Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section.

Use the multiple choice answer sheet for Questions 1-10.

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1 Which expression is the correct factorisation of $8x^3+125$?

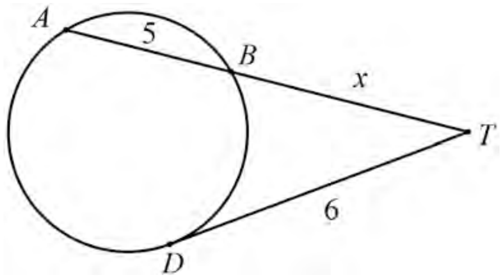
(A) $(2x+5)(4x^2+10x+25)$

(B) $(2x+5)(4x^2-10x+25)$

(C) $(2x-5)(4x^2+10x+25)$

(D) $(2x-5)(4x^2-10x+25)$

2



The line DT is a tangent to the circle at D and AT is a secant meeting the circle at A and B. Given that $DT=6$, $AB=5$ and $BT=x$, which of the following is the value of x?

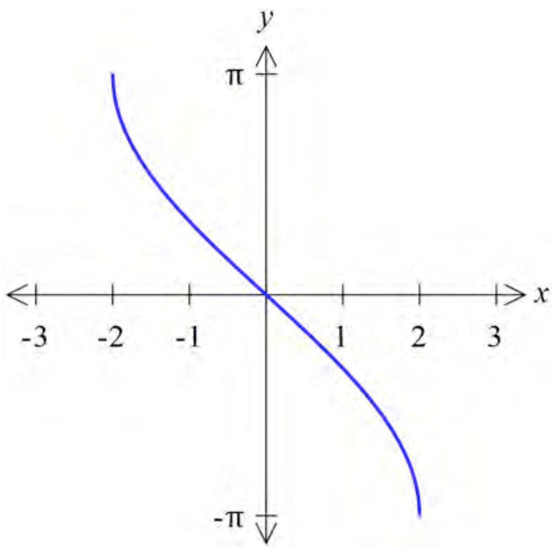
(A) $x=4$

(B) $x=5$

(C) $x=6$

(D) $x=9$

3 Which function does the following graph represent?



(A) $y = \frac{1}{2}\sin^{-1} 2x$

(B) $y = -\frac{1}{2}\sin^{-1} 2x$

(C) $y = 2\sin^{-1} \frac{x}{2}$

(D) $y = -2\sin^{-1} \frac{x}{2}$

4 What is the coefficient of x^5 in the expansion of $(2x+5)^8$?

(A) 1400000

(B) 224000

(C) 25000

(D) 4000

5 Which of the following is the solution to $\sin\theta = \cos 2\theta,\ 0 \le \theta \le 2\pi$?

(A) $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(B) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

(C) $\frac{4\pi}{3}, \frac{5\pi}{3}$

(D) $\frac{\pi}{3}, \frac{2\pi}{3}$

6 A is the point $(-5,6)$ and B is the point $(4,3)$. Which of the following points divides the interval AB externally in the ratio $3:2$?

(A) $(-2,-21)$

(B) $(-27,12)$

(C) $(22,-3)$

(D) $\left(\frac{2}{5},4\frac{1}{5}\right)$

7 What is the primitive of $\cos^2 x$?

(A) $\dfrac{\sin 2x+2x}{4}+C.$

(B) $\dfrac{1}{2}\sin 2x+x+C.$

(C) $\dfrac{4x-\sin 4x}{4}+C.$

(D) $x-\dfrac{1}{4}\sin 2x+C.$

8 What is the approximate size of the acute angle between the tangents to the curve $y=\ln(2x+1)$ at the points where $x=0$ and $x=\frac{1}{2}$?

(A) $11°$

(B) $18°$

(C) $72°$

(D) $79°$

9 A particle is projected so that at any time t, its position (x,y) is given by

$x = 36t,\ y = 15t - \frac{1}{2}gt^2,$ where distances are in metres and time is in seconds.

