PYMBLE LADIES' COLLEGE

MATHEMATICS EXTENSION 1

TRIAL HSC EXAMINATION

2005

Reading Time: 5 minutes
Working Time: 2 hours

Instructions to students:

- Write using blue or black biro.
- All questions may be attempted.
- Diagrams are not to scale.
- All necessary working should be shown in every question.
- Your name and your teacher's name may be written before you begin the assessment.
- Start each question in a new booklet.
- Marks may be deducted for careless or untidy work.
- Board-approved calculators may be used.
- A table of standard integrals is provided.

Question 1 (12 marks) Use a separate writing booklet

MARKS

(a) Differentiate $\tan^{-1}(2x)$. **2**

(b) Solve $\dfrac{3}{2-x} < 6$. **3**

(c) Evaluate $\lim\limits_{x\to 0} \dfrac{\sin\frac{x}{5}}{2x}$ **2**

(d) Find the coordinates of the point $P(x,y)$ which divides the interval joining the points $A(-6,2)$ and $B(4,7)$ externally in the ratio 3: 2. **2**

(e) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}}\,dx$, using the substitution $u = 1 + x^2$. **3**

Question 2 (12 marks) Use a separate writing booklet

MARKS

(a) (i) Show that $(x-3)$ is a factor of $x^3+3x^2-9x-27$. **1**

(ii) Hence, or otherwise, factorise $x^3+3x^2-9x-27$ completely. **2**

(b) Find the acute angle between the lines $x-2y+1=0$ and $x+3y+2=0$. **2**

(c) (i) Express $\sqrt{3}\sin x-\cos x$ in the form $A\sin(x-\alpha)$, where $A>0$ and $0\le\alpha\le\frac{\pi}{2}$. **2**

(ii) Hence, or otherwise, solve $\sqrt{3}\sin x-\cos x=1$ in the domain $0\le x\le 2\pi$. **2**

(d) Find the term independent of x in the expansion of $\left(3x-\frac{1}{2x^2}\right)^9$. **3**

Question 3 (12 marks) Use a separate writing booklet

MARKS

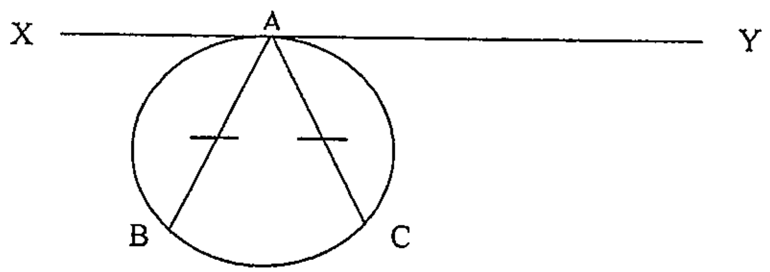
(a) Evaluate $\sin^{-1}\left(\cos\frac{2\pi}{3}\right)$. **1**

(b) State the domain and range of $y = 2\sin^{-1}3x$. **2**

(c) (i) Differentiate with respect to x : $x\sin^{-1}x + \sqrt{1-x^2}$. **2**

(ii) Hence evaluate $\int_0^1 \sin^{-1}x \ dx$. **2**

(d) **2**



Given that AB = AC and XY is a tangent to circle ABC at A, prove that XY is parallel to BC.

(e) Use mathematical induction to prove that for all positive integers n **3**

$1^2 + 3^2 + 5^2 + \ldots\ldots\ldots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

Question 4 (12 marks) Use a separate writing booklet

MARKS

Consider the function $f(x) = \dfrac{e^x}{e^x + 2}$.

(i) Show that $f(x)$ has no stationary points. 3

(ii) Find the coordinates of the point of inflexion given that 2

$$f''(x) = \frac{2e^x (2 - e^x)}{(e^x + 2)^3}$$

(iii) Show that $0 < f(x) < 1$ for all x. 2

(iv) Sketch the curve $f(x)$. 2

(v) Explain why $f(x)$ has an inverse function. 1

(vi) Find the inverse function $y = f^{-1}(x)$. 2

Question 5 (12 marks) Use a separate writing booklet

MARKS

(a) The points P $(4p, 2p^2)$ and Q $(4q, 2q^2)$ lie on the parabola $x^2 = 8y$.

