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Name: ....

Teacher: ....

2011
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Start each question in a new booklet
- Marks may be deducted for careless or untidy work

Total Marks – 84

- Attempt Questions 1-7
- All questions are of equal value

Mark	**/84**
Rank	**/**
Highest Mark	**/84**

Question 1. (12 Marks) Use a **SEPARATE** Writing Booklet. **Marks**

(a) Find $\int \frac{2}{\sqrt{36-x^2}}\,dx.$ **1**

(b) Differentiate $\tan^{-1}(\ln x)$ with respect to x. **1**

(c) Find the coordinates of the point P that divides the interval joining A (-3, 8) and B (7, -3) internally in the ratio 2 : 3. **2**

(d) Solve $\frac{4}{x-2} \leq 2.$ **3**

(e) The acute angle between the lines $2x - y = 4$ and $y = mx + 3$ is 45°. Find the two possible values of m. **2**

(f) Use the substitution $u = x - 3$ to evaluate $\int_4^5 \frac{x}{\sqrt{x-3}}\,dx.$ **3**

Question 2 (12 Marks) Use a **SEPARATE** Writing Booklet. **Marks**

(a) Let $f(x) = 3\sin^{-1} 2x$. **2**

Sketch the graph of $y = f(x)$, clearly indicating the endpoints for the domain and the range.

(b) (i) Differentiate $x\cos^2 x$ with respect to x. **2**

(ii) Hence, or otherwise, find $\int x\sin 2x\, dx$. **2**

(c) The polynomial $P(x) = x^3 + ax^2 - 2x + b$ has $(x + 1)$ as a factor. $P(x)$ has a remainder of 4 when divided by $(x\text{-}3)$. **3**

Find the values of a and b.

(d) The function $f(x) = x - e^{-2x}$ has one root between $x = 0$ and $x = 1$. **3**

Use one application of Newtons' method, starting at $x = 0.3$, to find another approximation for this root.

Write your answer correct to 2 decimal places.

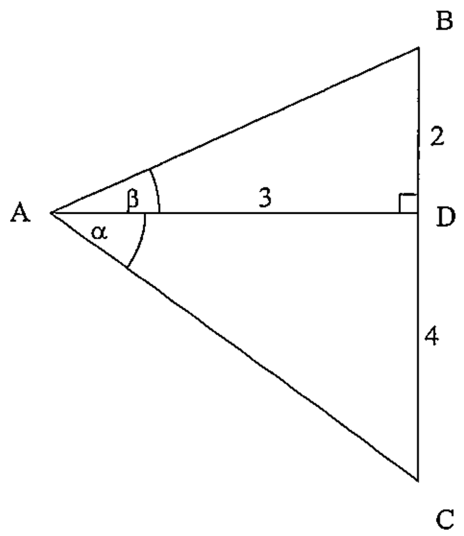
Question 3 (12 Marks) Use a **SEPARATE** Writing Booklet. **Marks**

(a) Evaluate $\lim_{x \to \infty} \dfrac{3x^2+1}{2-7x-5x^2}$. **2**

(b) (i) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $r\cos(\theta+\alpha)$, where $r > 0$, and $0 < \alpha < \dfrac{\pi}{2}$, giving r and α as exact values. **2**

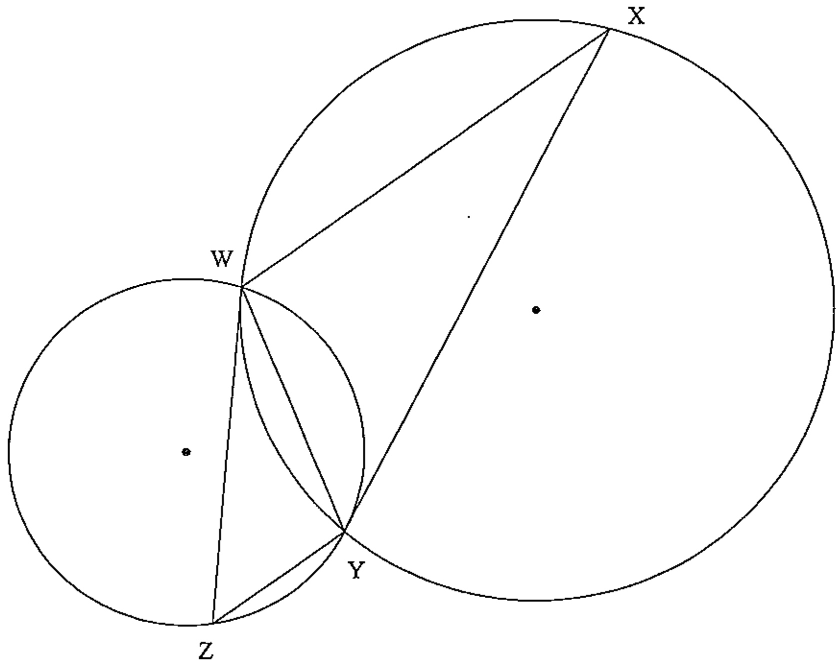
(ii) Solve $\sqrt{3}\cos\theta - \sin\theta = -1$, for $0 \le \theta \le 2\pi$, giving your answers as exact values. **2**

(c) In the diagram below, AD is perpendicular to BC. **3**
CD = 4, BD = 2 and AD = 3. $\angle CAD = \alpha$ and $\angle BAD = \beta$.



