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Name : ______________

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Pymble Ladies' College

Year 12

Extension I Mathematics Trial

11th August 2003

Time allowed : 2 hours plus 5 minutes reading time

Marking guidelines : The marks for each part are indicated beside the question

Instructions :

- **All questions should be attempted**
- **All necessary working must be shown**
- **Start each question on a new page**
- **Put your name and your teacher's name on each page**
- **Marks may be deducted for careless or untidy work**
- **Only approved calculators may be used**
- **All questions are of equal value**
- **Diagrams are not drawn to scale**
- **A standard integral sheet is attached**
- **DO NOT staple different questions together**
- **All rough working paper must be attached to the end of the last question**
- **Staple a coloured sheet of paper to the back of each question**
- **Hand in this question paper with your answers**
- **There are seven (7) questions and eight (8) pages in this paper**

Question 1

a) If P is the point (-3, 5) and Q is the point (1, -2), find the coordinates of the point R which divides the interval PQ externally in the ratio of 3 : 2. **2**

b) When $(x+3)(x-2)+2$ is divided by $x-k$, the remainder is k^2. Find the value of k. **2**

c) Solve $\dfrac{x}{x-3} \geq 1$. **3**

d) Find the general solution of $\sin\theta = \cos\theta$. **2**

e) Find the exact value of $\int_0^{\frac{\pi}{8}} 2\sin^2 x \, dx$. **3**

Question 2 (Start a new page)

a) i) Show that $x^2+4x+13=(x+2)^2+9$. **1**

ii) Hence find $\int \frac{1}{x^2+4x+13}\,dx$. **2**

b) A stone is projected from the ground with a velocity of $20\,ms^{-1}$ at an angle of 30°. Assume that $\ddot{x}=0$ and $\ddot{y}=-10$.

i) Prove that :

(1) $x=10\sqrt{3}t$ **2**

(2) $y=-5t^2+10t$ **2**

ii) Hence find the :

(1) time of flight **1**

(2) horizontal range **1**

(3) greatest height reached **1**

(4) velocity of the particle after $1\frac{1}{2}$ seconds **2**

Question 3 (Start a new page)

a) Evaluate $\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx$ using the substitution that $u=x^2+1$. **3**

b) i) Express $\cos\theta+\sqrt{3}\sin\theta$ in the form $r\cos(\theta-\alpha)$ where $r>0$ and $0<\alpha<\frac{\pi}{2}$. **2**

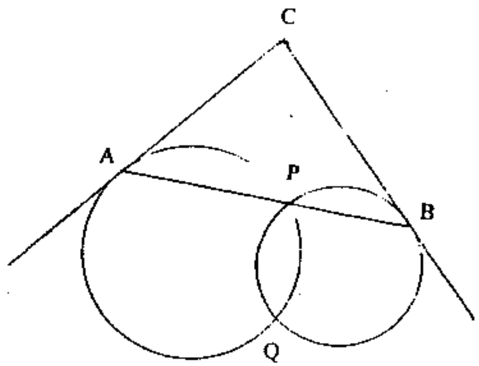
ii) Hence solve $\cos\theta+\sqrt{3}\sin\theta=1$ for $-2\pi\le\theta\le2\pi$. **2**

c) Given $f(x)=\frac{x-1}{x+2}$.

i) Write an expression for the inverse function $f^{-1}(x)$. **1**

ii) Write down the domain and range of $f^{-1}(x)$. **1**

d) Two circles meet at P and Q. A line APB is drawn through P and the tangents at A and B meet at C. Prove that ACBQ is a cyclic quadrilateral. **3**



Question 4 (**Start a new page**)

a) Assume that the rate at which a body warms in air is proportional to the difference between its temperature T and the constant temperature A of the surrounding air. This rate can be expressed by the differential equation $\frac{dT}{dt} = -k(T-A)$ where t is the time in minutes and k is a constant.

i) Show that $T = A - Ce^{-kt}$ is a solution of the differential equation where C is a constant. **1**

ii) A body warms from 3°C to 10°C in 15 minutes. The air temperature around the body is 30°C. Find the temperature of this body after a further 15 minutes have elapsed. Answer correct to the nearest $^{\circ}$C. **4**

iii) With the aid of the graph of T against t, explain the behaviour of T as t becomes large. **1**

b) The acceleration of a particle moving in a straight line is given by $\ddot{x} = -4x+8$ where x is the displacement, in metres, from the origin O and t is the time in seconds.