(i) Show that the equation of the tangent to the parabola at P is $y = px - 2p^2$. **2**

(ii) The tangent at P and the line passing through Q parallel to the y axis intersect at T. Show that the coordinates of T are $(4q, 4pq - 2p^2)$. **1**

(iii) Find coordinates of M, the midpoint of PT. **2**

(iv) Find the cartesian equation of the locus of M when $pq = 1$. **1**

(b) The acceleration of a particle P moving in a straight line is given by $\frac{d^2x}{dt^2} = 3x(x-4)$, where x metres is the displacement of the particle from the origin O and t is the time in seconds. Initially the particle is at O and its velocity is $4\sqrt{2}$ m/s.

(i) Show that $v^2 = 2(x^3 - 6x^2 + 16)$ where v is the velocity of P. **3**

(ii) Calculate the velocity and acceleration of P at $x = 2$. **2**

(iii) In which direction does P move from $x = 2$? Give a reason for your answer. **1**

Question 6 (12 marks) Use a separate writing booklet

MARKS

(a) Using the fact that $(1+x)^3(1+x)^8=(1+x)^{11}$, show that **2**

$${}^3C_0\,{}^8C_3+{}^3C_1\,{}^8C_2+{}^3C_2\,{}^8C_1+{}^3C_3\,{}^8C_0={}^{11}C_3$$

(b) A sphere of ice cream sits on top of a cone. As the ice cream melts, maintaining its spherical shape, the radius reduces at a constant rate of 0.25 centimetres per minute. [The volume of a sphere is given by the formula $V=\frac{4}{3}\pi r^3$.]

Find the rate of change of the volume of the ice cream with respect to time when the radius is 4cm. **2**

(c) **3**

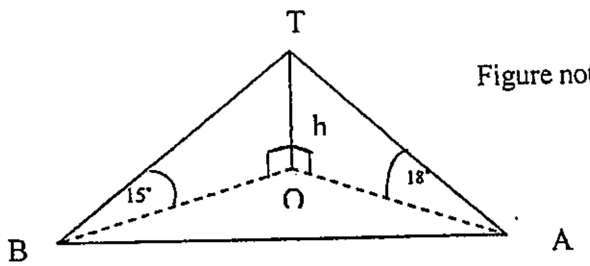


Figure not to scale

The angle of elevation of a hill OT, at the point A due south of the hill, is 18°. From B, due west of A, the angle of elevation is 15°. If AB is 6 km, find the height, h of the hill. Give your answer to the nearest 10 metres.

(d) The displacement x (in metres) of a particle is given by $x=5\cos(4\pi t)$, where t is in seconds.

(i) Show that the acceleration of the particle can be expressed in the form $\ddot{x}=-n^2x$. **2**

(ii) Determine the maximum speed of the particle and when it first occurs. **2**

(iii) Express v^2 in terms of x, where v is the velocity of the particle. **1**

Question 7 (12 marks) Use a separate writing booklet

MARKS

(a) A particle P is projected from a point on horizontal ground with velocity V at an angle of projection α. You may assume that the equations of motion are:

$$\ddot{x}=0 \qquad \ddot{y}=-g$$

$$\dot{x}=V\cos\alpha \qquad \dot{y}=V\sin\alpha-gt$$

$$x=Vt\cos\alpha \qquad y=Vt\sin\alpha-\frac{1}{2}gt^2$$

(i) Show that the particle's maximum height is $\dfrac{V^2\sin^2\alpha}{2g}$. **2**

(ii) A second particle Q is projected from the same point on horizontal ground with velocity $V\times\sqrt{\dfrac{5}{2}}$ at an angle $\dfrac{\alpha}{2}$ to the horizontal. If the maximum height is the same for both particles, show that $\alpha=\cos^{-1}\dfrac{1}{4}$. **4**

(b) (i) Prove that the graph of $y=\ln x$ is concave down for all $x>0$. **2**

(ii) Sketch the graph of $y=\ln x$. **2**
Suppose $0<a<b$ and consider the points A $(a, \ln a)$ and B $(b, \ln b)$ on the graph of $y=\ln x$. Find the coordinates of the point P that divides the line segment AB in the ratio 2:1.