Find the exact value of $\sin(\alpha - \beta)$.

(d) WZ and XY are tangents to the circles WXY and WYZ respectively. The circles share two common points, W and Y. **3**
Copy or trace the diagram into your Writing Booklet.



Prove that WX || YZ.

Question 4 (12 Marks) Use a **SEPARATE** Writing Booklet. **Marks**

(a) Find the exact value of $\int_0^{\ln 3} \frac{e^x}{e^x + 9}dx$. **2**

(b) Find the constant term in the expansion $\left(2x - \frac{1}{x}\right)^6$. **2**

(c) Prove by induction that $4p + 3p^2 + 2p^3$ is divisible by 3 for $p = 1, 2, 3...$ **3**

(d) The temperature (T°C) of steel, after it has been removed from a hot furnace, after t minutes, satisfies the differential equation:

$$\frac{dT}{dt} = k(T - 22) \quad \text{where } k \text{ is a constant.}$$

Initially, the temperature (T) of the steel is 100°C and when t=15minutes, T=70°C.

(i) Use this information to find the exact values of A and k. **2**

(ii) Hence find the value of t when $T = 40$°C to the nearest minute. **1**

(e) Show that: **2**

$$\sqrt{\frac{1+\sin 2\theta}{1-\sin 2\theta}} = \frac{1+\tan\theta}{1-\tan\theta}.$$

Question 5 (12 marks) Use a **SEPARATE** Writing Booklet. **Marks**

(a) The speed V cm/s of a particle moving along the x – axis is given by

$$V^2 = 27 + 18x - 9x^2 \text{ where } x \text{ is in cm.}$$

(i) Prove that the motion is Simple Harmonic. Find the period and amplitude of the motion. **3**

(ii) Find the acceleration of the particle when it is 1cm away from the centre of motion. **1**

(b) A stone is projected upwards from the edge of a cliff with a speed of 30m/s. It hits an object 120 m horizontally from the edge and 35 m vertically below it.

Assume that t seconds after the release, the position of the stone is given by

$$x = 30t\cos\alpha \quad \text{and} \quad y = -5t^2 + 30t\sin\alpha \ .$$

(i) Find α, the angle of projection, to the nearest minute. **3**

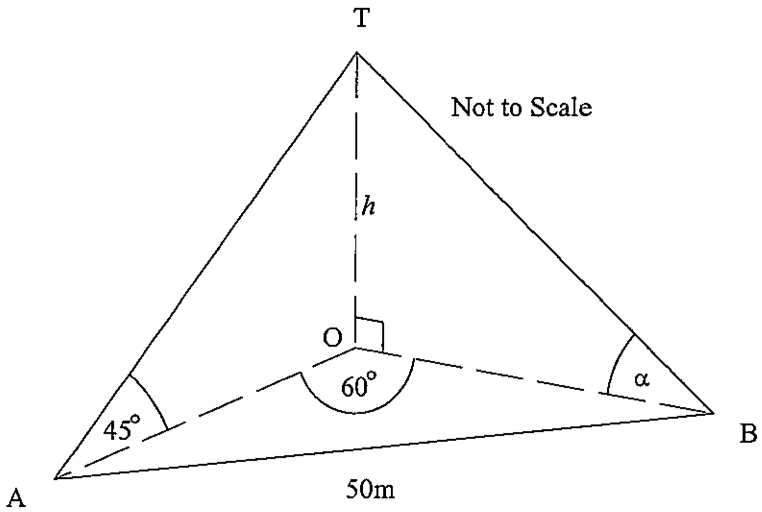
(ii) Find the time taken for the stone to hit the object. **1**

(c) (i) Sketch the curve $y = x + \dfrac{4}{x}$ showing clearly all stationary points and asymptotes. **3**

(ii) Hence, find the values of k such that $x + \dfrac{4}{x} = k$ has no real roots. **1**

Question 6 (12 marks) Use a **SEPARATE** Writing Booklet. **Marks**

(a)



In the above diagram, the points A, B and O are in the same horizontal plane. A and B are 50 m apart and $\angle AOB = 60^{\circ}$. OT is a vertical tower of height h metres.
The angles of elevation of T from A and B respectively are 45° and α° (α is acute).

(i) Explain why AO = h. **1**

(ii) Prove $h^2 \cot^2 \alpha - h^2 \cot \alpha + h^2 = 50^2$. **2**

(iii) Given that the tower is 30m high, find the angle α correct to the nearest degree. **3**

(b) (i) Verify that $\frac{d}{dx}\left(x\sin^{-1}x+\sqrt{1-x^2}\right)=\sin^{-1}x$. **1**

(ii) Hence, using a similar expression, find a primitive of $\cos^{-1}x$. **1**

(iii) The curves $y=\sin^{-1}x$ and $\cos^{-1}x$ intersect at P. **1**

The curve $y=\cos^{-1}x$ also intersects the x axis at Q.