i) Show that the particle is moving in simple harmonic motion. **1**

ii) Write down the centre of motion. **1**

iii) Show that $v^2 = 20+16x-4x^2$ given, that the particle is initially at rest at $x=5$. **2**

iv) Write down the amplitude of the motion. **1**

v) Find the maximum speed of the particle. **1**

Question 5 (**Start a new page**)

a) Consider the curve $f(x) = \ln(x+1)$. Find the gradient(s) of the possible tangent(s) to $f(x)$ which makes an angle of 45° with the tangent to $f(x)$ at the point where $x=1$. **3**

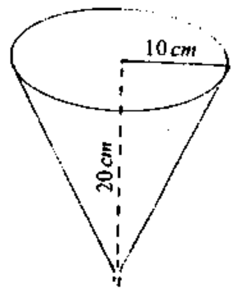
b) i) Use the table of standard integrals given to find $\frac{d}{dx}\left\{\ln\left(x+\sqrt{x^2+9}\right)\right\}$. **1**

ii) Hence use Newton's method to find a second approximation to the root of $x = \ln\left(x+\sqrt{x^2+9}\right)$. Take the first approximation as $x=-4.5$. **2**

c) Water is running out of a filled conical funnel at the rate of $5\,cm^3s^{-1}$. The radius of the funnel is $10\,cm$ and the height is $20\,cm$.

i) How fast is the water level dropping when the water is $10\,cm$ deep? **4**

ii) How long does it take for the water to drop to $10\,cm$ deep? **2**



Question 6 (Start a new page)

a) Given θ is acute.

i) Write $\sin\frac{\theta}{2}$ in terms of $\cos\theta$. **1**

ii) Prove that $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$. **2**

iii) If $\sin\theta = \frac{4}{5}$, find the value of $\tan\frac{\theta}{2}$. **2**

b) Find $\frac{d}{dx}\cos^{-1}(\sin x)$ **3**

c) Suppose the roots of the equation $x^3 + px^2 + qx + r = 0$ are real. Show that the roots are in a geometric progression if $q^3 = p^3 r$. **4**

Hint : let the roots be $\frac{a}{b}$, a and ab.

Question 7 (Start a new page)

a)i) Prove by mathematical induction that **4**

$$\frac{12}{1\cdot3\cdot4}+\frac{18}{2\cdot4\cdot5}+\frac{24}{3\cdot5\cdot6}+\dots+\frac{6(n+1)}{n(n+2)(n+3)} = \frac{17}{6}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{4}{n+3}.$$

ii) Hence find $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{6(r+1)}{r(r+2)(r+3)}$. **1**

b) Consider the variable point $P(x, y)$ on the parabola $x^2 = 2y$.
The x value of P is given by $x = t$;

i) write its y value in terms of t **1**

ii) write an expression, in terms of t, for the square of the distance, m, from P to the point (6, 0) **1**

iii) hence find the coordinates of P such that P is the closest to the point (6, 0). **5**