(iii) By using part (ii) deduce that $\frac{1}{3}\ln a+\frac{2}{3}\ln b < \ln\left(\frac{1}{3}a+\frac{2}{3}b\right)$ **2**

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

QUESTION 1

a. $\frac{d}{dx} \tan^{-1}(2x)$

$= \frac{1}{1+(2x)^2} \cdot 2$

$= \frac{2}{1+4x^2}$

b. $\frac{3}{2-x} < 6$

METHOD 1 : Multiply by the square of the Denominator

$\frac{3}{2-x} \cdot (2-x)^2 < 6(2-x)^2$

$3(2-x) < 6(4-4x+x^2)$

$2-x < 2(4-4x+x^2)$

$2-x < 8-8x+2x^2$

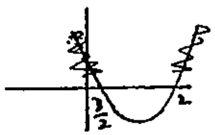
$0 < 2x^2-7x+6$

$2x^2-7x+6 > 0$

$(2x-3)(x-2) > 0$

Let $(2x-3)(x-2) = y$

WANT $y > 0$



Ans. $x < 1\frac{1}{2}$ or $x > 2$

METHOD 2 : Consider denominator POSITIVE and NEGATIVE

Consider $2-x > 0$ or $2-x < 0$

i.e. $x < 2$ $\quad$ $x > 2$

Q1

then $3 < 6(2-x)$ $\quad$ or $\quad$ $3 > 6(2-x)$

$3 < 12-6x$ $\quad$ $x > 1\frac{1}{2}$

$6x < 9$

$x < 1\frac{1}{2}$ $\quad$ and since $x > 2$ then

and Since $x < 2$ then

$x < 1\frac{1}{2}$ or $x > 2$

METHOD 3 : USING "CRITICAL" POINTS

c. $\lim_{x \to 0} \frac{\sin \frac{x}{5}}{2x}$

$= \frac{1}{2} \cdot \frac{1}{5} \lim_{x \to 0} \frac{\sin \frac{x}{5}}{\frac{x}{5}}$

$= \frac{1}{10} \lim_{x \to 0} \frac{\sin \frac{x}{5}}{\frac{x}{5}}$

$= \frac{1}{10}$ $\quad$ N.B. As $x \to 0$ then $\frac{x}{5} \to 0$

d. $A(-6, 2)$ $\quad$ $B(4, 7)$ $\quad$ $3 : -2$

$x_1 \ y_1$ $\quad$ $x_2 \ y_2$ $\quad$ $k : l$

$x = \frac{lx_1 + kx_2}{k+l}$ $\quad$ $y = \frac{ly_1 + ky_2}{k+l}$

$= \frac{-2 \times -6 + 3 \times 4}{3-2}$ $\quad$ $= \frac{-2 \times 2 + 3 \times 7}{3-2}$

$= 24$ $\quad$ $= 17$

$\therefore$ Required Point $P(x, y)$ is $(24, 17)$

Q3

d. $\angle XAB = \angle ACB$ — Angle between the chord and the tangent is equal to the angle at the circumference in the alternate segment

$\angle ACB = \angle ABC$ — Base angles of isosceles triangle ABC

or

Angles opposite equal sides of ΔABC

$\therefore \angle XAB = \angle ABC$ which represent alternate angles

As the alternate angles are equal then $XY \parallel BC$

e. To prove $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

STEP 1: To prove it true for $n=1$

$LHS = 1^2$
$= 1$

$RHS = \frac{1}{3} \times 1 \times (2\times1-1)(2\times1+1)$
$= 1$
$= LHS$

$\therefore$ True for $n=1$

STEP 2: Assume it true for $n=k$

i.e. $1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$

STEP 3: To prove it true for $n=k+1$

Q3

e. cont'd $LHS = \underbrace{1^2 + 3^2 + 5^2 + \dots + (2k-1)^2}_{\text{from Step 2}} + [2(k+1)-1]^2$

$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$

$= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)]$

$= \frac{1}{3}(2k+1)(2k^2+5k+3)$

$= \frac{1}{3}(2k+1)(2k+3)(k+1)$

$RHS = \frac{1}{3}(k+1)[2(k+1)-1][2(k+1)-1]$
$= \frac{1}{3}(k+1)(2k+1)(2k+3)$
$= LHS$

$\therefore$ True for $n=k+1$

STEP 4: Since it is true for $n=1$ and proven true for $n=1+1$ i.e. $n=2$ and for $n=2+1$ i.e. $n=3$ and so on, it is true for all positive integer values of n.