Show that P has co-ordinates $\left(\frac{1}{\sqrt{2}}\,,\,\frac{\pi}{4}\right)$.

(iv) Find the area enclosed by the x – axis and the arcs OP and PQ. **3**

Question 7(12 marks) Use a **SEPARATE** Writing Booklet. **Marks**

(a) A particle is moving in a straight line with acceleration given by

$$\frac{d^2x}{dt^2} = 9(x-2).$$

where x is the displacement in metres, from the origin O after t seconds. Initially the particle is 4m to the right of O and it has a velocity of $V = 6$m/s.

(i) Show that $V^2 = 9(x-2)^2$. **3**

(ii) Find an expression for V and hence find x as a function of t. **3**

(b) PQ is a variable chord of the parabola $x^2 = 4ay$ which subtends a right angle at the vertex.

(i) If p and q are the parameters corresponding to the points P and Q, prove that $pq = -4$. **2**

(ii) Show that the equation of the normal at P is
$$x + py = 2ap + ap^3.$$ **2**

(iii) Hence prove that the locus of the point of intersection of the normals at P and Q is the parabola $x^2 = 16a(y - 6a)$. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

a) $\int \frac{2}{\sqrt{36-x^2}}\,dx$

$= 2\sin^{-1}\frac{x}{6} + c$ 1 r/w

b) $\frac{d}{dx}\left(\tan^{-1}(\ln x)\right)$

$= \frac{1/x}{1+(\ln x)^2}$

$= \frac{1}{x + x(\ln x)^2}$ 1 r/w. Subsequent errors ignored

c) $A\overset{x_1,\ y_1}{(-3, 8)}$ and $B\overset{x_2,\ y_2}{(7, -3)}$ $\overset{m\ \ n}{2 : 3}$

$x = \frac{nx_1 + mx_2}{m+n}$

$\therefore = \frac{3\times(-3) + 2\times 7}{2+3}$

$= \frac{5}{5}$ ①

$= 1$

$y = \frac{ny_1 + my_2}{m+n}$

$= \frac{3\times 8 + 2\times(-3)}{2+3}$

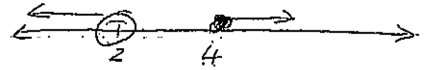
$= 18/5$ ①

$\therefore$ Point is $(1, 18/5)$

①

d) $\frac{4}{x-2} \leq 2$ C.P $x = 2$

Consider $\frac{4}{x-2} = 2$



$2x - 4 = 4$

$2x = 8$

$x = 4$

Test

$x = 1$ $\frac{4}{1-2} \leq 2$ True Testing ①

$x = 3$ $\frac{4}{3-2} \leq 2$ False

$x = 5$ $\frac{4}{5-2} \leq 2$ True

Sol^n $x < 2$ ① and $x \geq 4$ ①

f) $u = x - 3$ $\int_4^5 \frac{x}{\sqrt{x-3}}\,dx$

When $x = 5$, $u = 5 - 3 = 2$

$x = 4$, $u = 4 - 3 = 1$

$u = x - 3$

$du = dx$

$= \int_1^2 \frac{u+3}{\sqrt{u}}\,du$

$= \int_1^2 u^{1/2} + 3u^{-1/2}\,du$

$= \left[\frac{2u^{3/2}}{3} + \frac{3u^{1/2}}{1/2}\right]_1^2$ ① Correct integral and limits

$= \frac{2}{3}\times 2^{3/2} + 6\sqrt{2} - \frac{2}{3} - 6$ ① Correct integration

$= \frac{2}{3}\times(\sqrt{2})^3 + 6\sqrt{2} - \frac{2}{3} - 6$ ② Correct substitution and attempt at tidying answer

$= \frac{2}{3}\times 2\sqrt{2} + 6\sqrt{2} - \frac{20}{3}$

$= \frac{4}{3}\sqrt{2} + 6\sqrt{2} - \frac{20}{3}$

$= 22\sqrt{2} - 20 = 2(11\sqrt{2} - 10)$

e) $2x - y - 4 = 0 \qquad y = mx + 3 \qquad \theta = 45^\circ$

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| \qquad y = 2x - 4$

$\therefore \tan 45 = \left|\dfrac{2 - m}{1 + 2m}\right|$

$\therefore \left|\dfrac{2-m}{1+2m}\right| = 1$ ①

$\therefore \dfrac{2-m}{1+2m} = 1$ or $\dfrac{2-m}{1+2m} = -1$

$2 - m = 1 + 2m$ or $2 - m = -1 - 2m$

$3m = 1 \qquad 3 = -m$

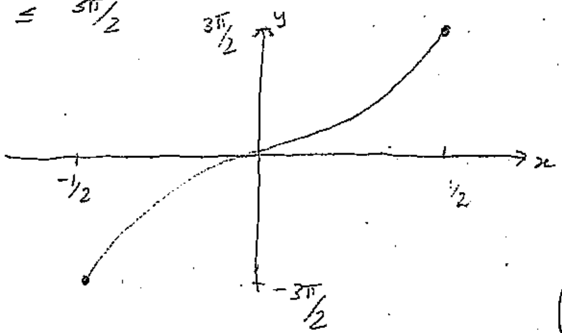
$\underline{m = 1/3}$ ① $\qquad \underline{m = -3}$