*** **End of Paper** ***

Question 1

a) $P(-3, 5)$ $Q(1, -2)$ $-3 : 2$

$x = \frac{-6-3}{-3+2} = 9$ 1

$y = \frac{10+6}{-3+2} = -16$ 1 ②

$\therefore R(9, -16)$

b) $P(k) = (k+3)(k-2)+2 = k^2$ ½

$k^2 + k - 6 + 2 = k^2$ ½

$k - 4 = 0$ ½

$\therefore k = 4$ ½ ②

c) $\frac{x}{x-3} \geq 1$; $x \neq 3$

$x(x-3) \geq (x-3)^2$ ½

$x^2 - 3x \geq x^2 - 6x + 9$ ½

$3x \geq 9$ ½

$x \geq 3$ ½ ③

However $x \neq 3$, $\therefore x > 3$. 1

d) $\sin\theta = \cos\theta$

$\tan\theta = 1$ 1

$\theta = \tan^{-1} 1 + n\pi$ ②

$\theta = \frac{\pi}{4} + n\pi$ 1

e) $\int_0^{\frac{\pi}{8}} 2\sin^2 x \, dx$

$= \int_0^{\frac{\pi}{8}} (1 - \cos 2x)\, dx$ 1

$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{8}}$ 1

$= \frac{\pi}{8} - \frac{1}{2}\sin\frac{\pi}{4} - 0$ ½

$= \frac{\pi}{8} - \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)$ ③

$= \frac{\pi}{8} - \frac{\sqrt{2}}{4}$ ½

Question 2

a) i) $RHS = (x+2)^2 + 9$

$= x^2 + 4x + 4 + 9$ 1 ①

$= x^2 + 4x + 13$

$= LHS$

ii) $\int \frac{1}{x^2+4x+13}\, dx = \int \frac{1}{(x+2)^2+9}\, dx$ ½

$= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C$ ②

½ 1

b) i) (1) $\ddot{x} = 0$

$\dot{x} = C_1$ ½

When $t = 0$, $\dot{x} = 20\cos 30°$; $C_1 = 20\left(\frac{\sqrt{3}}{2}\right) = 10\sqrt{3}$ ½

$\Rightarrow \dot{x} = 10\sqrt{3}$

$x = 10\sqrt{3}\, t + C_2$ ½

When $t = 0$, $x = 0$; $C_2 = 0$ ½

$\Rightarrow x = 10\sqrt{3}\, t$ ②

(2) $\ddot{y} = -10$

$\dot{y} = -10t + C_3$ ½

When $t = 0$, $\dot{y} = 20\sin 30°$; $C_3 = 20\left(\frac{1}{2}\right) = 10$ ½

$\Rightarrow \dot{y} = -10t + 10$

$y = -5t^2 + 10t + C_4$ ½

When $t = 0$, $y = 0$; $C_4 = 0$ ½

$\Rightarrow y = -5t^2 + 10t$ ③

ii) (1) When $y = 0$; $-5t^2 + 10t = 0$

$-5t(t-2) = 0$ ½

$t = 0$ or $t = 2$

$\therefore$ Time of flight $= 2$s ½ ①

(2) When $t = 2$, $x = 10\sqrt{3}(2) = 20\sqrt{3}$ ½

$\therefore$ Horizontal range $= 20\sqrt{3}$ m. ½ ①

(3) When $t = 1$, $y = -5(1)^2 + 10(1) = 5$.

$\therefore$ Greatest height $= 5$m. ½ ①

(4) When $t = 1\frac{1}{2}$,

$\dot{x} = 10\sqrt{3}$ $\frac{1}{2}$ & $\dot{y} = -10(1\frac{1}{2}) + 10 = -5$ $\frac{1}{2}$

$\Rightarrow v = \sqrt{(10\sqrt{3})^2 + (-5)^2}$ $\frac{1}{2}$

$= \sqrt{100 \times 3 + 25}$

$= \sqrt{325}$

$= 5\sqrt{13}$ $\frac{1}{2}$ ms^{-1} (going down). (2)

Question 3

a) $\int_0^{\sqrt{3}} x\sqrt{x^2+1}\, dx$

$u = x^2 + 1$

$du = 2x\, dx$ $\frac{1}{2}$

$x = \sqrt{3},\ u = 3 + 1 = 4$ $\frac{1}{2}$

$x = 0,\ u = 0 + 1 = 1$ $\frac{1}{2}$

$= \int_1^4 \frac{1}{2}\sqrt{u}\, du$ $\frac{1}{2}$

$= \left.\frac{1}{3}u^{3/2}\right]_1^4$ $\frac{1}{2}$

$= \frac{1}{3}[4^{3/2} - 1]$

$= \frac{7}{3}$ $\frac{1}{2}$

(3)

b) i) $\cos\theta + \sqrt{3}\sin\theta = r\cos(\theta - \alpha)$

$\cos\theta + \sqrt{3}\sin\theta = r\cos\theta\cos\alpha + r\sin\theta\sin\alpha$

$\left[\begin{array}{l} r\cos\alpha = 1 \\ r\sin\alpha = \sqrt{3} \end{array}\right.$ $\Rightarrow \tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$ 1

$\left[\begin{array}{l} r^2\cos^2\alpha = 1 \\ r^2\sin^2\alpha = 3 \end{array}\right.$ $\Rightarrow r^2 = 4 \Rightarrow r = 2$ 1

$\therefore \cos\theta + \sqrt{3}\sin\theta = 2\cos(\theta - \frac{\pi}{3})$ (2)

ii) $\cos\theta + \sqrt{3}\sin\theta = 1$; $-2\pi \le \theta \le 2\pi$

$2\cos(\theta - \frac{\pi}{3}) = 1$; $\frac{-7\pi}{3} \le \theta - \frac{\pi}{3} \le \frac{5\pi}{3}$ $\frac{1}{2}$