QUESTION 4

i. $f'(x) = \dfrac{e^x(e^x+2) - e^x \cdot e^x}{(e^x+2)^2}$

$= \dfrac{e^{2x} + 2e^x - e^{2x}}{(e^x+2)^2}$

$= \dfrac{2e^x}{}$

Q4

i. cont'd · $f'(x) \neq 0$ as $2e^x \neq 0$ since $e^x > 0$ for all x

ii. $f''(x) = \dfrac{2e^x(2-e^x)}{(e^x+2)^3}$

Points of Inflexion occur when $f''(x)=0$

i.e. $2e^x(2-e^x) = 0$

Either $e^x = 0$ or $2-e^x = 0$

No sol'n as $e^x > 0$ | $e^x = 2$, $x = \ln 2$

$f''(0) = \dfrac{2e^0(2-e^0)}{(e^0+2)^3}$ (ln 1) > 0

$f''(\ln 3) = \dfrac{2e^{\ln 3}(2-e^{\ln 3})}{(e^{\ln 3}+2)^3}$

$= \dfrac{3 \times -1}{5^3}$

< 0

As there is a change in concavity then $x = \ln 2$ must be a point of inflexion.

$f(\ln 2) = \dfrac{e^{\ln 2}}{e^{\ln 2}+2}$

$= \dfrac{2}{2+2}$

$= \frac{1}{2}$

Q4

iii. $f(x) = \dfrac{e^x}{e^x+2}$

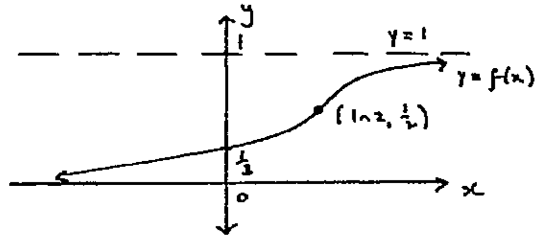
As $e^x > 0$ then $\dfrac{e^x}{e^x+2} > 0$ for all x

As $x \to \infty$, then $\dfrac{e^x}{e^x+2} \to 1$

As $x \to -\infty$, then $\dfrac{e^x}{e^x+2} \to 0$

$\therefore 0 < f(x) < 1$ for all x

iv.



$f(0) = \dfrac{e^0}{e^0+2} = \dfrac{1}{3}$

v. For every y-value there is only 1 x-value

OR ANY HORIZONTAL LINE CAN BE DRAWN AND IT CUTS $f(x)$ AT ONLY 1 POINT.

OR INCREASING FUNCTION OVER ENTIRE DOMAIN

Q4

vi. $$y = \frac{e^x}{e^x + 2}$$

Inverse function is given by

$$x = \frac{e^y}{e^y + 2}$$

$$x e^y + 2x = e^y$$

$$e^y(1-x) = 2x$$

$$e^y = \frac{2x}{1-x}$$

$$y = \ln\left(\frac{2x}{1-x}\right)$$

QUESTION 5

a. i. $$x^2 = 8y$$

$$y = \frac{x^2}{8}$$

$$\frac{dy}{dx} = \frac{x}{4}$$

when $x = 4p$, $\frac{dy}{dx} = \frac{4p}{4}$

$$= p$$

Eqn of tangent at $P(4p, 2p^2)$ is given by

$$y - y_1 = m(x - x_1)$$

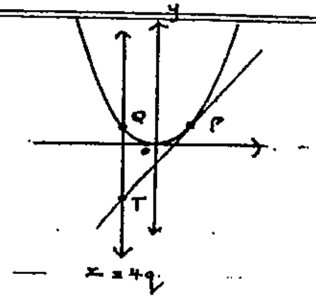
$$y - 2p^2 = p(x - 4p)$$

$$y - 2p^2 = px - 4p^2$$

$$\therefore\ y = px - 2p^2 \qquad (1)$$

Q5

a. ii.