Question 2

$f(x) = 3\sin^{-1} 2x$

a) $-\pi/2 \le \dfrac{y}{3} \le \pi/2 \qquad -1 \le 2x \le 1$

$-\dfrac{3\pi}{2} \le y \le 3\pi/2 \qquad -1/2 \le x \le 1/2$



① shape

① domain & range

b) i) $\frac{d}{dx}(x\cos^2 x)$

$= x \times 2\cos x(-\sin x) + \cos^2 x$

$= \underline{-x\sin 2x + \cos^2 x}$

① Correct product rule

① Correct derivative of $\cos^2 x$

ii) $\int x \sin 2x \, dx$

Now

$\int(\cos^2 x - x\sin 2x)\,dx = x\cos^2 x$ from i)

$\therefore \int \cos^2 x - \int x \sin 2x \, dx = x\cos^2 x$

$\therefore \int x\sin 2x\,dx = \int \cos^2 x\,dx - x\cos^2 x$ ① Ex[illegible] for $\int$[illegible]

$= \int \dfrac{1+\cos 2x}{2}\,dx - x\cos^2 x$

$= \underline{\dfrac{x}{2} + \dfrac{\sin 2x}{4} - x\cos^2 x + C}$

c) $P(x) = x^3 + ax^2 - 2x + b \qquad x + 1$

$P(-1) = 0 \qquad \therefore (-1)^3 + a(-1)^2 - 2(-1) + b = 0$

$-1 + a + 2 + b = 0$

$\therefore a + b = -1$ — ①

$P(3) = 4 \qquad \therefore (3)^3 + a(3)^2 - 2(3) + b = 4$

$\therefore 27 + 9a - 6 + b = 4$

$9a + b = -17$ — ②

② − ①

d) $f(x) = x - e^{-2x}$ $\quad x = 0.3$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $\quad$ (1) $f'(x) = 1 + 2e^{-2x}$

$f(0.3) = 0.3 - e^{-0.6}$

$f'(0.3) = 1 + 2e^{-0.6}$

$\therefore x_1 = 0.3 - \frac{0.3 - e^{-0.6}}{1 + 2e^{-0.6}}$ (1)-sub

$\doteqdot 0.418615978$

$= 0.42$ (2 dp.) $\quad$ (1) - correctly rounded answer

Question 3

a) $\lim_{x \to \infty} \frac{3x^2 + 1}{2 - 7x - 5x^2}$ $\quad$ (2) Marks: Correct with correct notation

$= \lim_{x \to \infty} \frac{3x^2/x^2 + 1/x^2}{2/x^2 - 7x/x^2 - 5x^2/x^2}$ $\quad$ (1) - Incorrect use of notation with correct answer. Incorrect answer with minor error

$= \lim_{x \to \infty} \frac{3 + 1/x^2}{2/x^2 - 7/x - 5}$

$= -3/5$ $\quad$ since $\lim_{x \to \infty} \frac{1}{x} = 0$

b) i) $\sqrt{3}\cos\theta - \sin\theta$ $\quad$ $r\cos(\theta + \alpha)$

$r\cos(\theta + \alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$

$\sqrt{3}\cos\theta - \sin\theta = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$

Equating coefficients

$r\cos\alpha = \sqrt{3}$ $\quad$ $r\sin\alpha = 1$

Squaring and adding

$r^2\cos^2\alpha + r^2\sin^2\alpha = 3 + 1$

$\therefore r^2 = 4$

$r = 2, \quad r > 0$

$\therefore 2\cos\alpha = \sqrt{3}$ $\quad$ $2\sin\alpha = 1$

$\cos\alpha = \sqrt{3}/2$ $\quad$ $\sin\alpha = 1/2$

Since $0 < \alpha < \pi/2$

$\alpha = \pi/6$ $\quad$ (1) $\quad$ (1)

$\therefore \sqrt{3}\cos\theta - \sin\theta = 2\cos(\theta + \pi/6)$

ii) $2\cos(\theta + \pi/6) = -1$

$\cos(\theta + \pi/6) = -1/2$

$\therefore \theta + \pi/6 = 2\pi/3, \; 4\pi/3$ $\quad$ 2nd and 3rd quadrants

$\therefore \theta = \pi/2, \; 7\pi/6$

(2) - Correct angles and setting out

(1) - 1 error, too many angles, one incorrect angle

c) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

In $\triangle ADC$ $AC = 5$ by Pythagoras' Thm since $\triangle ADC$ is right angled.