$\cos(\theta - \frac{\pi}{3}) = \frac{1}{2}$ $\frac{1}{2}$

$\theta - \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{-\pi}{3}, \frac{-5\pi}{3}$ OR $\frac{-7\pi}{3}$ $\frac{1}{2}$

$\theta = \frac{2\pi}{3}, 2\pi, 0, \frac{-4\pi}{3}$ OR -2π $\frac{1}{2}$

$\therefore \theta = -2\pi, \frac{-4\pi}{3}, 0, \frac{2\pi}{3}$ OR 2π (2)

c) i) $f^{-1}(x) \Rightarrow x = \frac{y-1}{y+2}$

$xy + 2x = y - 1$

$xy - y = -2x - 1$ $\frac{1}{2}$

$y(1 - x) = 2x + 1$

$y = \frac{2x+1}{1-x}$

$f^{-1}(x) = \frac{2x+1}{1-x}$ $\frac{1}{2}$ (1)

ii) Domain : all real x ; $x \neq 1$ $\frac{1}{2}$ (1)

Range : all real y ; $y \neq -2$ $\frac{1}{2}$

d) Prove that ACBQ is a cyclic quad;
i.e. $\angle ACB + \angle AQB = 180^\circ$

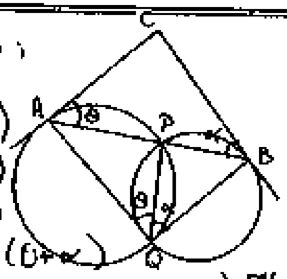
$\angle CAB = \angle AQP = \theta$ ($\angle$s in alt. segment)

$\angle CBA = \angle BQP = \alpha$ ($\angle$s in alt. segment)

$\angle ACB = 180^\circ - \theta - \alpha$ ($\angle$s sum of $\triangle ABC$)

$\angle ACB + \angle AQB = (180^\circ - \theta - \alpha) + (\theta + \alpha)$
$= 180^\circ$

$\therefore$ ACBQ is a cyclic quadrilateral. (3)



$-\frac{1}{2}$ per mistake

Question 4

a) i) $T = A - Ce^{-kt}$ $\Rightarrow$ $T - A = -Ce^{-kt}$ $\frac{1}{2}$

$\frac{dT}{dt} = -k(-Ce^{-kt})$ $\frac{1}{2}$

$= -k(T - A)$ (1)

ii) $T = 30 - Ce^{-kt}$ $\frac{1}{2}$

When $t = 0$, $T = 3$;

$3 = 30 - Ce^{0}$

$C = 27$ $\frac{1}{2}$

$T = 30 - 27e^{-kt}$

When $t = 15$, $C = 10$;

$10 = 30 - 27e^{-k \cdot 15}$ $\frac{1}{2}$

$27e^{-15k} = 20$

$e^{-15k} = \frac{20}{27}$ $\frac{1}{2}$

$-15k = \ln\left(\frac{20}{27}\right)$

$k = \frac{-1}{15}\ln\left(\frac{20}{27}\right)$ $\frac{1}{2}$

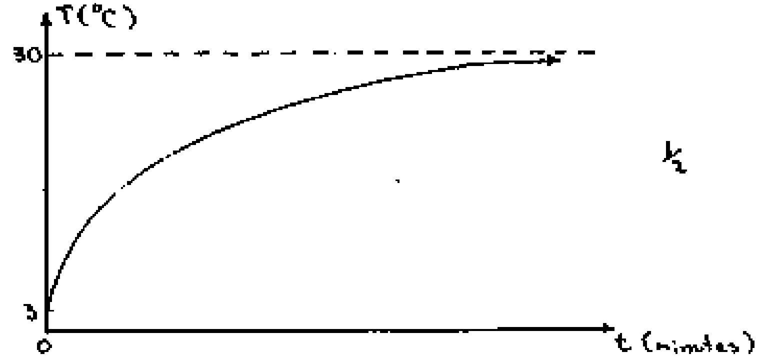
When $t = 30$; $\frac{1}{2}$

$T = 30 - 27e^{-30k}$ $\frac{1}{2}$

$= 15.18\ldots$

$= 15^\circ C$ $\frac{1}{2}$ (4)

iii)