As parallel to the y-axis then eqn of the line must be $x = 4q$. (2)

As T lies on this line then x-value of T must be $4q$.

Sub (2) into (1)

$$y = p(4q) - 2p^2$$

$$y = 4pq - 2p^2$$

$\therefore$ Coords of T is $(4q,\ 4pq - 2p^2)$

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iii. $P(4p, 2p^2)$ $\quad T(4q, 4pq - 2p^2)$

$\quad x_1 \quad y_1 \qquad\qquad x_2 \quad y_2$

Midpoint M is given by

$$x = \frac{x_1 + x_2}{2} \qquad y = \frac{y_1 + y_2}{2}$$

$$= \frac{4p + 4q}{2} \qquad = \frac{2p^2 + 4pq - 2p^2}{2}$$

$$= \frac{4(p+q)}{2} \qquad = \frac{4pq}{2}$$

$$= 2(p+q) \qquad = 2pq$$

$\therefore$ M is $(2(p+q),\ 2pq)$

Q5

a. iv. $x = 2(p+q) \quad y = 2pq$

Sub $pq = 1$

$x = 2(p+q) \quad y = 2$

As y is independent of p and q then this represents the locus of M i.e. the horizontal line $y = 2$.

b. i. $a = 3x(x-4)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3x^2 - 12x$

$\frac{1}{2}v^2 = \frac{3x^3}{3} - \frac{12x^2}{2} + c$

$\frac{1}{2}v^2 = x^3 - 6x^2 + c$

Sub $x = 0$, $v = 4\sqrt{2}$

$\frac{1}{2}[16 \times 2] = c$

$\therefore c = 16$

$\frac{1}{2}v^2 = x^3 - 6x^2 + 16$

$v^2 = 2(x^3 - 6x^2 + 16)$

ii. when $x = 2$, $v^2 = 2(2^3 - 6 \times 2^2 + 16)$

$v^2 = 0$

$\therefore v = 0$ At rest

$a = 3 \times 2(2-4)$

$= -12 \text{ m s}^{-2}$

Q5

b. iii. Moves to the left (towards Origin) as $a < 0$ and $v = 0$.

The force will now push it backwards as the particle stopped momentarily at $x = 2$.

Question 6

a. $(1+x)^3(1+x)^8 = (1+x)^{11}$

$\left[\sum_{r=0}^{3} {}^3C_r x^r\right]\left[\sum_{r=0}^{8} {}^8C_r x^r\right] = \sum_{r=0}^{11} {}^{11}C_r x^r$

To prove ${}^3C_0\,{}^8C_3 + {}^3C_1\,{}^8C_2 + {}^3C_2\,{}^8C_1 + {}^3C_3\,{}^8C_0 = {}^{11}C_3$

RHS $= {}^{11}C_3$ represents the coefficient of x^3 term

Q6

a. cont'd

LHS: Obtain x^3 term by

${}^3C_0\,{}^8C_3x^3 + {}^3C_1x\,{}^8C_2x^2 + {}^3C_2x^2\,{}^8C_1x + {}^3C_3x^3\,{}^8C_0$

$= x^3\left[{}^3C_0\,{}^8C_3 + {}^3C_1\,{}^8C_2 + {}^3C_2\,{}^8C_1 + {}^3C_3\,{}^8C_0\right]$

Hence co-efficient of x^3 term is

${}^3C_0\,{}^8C_3 + {}^3C_1\,{}^8C_2 + {}^3C_2\,{}^8C_1 + {}^3C_3\,{}^8C_0$

Hence equating co-efficients of x^3 term

$\therefore\ {}^3C_0\,{}^8C_3 + {}^3C_1\,{}^8C_2 + {}^3C_2\,{}^8C_1 + {}^3C_3\,{}^8C_0 = {}^{11}C_3$

b.