In $\triangle ADB$ $AB = \sqrt{2^2 + 3^2}$

$= \sqrt{13}$ also by Pythagoras' Thm since $\triangle ACD$ is right angled

$\therefore \sin\alpha = 4/5$ $\quad$ $\cos\alpha = 3/5$ $\quad$ (1)

[illegible]

$$\sin(\alpha - \beta) = \frac{4}{5} \times \frac{3}{\sqrt{13}} - \frac{3}{5} \times \frac{2}{\sqrt{13}}$$ (1)

$$= \frac{12 - 6}{5\sqrt{13}}$$

$$= \frac{6}{5\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}}$$

$$= \frac{6\sqrt{13}}{65}$$ (1)

d) i. $\angle WYX = \angle WZY$ Angle between tangent and chord equals the angle at the circumference in the alternate segment.

Similarly $\angle ZWY = \angle WXY$ (1) For both angle justifications.

$\therefore \angle WYZ = \angle XWY$ angle sum triangles WXY and WZY with two equal angles in both triangles (1)

but $\angle WYZ$ and $\angle XWY$ are alternate and equal $\therefore WX \parallel YZ$ (1)

QUESTION 4

a) $$\int_0^{\ln 3} \frac{e^x}{e^x + 9}\,dx$$

$$= \left[\ln(e^x + 9)\right]_0^{\ln 3}$$ (1)

$$= \ln(e^{\ln 3} + 9) - \ln(e^0 + 9)$$

$$= \ln 12 - \ln 10$$

$$= \ln 1.2$$ (1)

b). $\binom{6}{r} 2^{6-r} x^{6-r}, (-1)^r, x^{-r}$

where $6 - 2r = 0$

$r = 3$ (1)

$\therefore {}^6C_3\ 8x^3 \times -\frac{1}{x^3}$

$= -160$ (1)

(c) Show statement is true for $p = 1$

$\therefore 4(1) + 3(1)^2 + 2(1)^3 = 9$

which is div by 3

$\therefore$ Statement is true for $p = 1$ (1)

Assume statement is true for $n = k$

$\therefore 4k + 3k^2 + 2k^3 = 3M$ where M is a +ve integer.

RTP the statement is true for $p = k + 1$

Now $4(k+1) + 3(k+1)^2 + 2(k+1)^3$

$= (k+1)\left(4 + 3(k+1) + 2(k+1)^2\right)$

$= (k+1)\left(4 + 3k + 3 + 2k^2 + 4k + 2\right)$

$= (k+1)\left(2k^2 + 7k + 9\right)$

$= 2k^3 + 7k^2 + 9k + 2k^2 + 7k + 9$

$= 2k^3 + 9k^2 + 16k + 9$ — (1)

$= (2k^3 + 3k^2 + 4k) + 6k^2 + 12k + 9$

$= 3M + 3(2k^2 + 6k + 3)$ — from assumption.

$= 3(M + 2k^2 + 6k + 3)$ — (1)

where $(M + 2k^2 + 6k + 3)$ is integral.

If statement is true for $p = k$, then it is true for $p = k+1$. Since it is true for $p = 1$, then true for $p = 2$ & so on. It is true for all +ve integers p.

ii) When $t = 0$ $T = 100°C$.

$t = 15$ $T = 70°C$.

$100 = Ae^{k \times 0} + 22$

$100 = A + 22$

$A = 78$

$70 = 78e^{15k} + 22$ ← (1)

$78e^{15k} = 48$

$e^{15k} = 8/13$ $15k = \ln 8/13$ ← (1)

e) $LHS = \sqrt{\dfrac{1 + \sin 2\theta}{1 - \sin 2\theta}}$

$= \sqrt{\dfrac{1 + 2\cos\theta\sin\theta}{1 - 2\cos\theta\sin\theta}}$ $\div \cos^2\theta$

$= \sqrt{\dfrac{\sec^2\theta + \dfrac{2\cos\theta\sin\theta}{\cos^2\theta}}{\sec^2\theta - \dfrac{2\cos\theta\sin\theta}{\cos^2\theta}}}$

$= \sqrt{\dfrac{\sec^2\theta + 2\tan\theta}{\sec^2\theta - 2\tan\theta}}$ (1)

$= \sqrt{\dfrac{1 + \tan^2\theta + 2\tan\theta}{1 + \tan^2\theta - 2\tan\theta}}$

$= \sqrt{\dfrac{(1 + \tan\theta)^2}{(1 - \tan\theta)^2}}$ (1)

$= \dfrac{1 + \tan\theta}{1 - \tan\theta}$

$= RHS$.