$\frac{1}{2}$

As t becomes large, T approaches $30^\circ C$. $\frac{1}{2}$ (1)

b) i) $\ddot{x} = -4x + 8$

$\ddot{x} = -4(x - 2)$ 1 ①

$\Rightarrow \ddot{x} = -n^2 X \Rightarrow$ S.H.M

ii) Centre of motion = 2 1 ①

iii) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x + 8$

$\frac{1}{2}v^2 = -2x^2 + 8x + C$ $\frac{1}{2}$

When $x = 5$, $v = 0$; $\frac{1}{2}$

$0 = -2(25) + 40 + C$

$C = 10$ $\frac{1}{2}$

$\frac{1}{2}v^2 = -2x^2 + 8x + 10$ $\frac{1}{2}$

$v^2 = -4x^2 + 16x + 20$

$v^2 = 20 + 16x - 4x^2$ ②

iv) Amplitude $= 5 - 2 = 3$ m 1 ①

v) Max. velocity when $x = 2$;

$v^2 = 20 + 16(2) - 4(4)$ $\frac{1}{2}$

$v^2 = 36$

$v = \pm 6$

$\therefore$ max. speed $= 6\,ms^{-1}$. $\frac{1}{2}$ ①

Question 5

a) $f(x) = \ln(x+1)$

$f'(x) = \frac{1}{x+1}$ $\frac{1}{2}$

$f'(1) = \frac{1}{2}$ $\frac{1}{2}$ 1½ if only find 1 value of m.

$\tan 45° = 1 = \left|\frac{\frac{1}{2} - m}{1 + \frac{1}{2}m}\right|$ $\frac{1}{2}$

$1 + \frac{1}{2}m = \frac{1}{2} - m$ OR $-1 - \frac{1}{2}m = \frac{1}{2} - m$

$\frac{3}{2}m = \frac{-1}{2}$ $\quad\quad$ $\frac{1}{2}m = \frac{3}{2}$

$m = -3$ $\frac{1}{2}$ $\quad\quad$ $m = 3$ $\frac{1}{2}$

For all x, $x > -1$, $f'(x) > 0$, $\frac{1}{2}$

$\therefore m = 3$. ③

b) i) $\frac{d}{dx}\left[\ln(x + \sqrt{x^2+9})\right]$

$= \frac{1}{\sqrt{x^2+9}}$ 1 ①

ii) $x - \ln(x + \sqrt{x^2+9}) = 0$ $\frac{1}{2}$

$x_1 = -4.5 - \frac{(-4.5) - \ln(-4.5 + \sqrt{(-4.5)^2+9})}{1 - \frac{1}{\sqrt{(-4.5)^2+9}}}$ $\frac{1}{2}$

$= 0.9028$ 1

$= 0.90$ ②

c) i) $V = \frac{1}{3}\pi r^2 h$ $\frac{1}{2}$ $\quad$ $r = \frac{1}{2}h$ $\frac{1}{2}$

$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$= \frac{1}{12}\pi h^3$ $\frac{1}{2}$

$\frac{dV}{dh} = \frac{1}{4}\pi h^2$ $\frac{1}{2}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ $\frac{1}{2}$

$= \frac{4}{\pi h^2} \times (-5)$ $\frac{1}{2}$

$= \frac{-20}{\pi h^2}$ $\frac{1}{2}$

When $h = 10$, $\frac{dh}{dt} = \frac{-1}{5\pi}$ $\frac{1}{2}$ cm/s ④

ii) $\frac{dV}{dt} = -5$

$V = -5t + C$ $\quad \frac{1}{2}$

When $t = 0$, $V = \frac{1}{12}\pi(20)^3 = \frac{2000\pi}{3}$;

$\frac{2000\pi}{3} = 0 + C$

$\Rightarrow V = -5t + \frac{2000\pi}{3}$ $\quad \frac{1}{2}$

When $h = 10$, $V = \frac{1}{12}\pi(10)^3 = \frac{250\pi}{3}$;

$\frac{250\pi}{3} = -5t + \frac{2000\pi}{3}$ $\quad \frac{1}{2}$

$5t = \frac{1750\pi}{3}$

$t = \frac{350\pi}{3}$ $\quad \frac{1}{2}$

$= 367$ s (nearest whole number) $\quad$ (2)

Question 6

a) i) $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$2\sin^2\frac{\theta}{2} = 1 - \cos\theta$ $\quad \frac{1}{2}$

$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$

$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$ $\frac{1}{2}$; θ is acute $\quad$ (1)

ii) Prove that $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$:

RHS $= \frac{\sin\theta}{1+\cos\theta}$

$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+(2\cos^2\frac{\theta}{2}-1)}$ $\frac{1}{2}$ $\frac{1}{2}$

$= \frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}$ $\frac{1}{2}$

$= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$ $\quad \frac{1}{2}$

$= \tan\frac{\theta}{2}$

$=$ LHS

LHS $= \tan\frac{\theta}{2}$

$= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

$= \sqrt{\frac{1-\cos\theta}{2} \times \frac{2}{1+\cos\theta}}$ $\frac{1}{2}$

$= \sqrt{\frac{1-\cos\theta}{1+\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}}$ $\frac{1}{2}$

$= \sqrt{\frac{1-\cos^2\theta}{(1+\cos\theta)^2}}$ $\quad \frac{1}{2}$

$= \sqrt{\frac{\sin^2\theta}{(1+\cos\theta)^2}}$ $\quad \frac{1}{2}$

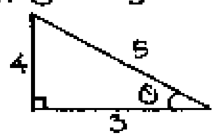
$= \frac{\sin\theta}{1+\cos\theta}$

$=$ RHS $\quad$ (2)

iii) $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$ $\qquad \sin\theta = \frac{4}{5}$

$= \frac{4}{5} \div \left(1 + \frac{3}{5}\right)$ $\quad 1$

$= \frac{1}{2}$ $\quad \frac{1}{2}$



$\frac{1}{2}$ $\quad$ (2)

b) $\frac{d}{dx}\cos^{-1}(\sin x)$

$= \frac{-\frac{d}{dx}\sin x}{\sqrt{1-(\sin x)^2}}$ $\quad 1$ $\qquad$ ½ if just give -1 as answer

$= \frac{-\cos x}{\sqrt{\cos^2 x}}$ $\quad \frac{1}{2}$

$= \begin{cases} -1 & \text{for } x \text{ in 1st \& 4th quadrant } \frac{1}{2} \\ 1 & \text{for } x \text{ in 2nd \& 3rd quadrant } \frac{1}{2} \end{cases}$; $\cos x \neq 0$ $\frac{1}{2}$ $\quad$ (3)

c) $x^3 + px^2 + qx + r = 0$

Let the roots be $\frac{a}{b}$, a and ab,

then $\begin{cases} \frac{a}{b} + a + ab = -p & \text{— (1)} \quad \frac{1}{2} \\ (\frac{a}{b})(a)(ab) = -r \Rightarrow a^3 = -r & \text{— (2)} \quad \frac{1}{2} \\ (\frac{a}{b})(a) + (a)(ab) + (\frac{a}{b})(ab) = q & \text{— (3)} \quad \frac{1}{2} \end{cases}$

(3) $\Rightarrow \frac{a^2}{b} + a^2 b + a^2 = q$

$a^2(\frac{1}{b} + b + 1) = q$

$q^3 = [a^2(\frac{1}{b} + b + 1)]^3$

$q^3 = a^6(\frac{1}{b} + b + 1)^3$ $\quad \frac{1}{2}$

From (1) & (2), $p^3 r = (-p)^3(-r)$ $\quad \frac{1}{2}$

$= (\frac{a}{b} + a + ab)^3(a^3)$

$= [a(\frac{1}{b} + 1 + b)]^3(a^3)$ $\quad \frac{1}{2}$

$= [a^3(\frac{1}{b} + b + 1)^3](a^3)$ $\quad \frac{1}{2}$

$= a^6(\frac{1}{b} + b + 1)^3$

$= q^3$

(4)

Question 7

a) i) Prove by induction that the statement

$$\frac{12}{1\cdot3\cdot4} + \frac{18}{2\cdot4\cdot5} + \frac{24}{3\cdot5\cdot6} + \dots + \frac{6(n+1)}{n(n+2)(n+3)} = \frac{17}{6} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{4}{n+3}$$

is true.

Step 1: Prove that the statement is true for $n = 1$;

LHS $= \frac{6(1+1)}{1(1+2)(1+3)} = \frac{\cancel{6}(2)}{1(3)(\cancel{4})} = 1$

RHS $= \frac{17}{6} - \frac{1}{1+1} - \frac{1}{1+2} - \frac{4}{1+3} = \frac{17}{6} - \frac{1}{2} - \frac{1}{3} - 1 = 1 = \text{LHS}$ $\quad \frac{1}{2}$

Step 2: Assume the statement is true for $n = k$.