$\frac{dr}{dt} = -0{\cdot}25 \qquad V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

Want $\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$

$= 4\pi r^2 \cdot -0{\cdot}25$

$= -\pi r^2$

Sub $r = 4$

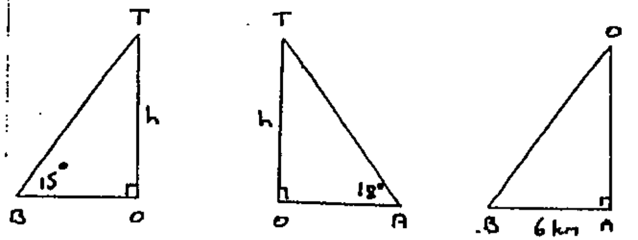
$\frac{dV}{dt} = -4^2\cdot\pi$

$= -16\pi$ cm³/minute

The volume is decreasing at the rate of 16π cm³/minute.

Q6

c. In ΔBOT In ΔAOT In ΔAOB



$\tan 15° = \frac{h}{OB}$ $\tan 18° = \frac{h}{OA}$ $OA^2 + 6^2 = OB^2$ Pythagoras' Thm

$OB = \frac{h}{\tan 15°}$ $OA = \frac{h}{\tan 18°}$

or

$\tan 75° = \frac{OB}{h}$ $\tan 72° = \frac{OA}{h}$

$OB = h\tan 75°$ $OA = h\tan 72°$

From ΔAOB

$OB^2 - OA^2 = 36$ Pythagoras' Thm

$h^2\left[\tan^2 75 - \tan^2 72\right] = 36$

$h^2 = \frac{36}{\tan^2 75 - \tan^2 72}$

$h = \frac{6}{\sqrt{\tan^2 75 - \tan^2 72}}$

Height of hill $\doteqdot 2{\cdot}84$ km to nearest 10m

d.i.

$x = 5\cos(4\pi t)$

$\dot{x} = -5.\sin(4\pi t)\ .\ 4\pi$

$\dot{x} = -20\pi\ \sin(4\pi t)$

$\ddot{x} = -20\pi\ .\ 4\pi\cos(4\pi t)$

$\ddot{x} = -16\pi^2\ .\ 5\cos(4\pi t)$

$\therefore\ \ddot{x} = -(4\pi)^2 x$

This is of the form $\ddot{x} = -n^2x$ where $n = 4\pi$

Q6

d. ii. From part (i) $\dot{x} = -20\pi \sin(4\pi t)$

Max Speed occurs when $\sin(4\pi t) = 1$

i.e. $|\dot{x}| = 20\pi$ m/s

OR $v^2 = n^2(a^2 - x^2)$

Sub $n = 4\pi$, $a = 5$

$v^2 = 16\pi^2 (25 - x^2)$

Max Speed occurs at the centre of motion

i.e. sub $x = 0$

$v^2 = 16 \times 25 \times \pi^2$

Speed $= |v| = 4 \times 5 \times \pi$

$= 20\pi$ m/s

OR Max Speed occurs at the centre of motion i.e. at $x = 0$

$\therefore 5\cos(4\pi t) = 0$

$\cos(4\pi t) = 0$

$4\pi t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$t = \frac{1}{8}$ seconds

Sub into ①

$\dot{x} = -20\pi \sin(4\pi \cdot \frac{1}{8})$

$= -20\pi \sin \frac{\pi}{2}$

$\therefore$ Max Speed $= 20\pi$ m/s

ANSWER: MAX SPEED IS 20π m/s and first occurs after $\frac{1}{8}$ seconds

iii. $v^2 = n^2(a^2 - x^2)$

Sub $n = 4\pi$, $a = 5$

$v^2 = 16\pi^2(25 - x^2)$

QUESTION 7

a. i. Maximum Height occurs when $\dot{y} = 0$

i.e. $V\sin\alpha - gt = 0$

$gt = V\sin\alpha$

$t = \frac{V}{g}\sin\alpha$

Sub into $y = Vt\sin\alpha - \frac{1}{2}gt^2$

$y = V \cdot \frac{V}{g}\sin\alpha \cdot \sin\alpha - \frac{1}{2}g\left[\frac{V}{g}\sin\alpha\right]^2$