Question 5

i) $v^2 = 27 + 18x - 9x^2$

$\frac{1}{2}v^2 = \frac{1}{2}(27 + 18x - 9x^2)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left\{\frac{1}{2}(27 + 18x - 9x^2)\right\}$

$= 9 - \frac{9}{2} \times 2x$

$= 9 - 9x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 9(1-x)$

$= -9(x-1)$ (1)

Since $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\ddot{x} = 9(1-x)$. since this is a multiple of the displacement

$\ddot{x} = -3^2(x-1)$

of the form $\ddot{x} = -n^2(x-a)$ the motion is S.H

The centre of motion is 1 and the period is $\frac{2\pi}{3}$ (1)

When $v = 0$

$27 + 18x - 9x^2 = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3$ or $x = -1$

$$-1 \qquad 1 \qquad 3$$

Particle oscillates between -1 and 3

$\therefore$ amplitude is 2 (1)

ii) $\ddot{x} = -3^2(x-1)$

When $x = 2$ (1cm from centre of motion)

$\ddot{x} = -9(2-1)$

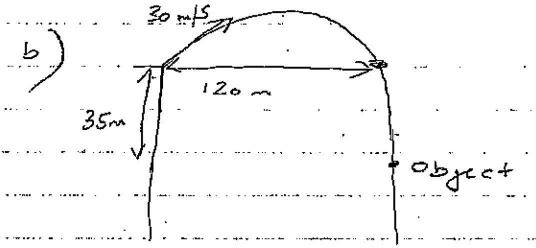
$= -9$ cm/s^2

When $x = 0$

$\ddot{x} = -9(-1)$

$= 9$ cm/s^2 (1) for both cases

b)



$x = 30t\cos\alpha$ — ①

$y = -5t^2 + 30t\sin\alpha$ — ②

From ① $t = \frac{x}{30\cos\alpha}$ subs into ②

$y = -5\left(\frac{x}{30\cos\alpha}\right)^2 + 30\left(\frac{x}{30\cos\alpha}\right)\sin\alpha$ (1)

$= \frac{-5x^2}{900\cos^2\alpha} + x\tan\alpha$

$y = \frac{-x^2}{180}\sec^2\alpha + x\tan\alpha$

When $x = 120$ $y = -35$

$\therefore -35 = \frac{-120^2}{180}\sec^2\alpha + 120\tan\alpha$ (1)

$-6300 = -14400(1 + \tan^2\alpha) + 21600\tan\alpha$

$14400\tan^2\alpha - 21600\tan\alpha + 8100 = 0$

$48\tan^2\alpha - 72\tan\alpha + 27 = 0$

$16\tan^2\alpha - 24\tan\alpha + 9 = 0$

$(4\tan\alpha - 3)^2 = 0$

$\therefore \tan\alpha = 3/4$ (1)

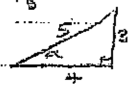
ii) When $x = 120$, $\alpha = 36°52'$

$\therefore 120 = 30t\cos 36°52'$

$t = \dfrac{120}{30\cos 36°52'}$

$= 5$ seconds (1)

or use $\tan\alpha = 3/4$, $\cos\alpha = 4/5$, $\therefore t = \frac{4}{\cos\alpha} = 5$



c) i) $y = x + \dfrac{4}{x}$

$dy/dx = 1 - \dfrac{4}{x^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{8}{x^3}$

When $\dfrac{dy}{dx} = 0$

$1 - \dfrac{4}{x^2} = 0$

$\dfrac{4}{x^2} = 1$

$x^2 = 4$

$x = \pm 2$

When $x = 2$, $y = 2 + \frac{4}{2} = 4$ $x = -2$, $y = -2 - \frac{4}{2} = -4$

$\therefore$ sps are at $(2, 4)$ and $(-2, -4)$

When $x = 2$, $\dfrac{d^2y}{dx^2} = \dfrac{8}{2^3} > 0$ $\therefore (2, 4)$ is a min s.p. When $x = -2$, $\dfrac{d^2y}{dx^2} = \dfrac{8}{(-2)^3} < 0$ $\therefore (-2, -4)$ is a max s.p

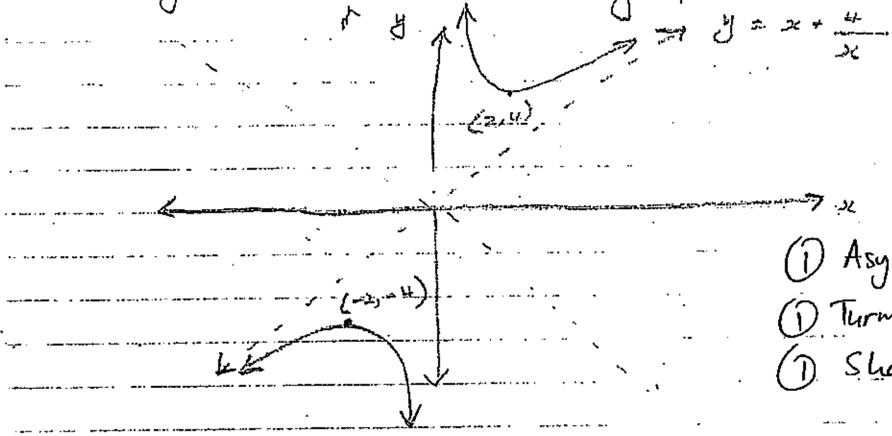
When $\dfrac{d^2y}{dx^2} = 0$, $\dfrac{8}{x^3} = 0$ which has no sol^{n}s

$\therefore$ there are no inf pts.