Step 3: Prove that the statement is true for $n = k+1$;

i.e. prove $\frac{12}{1\cdot3\cdot4} + \frac{18}{2\cdot4\cdot5} + \dots + \frac{6(k+2)}{(k+1)(k+3)(k+4)} = \frac{17}{6} - \frac{1}{k+2} - \frac{1}{k+3} - \frac{4}{k+4}$

$$\text{LHS} = \frac{12}{1\cdot3\cdot4} + \frac{18}{2\cdot4\cdot5} + \frac{24}{3\cdot5\cdot6} + \dots + \frac{6(k+1)}{k(k+2)(k+3)} + \frac{6(k+2)}{(k+1)(k+3)(k+4)}$$

$$= \frac{17}{6} - \frac{1}{k+1} - \frac{1}{k+2} - \frac{4}{k+3} + \frac{6(k+2)}{(k+1)(k+3)(k+4)} \quad \frac{1}{2}$$

$$= \frac{17}{6} - \frac{1}{k+2} - \frac{1}{k+3} - \frac{3}{k+3} - \frac{1}{k+1} + \frac{6(k+2)}{(k+1)(k+3)(k+4)} \quad \frac{1}{2}$$

$$= \frac{17}{6} - \frac{1}{k+2} - \frac{1}{k+3} + \frac{(-3)(k+1)(k+4) - 1(k+3)(k+4) + 6(k+2)}{(k+1)(k+3)(k+4)} \quad \frac{1}{2}$$

$$= \frac{17}{6} - \frac{1}{k+2} - \frac{1}{k+3} + \frac{-3k^2 - 15k - 12 - k^2 - 7k - 12 + 6k + 12}{(k+1)(k+3)(k+4)} \quad \frac{1}{2}$$

$$= \frac{17}{6} - \frac{1}{k+2} - \frac{1}{k+3} + \frac{-4k^2 - 16k - 12}{(k+1)(k+3)(k+4)}$$

$$= \frac{17}{6} - \frac{1}{k+2} - \frac{1}{k+3} + \frac{(-4)(k^2 + 4k + 3)}{(k+1)(k+3)(k+4)} \quad \frac{1}{2}$$

$$= \frac{17}{6} - \frac{1}{k+2} - \frac{1}{k+3} + \frac{(-4)\cancel{(k+1)}\cancel{(k+3)}}{\cancel{(k+1)}\cancel{(k+3)}(k+4)} \quad \frac{1}{2}$$

$$= \frac{17}{6} - \frac{1}{k+2} - \frac{1}{k+3} - \frac{4}{k+4}$$

$= \text{RHS}$ $\quad \frac{1}{2}$

Step 4: ...

(4)

iii) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r(r+2)(r+3)} = \frac{11}{6}$ 1 (1)

b) $x^2 = 2y$

i) $x = t \; ; \; t^2 = 2y \Rightarrow y = \frac{1}{2}t^2$ 1 (1)

ii) $m^2 = (t-6)^2 + (\frac{1}{2}t^2 - 0)^2$

$= (t-6)^2 + (\frac{t^2}{2})^2$ $\frac{1}{2}$

$= t^2 - 12t + 36 + \frac{t^4}{4}$ $\frac{1}{2}$ (1)

iii) $\frac{dm^2}{dt} = 2t - 12 + t^3 = 0$ — (1) 1

$(t-2)(t^2+2t+6) = 0$ — (2) $\frac{1}{2}$

$t - 2 = 0$

$t = 2$ $\frac{1}{2}$

$\therefore P(2, 2)$. $\frac{1}{2}$

(1) $\Rightarrow$ Let $P(t) = t^3 + 2t - 12$

$P(2) = 8 + 4 - 12 = 0$ $\frac{1}{2}$

$$\begin{array}{r|l} & t^2 + 2t + 6 \\ t-2 & t^3 + 0 + 2t - 12 \\ & \underline{t^3 - 2t^2} \\ & 2t^2 + 2t - 12 \\ & \underline{2t^2 - 4t} \\ & 6t - 12 \\ & \underline{6t - 12} \end{array}$$ 1

(2) $\Rightarrow$ $t^2 + 2t + 6 = 0$ $\frac{1}{2}$

$\Delta = (2)^2 - 4(1)(6) < 0$ $\frac{1}{2}$

So $t^2 + 2t + 6 = 0$ has no solutions. (5)