$y = \frac{V^2}{g}\sin^2\alpha - \frac{V^2}{2g}\sin^2\alpha$

$\therefore$ Max Height $y_P = \frac{V^2\sin^2\alpha}{2g}$

ii. Similarly Max Height for particle Q is

$y_Q = \frac{\left(V \times \sqrt{\frac{5}{2}}\right)^2 \sin^2\left(\frac{\alpha}{2}\right)}{2g}$

$y_Q = \frac{5V^2\sin^2\left(\frac{\alpha}{2}\right)}{4g}$

Since Max Height the same for both particles then $y_P = y_Q$

i.e. $\frac{V^2\sin^2\alpha}{2g} = \frac{5V^2\sin^2\left(\frac{\alpha}{2}\right)}{4g}$

$\sin^2\alpha = \frac{5}{2}\sin^2\left(\frac{\alpha}{2}\right)$

USING $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

THEN $\sin^2\alpha = \frac{5}{4}(1 - \cos\alpha)$

Q7

a. ii. cont'd

$$1 - \cos^2\alpha = \frac{5}{4}(1 - \cos\alpha)$$

$$4 - 4\cos^2\alpha = 5 - 5\cos\alpha$$

$$4\cos^2\alpha - 5\cos\alpha + 1 = 0$$

$$(4\cos\alpha - 1)(\cos\alpha - 1) = 0$$

Either $4\cos\alpha - 1 = 0$ or $\cos\alpha - 1 = 0$

$4\cos\alpha = 1$ $\quad$ $\cos\alpha = 1$

$\cos\alpha = \frac{1}{4}$ $\quad$ $\alpha = 0, \pi$

rejected as NOT APPLICABLE i.e. not projected up

$\therefore \alpha = \cos^{-1}\left(\frac{1}{4}\right)$

b. i. $y = \ln x$

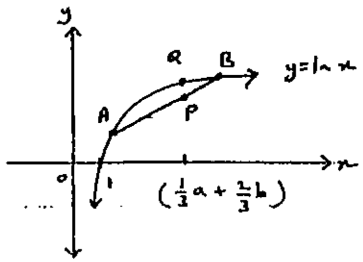
$$\frac{dy}{dx} = \frac{1}{x} = x^{-1}$$

$$\frac{d^2y}{dx^2} = -x^{-2} = \frac{-1}{x^2}$$

As $x^2 > 0$ for all x, then $\frac{-1}{x^2} < 0$

Hence $\frac{d^2y}{dx^2} < 0$ for all $x > 0$

ii.



Q7

b. ii. cont'd $A(a, \ln a)$ $\quad$ $B(b, \ln b)$ $\quad$ $2:1$

$x_1 \; y_1$ $\quad$ $x_2 \; y_2$ $\quad$ $k \; l$

Hence coordinates of P is given by:

$$x = \frac{x_1 l + x_2 k}{k + l} \qquad y = \frac{y_1 l + y_2 k}{k + l}$$

$$x = \frac{a + 2b}{3} \qquad y = \frac{\ln a + 2\ln b}{3}$$

or $\frac{1}{3}(a + 2b)$ $\quad$ or $\frac{1}{3}\ln(ab^2)$

$\therefore$ P is $\left(\frac{1}{3}a + \frac{2}{3}b, \frac{1}{3}\ln a + \frac{2}{3}\ln b\right)$

iii. Q (directly above P) is on the curve $y = \ln x$.

So for $y = \ln x$

Sub $x = \frac{1}{3}a + \frac{2}{3}b$

then $y_Q = \ln\left(\frac{1}{3}a + \frac{2}{3}b\right)$

As $0 < a < b$ and the curve is concave down for all $x > 0$, then the curve must lie above the line segment AB.

Hence the y-value of the point P must be below the y-value of the point Q on the curve.

i.e. $y_P < y_{Q(\ln x)}$

$$\frac{1}{3}\ln a + \frac{2}{3}\ln b < \ln\left(\frac{1}{3}a + \frac{2}{3}b\right)$$