$\lim\limits_{x\to\infty} x + 4/x$ $\lim\limits_{x\to-\infty} x + 4/x$

$= x$ $\therefore$ $= x$

$\therefore y = x$ is a skew asymptote



(1) Asymptotes
(1) Turning pts.
(1) Shape

$x = 0$ is a vertical asymptote

ii) $x + \dfrac{4}{x} = k$. Since $y = x + \dfrac{4}{x}$ does not exist when $-4 < x < 4$ {$(2,4)$ is a min $(-2,-4)$ is a max} (1)

$x + \dfrac{4}{x} = k$ will have no sol^n if $-4 < k < 4$

Question 6

a) i) In $\triangle AOT$ $\tan 45° = \dfrac{h}{AO}$

$\therefore AO = \dfrac{h}{\tan 45°}$ (1)

But $\tan 45° = 1$ $\therefore AO = h$ r/w

ii) In ΔBOT $\tan\alpha = \frac{h}{OB}$

$\therefore OB = \frac{h}{\tan\alpha}$

$= h\cot\alpha$ (1)

In ΔAOB by the cosine rule

$50^2 = OB^2 + OA^2 - 2\times OB\times OA\cos 60$

$50^2 = h^2\cot^2\alpha + h^2 - 2h\cot\alpha\times h\times\frac{1}{2}$ (1)

$50^2 = h^2\cot^2\alpha + h^2 - h^2\cot\alpha$

iii) When $h = 30$

$50^2 = 30^2\cot^2\alpha + 30^2 - 30^2\cot\alpha$

$900\cot^2\alpha - 900\cot\alpha + 900 - 2500 = 0$

$900\cot^2\alpha - 900\cot\alpha - 1600 = 0$

$9\cot^2\alpha - 9\cot\alpha - 16 = 0$

$\cot\alpha = \frac{9 \pm \sqrt{81 - 4\times 9\times -16}}{18}$ (1)

$= \frac{9 \pm \sqrt{657}}{18}$

* Since α is acute $\cot\alpha = \frac{9+\sqrt{657}}{18}$ (1)

$\tan\alpha = \frac{18}{9+\sqrt{657}}$ (1)

$\alpha = 27°$ to nearest degree.

b) i) $\frac{d}{dx}\left(x\sin^{-1}x + (1-x^2)^{\frac{1}{2}}\right)$

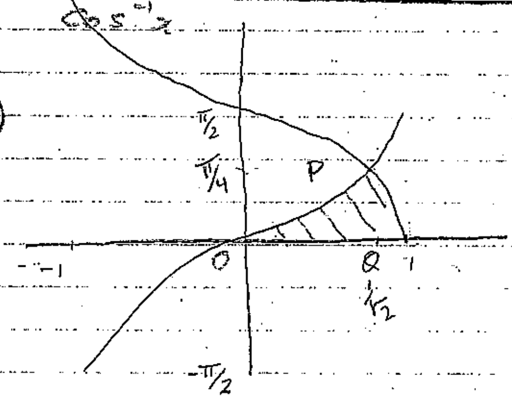
$= x\times\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x + \frac{1}{2}(1-x^2)^{-\frac{1}{2}}\times -2x$ (1)

$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}}$

$= \sin^{-1}x$

ii) $x\cos^{-1}x - \sqrt{1-x^2}$ is a primitive of $\cos^{-1}x$ (1)

iii)



$\sin^{-1}x = \cos^{-1}x$

when $x = 1/\sqrt{2}$

$\sin^{-1}(1/\sqrt{2}) = \pi/4$

$\cos^{-1}(1/\sqrt{2}) = \pi/4$

$\therefore \sin^{-1}x = \cos^{-1}x$ (1)

when $x = 1/\sqrt{2}$

iv) $\int_0^{1/\sqrt{2}}\sin^{-1}x\,dx + \int_{1/\sqrt{2}}^{1}\cos^{-1}x\,dx$

Method 1

from i) and ii)

$A = $ (1) $\left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^{1/\sqrt{2}} + \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_{1/\sqrt{2}}^{1}$

(1) $= \frac{1}{\sqrt{2}}\sin^{-1}\frac{1}{\sqrt{2}} + \sqrt{1-(1/\sqrt{2})^2} - \left(0\sin^{-1}0 + \sqrt{1-0^2}\right) + 1.\cos^{-1}1$

$- \sqrt{1-(1)^2} - \left(1/\sqrt{2}\cos^{-1}1/\sqrt{2} - \sqrt{1-(1/\sqrt{2})^2}\right)$ $\sqrt{2}-1$

$= \frac{2}{\sqrt{2}} - 1$

$= \sqrt{2} - 1$ sq units

OR $A = \int_0^{\pi/4} \cos y \, dy - \int_0^{\pi/4} \sin y \, dy$ (1)

$= \left[\sin y\right]_0^{\pi/4} - \left[-\cos y\right]_0^{\pi/4}$

$= \sin \pi/4 - \sin 0 + \cos \pi/4 - \cos 0$ (1)

