motion:

$\ddot{y} = -10$

Integrating w.r.t. t,

$\dot{y} = -10t + c_3$

When $t = 0$, $\dot{y} = V\sin\theta$

$\therefore c_3 = 5\sin\theta$

$\therefore \dot{y} = -10t + 5\sin\theta$

Integrating w.r.t. t,

$y = -5t^2 + 5t\sin\theta + c_4$

When $t = 0$, $y = 0$

$\therefore c_4 = 0$

$\therefore y = -5t^2 + 5t\sin\theta$ $\boxed{2}$

In ΔPOM, $\cos\frac{\pi}{4} = \frac{x}{R}$ and $\sin\frac{\pi}{4} = \frac{y}{R}$

$\therefore x = R\cos\frac{\pi}{4}$ and $y = R\sin\frac{\pi}{4}$

$= R \times \frac{1}{\sqrt{2}}$ $\qquad = R \times \frac{1}{\sqrt{2}}$ $\boxed{1}$

$= \frac{R}{\sqrt{2}}$ $\qquad = \frac{R}{\sqrt{2}}$

i.e. $x = y = \frac{R}{\sqrt{2}}$

$\sin\theta) = 0$

$\therefore t = 0$ (at O) or $t = \sin\theta - \cos\theta$ (at P)

Now, $x = 5t\cos\theta = \frac{R}{\sqrt{2}}$

$\therefore R = 5\sqrt{2}\,t\cos\theta$ $\boxed{2}$

$\therefore R = 5\sqrt{2}(\sin\theta - \cos\theta)\cos\theta$

$\therefore R = 5\sqrt{2}(\sin\theta\cos\theta - \cos^2\theta)$

(iv) $\frac{dR}{d\theta} = 5\sqrt{2}\left[\cos\theta.\cos\theta + \sin\theta.-\sin\theta - 2\cos\theta.-\sin\theta\right]$

$= 5\sqrt{2}\left[\cos^2\theta - \sin^2\theta + 2\sin\theta\cos\theta\right]$

$= 5\sqrt{2}\left[\cos 2\theta + \sin 2\theta\right]$

When $\frac{dR}{d\theta} = 0$, $\sin 2\theta = -\cos 2\theta$

$\tan 2\theta = -1$ $\boxed{6}$

$\therefore 2\theta = 3\pi/4$ (as $\pi/2 < 2\theta < \pi$)

$\therefore \theta = 3\pi/8$ (as $\pi/4 < \theta < \pi/2$)

When $\theta < 3\pi/8$, $\frac{dR}{d\theta} > 0$

When $\theta > 3\pi/8$, $\frac{dR}{d\theta} < 0$

$\therefore$ MAX. Distance R when $\theta = 3\pi/8$

(v) When $\theta = 3\pi/8$, $R = 5\sqrt{2}\left(\sin 3\pi/8 \cos 3\pi/8 - \cos^2 3\pi/8\right)$

$\therefore R = 1.464\ldots$ $\boxed{1}$

As $R < 1.8$, cat will need to run up the slope.