$= 1/\sqrt{2} - 0 + 1/\sqrt{2} - 1$

$= 2/\sqrt{2} - 1$

$= \sqrt{2} - 1$ sq units (1)

Question 7

a) $\ddot{x} = 9(x-2)$ When $t = 0$ $x = 4$ $\dot{x} = 6$

i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 9(x-2)$

$\frac{1}{2}v^2 = \int 9(x-2)\,dx$

$\frac{1}{2}v^2 = \frac{9x^2}{2} - 18x + C$

When $x = 4$ $v = 6$

$\therefore \frac{1}{2} \times 6^2 = \frac{9 \times 4^2}{2} - 18 \times 4 + C$

$18 = 72 - 72 + C$

$\therefore c = 18$

$\therefore \frac{1}{2}v^2 = \frac{9x^2}{2} - 18x + 18$

$v^2 = 9x^2 - 36x + 36$

$v^2 = 9(x^2 - 4x + 4)$

$v^2 = 9(x-2)^2$

Q7

a) $\ddot{x} = 9(x-2)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 9(x-2)$ 1

$\frac{1}{2}v^2 = 9\int(x-2)\,dx$

$\frac{1}{2}v^2 = \frac{9}{2}(x-2)^2 + C$ 1

When $x = 4$, $v = 6$:

$\frac{1}{2} \times 6^2 = \frac{9}{2}(4-2)^2 + C$

$18 = 18 + C$

$C = 0$ 1

$\frac{1}{2}v^2 = \frac{9}{2}(x-2)^2$

$v^2 = 9(x-2)^2$ (3)

ii) $v = \pm 3(x-2)$

But when $x = 4$, $v = +6 > 0$ 1

so $v = 3(x-2)$

$\frac{dx}{dt} = 3(x-2)$

$\frac{dt}{dx} = \frac{1}{3(x-2)}$

$t = \frac{1}{3}\int \frac{1}{x-2}\,dx$

$t = \frac{1}{3}\ln(x-2) + c$ 1

When $t = 0$, $x = 4$:

$0 = \frac{1}{3}\ln 2 + c$

$c = -\frac{1}{3}\ln 2$

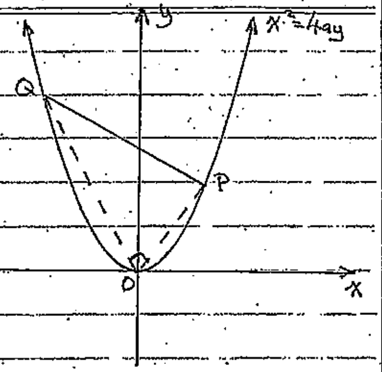
$t = \frac{1}{3}\ln(x-2) - \frac{1}{3}\ln 2$

$t = \frac{1}{3}\ln\left(\frac{x-2}{2}\right)$

$\frac{x-2}{2} = e^{3t}$

$x = 2e^{3t} + 2$ 1 (3)

b) $x^2 = 4ay$



Vertex = $(0, 0)$

$P = (2ap, ap^2)$

$Q = (2aq, aq^2)$

i) $m_{PO} = \dfrac{ap^2 - 0}{2ap - 0}$

$= \dfrac{p}{2}$ 1

$m_{QO} = \dfrac{q}{2}$

Since chord PQ subtends right angle at vertex,

$m_{PO} \times m_{QO} = -1$ 1

$\Rightarrow \dfrac{p}{2} \times \dfrac{q}{2} = -1$

$pq = -4$ (2)

ii) $x^2 = 4ay$

$y = \dfrac{x^2}{4a}$

$y' = \dfrac{x}{2a}$

$m_P = \dfrac{2ap}{2a} = p$

m of normal at P $= \dfrac{-1}{p}$ 1

Eq. of normal at P

$\Rightarrow \dfrac{-1}{p} = \dfrac{y - ap^2}{x - 2ap}$ 1

$-x + 2ap = py - ap^3$

$x + py = 2ap + ap^3$ — (1) (2)

iii) Similarly eq. of normal at Q

is $x + qy = 2aq + aq^3$ — (2)

(1) − (2) $\Rightarrow (p-q)y = 2a(p-q) + a(p^3 - q^3)$

$y = 2a + a(p^2 + pq + q^2)$ — (3)

Sub (3) into (1),

$x + 2ap + ap(p^2 + pq + q^2) = 2ap + ap^3$

$x + 2ap + ap^3 + ap^2q + apq^2 = 2ap + ap^3$

$x = -apq(p+q)$

Pt. of intersection of the normals at P & Q

$= (-apq(p+q),\ 2a + a(p^2 + pq + q^2))$ 1

Since $pq = -4$ from (i),

$x = 4a(p+q) \Rightarrow p + q = \dfrac{x}{4a}$

$y = 2a + a(p^2 + 2pq + q^2 - pq)$

$= 6a + a(p+q)^2$ 1

$y - 6a = a\left(\dfrac{x}{4a}\right)^2$

$y - 6a = \dfrac{x^2}{16a}$

$16a(y - 6a) = x^2$ (2)

(9.18)