If θ is the angle of projection and V is the initial velocity, which of the following statements is correct?

(A) $V = 36\,\mathrm{m\,s^{-1}}$ and $\theta = \sin^{-1}\left(\frac{5}{13}\right)$

(B) $V = 15\,\mathrm{m\,s^{-1}}$ and $\theta = \cos^{-1}\left(\frac{12}{13}\right)$

(C) $V = 39\,\mathrm{m\,s^{-1}}$ and $\theta = \tan^{-1}\left(\frac{5}{13}\right)$

(D) $V = 39\,\mathrm{m\,s^{-1}}$ and $\theta = \tan^{-1}\left(\frac{5}{12}\right)$

10 Which expression is equivalent to $8\sin x - 15\cos x$?

(A) $17\cos(x - 61°56')$

(B) $17\cos(x + 61°56')$

(C) $17\sin(x - 61°56')$

(D) $17\sin(x + 61°56')$

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11. (15 marks) Use a **Separate** Booklet. **Marks**

(a) The polynomial equation $-2x^3+5x-1=0$ has three roots α, β, and γ.
Evaluate the following

(i) $\beta\alpha+\beta\gamma+\alpha\gamma$. **1**

(ii) $\alpha^{-1}+\beta^{-1}+\gamma^{-1}$. **2**

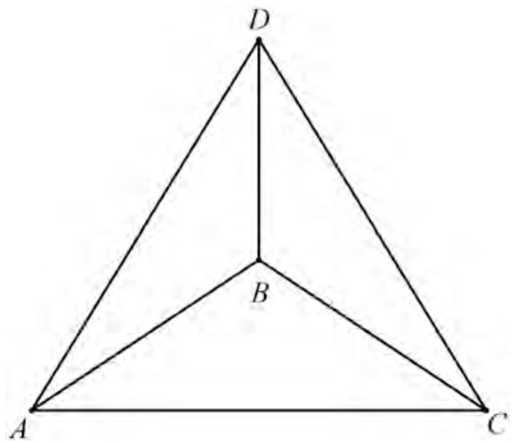
(b) Solve $\dfrac{5-x}{x}<3$ **3**

(c) A particle moves such that its velocity is given by $v=x-5$ m/s. **2**
Show that the acceleration is the same as the velocity for all x.

(d) Write $\tan\left(\cos^{-1}\left(-\dfrac{1}{3}\right)\right)$ in the form $a\sqrt{b}$ where a and b are rational. **2**

Question 11 continues on page 6

(e) Jane and John are competing in a sailing boat race. Jane (*C*) can see the top of a cliff (*D*) that is 1500 m above sea level. The cliff is on a bearing of $359°$ from her position and the angle of elevation of the top of the cliff is $16°$. John (*A*) can also see the cliff on a bearing of $059°$ with an angle of elevation of $23°$. The base of the cliff (*B*) is at sea level.



(i) Show that $\angle ABC = 60$. **1**

(ii) Copy the diagram and include all relevant information. **1**

(iii) Find the distance *AC* between the two sailing boats to the nearest metre. **3**

End of Question 11

Question 12. (15 marks) Use a **Separate** Booklet. **Marks**

(a) (i) Find the linear factors of $x^3 - 5x^2 + 8x - 4$. **2**

(ii) Hence solve $x^3 - 5x^2 + 8x - 4 > 0$. **1**

(b) Use the substitution $u = x^2 + 1$ to evaluate $\int_0^2 \frac{x}{\left(x^2+1\right)^3}\, dx$. **3**

(c) The volume of water in a tidal pool is given by the formula $V = 2\cos\frac{3\pi}{x}$, **3**

where x is the depth of water in the pool, in metres.

Find the exact rate at which the depth of the pool will be increasing when the volume of water is increasing at 12 m^3/h and the depth is 1.2 m.

(d) A parabola is given by the parametric equations $x = t$ and $y = t^2$.

(i) Sketch the parabola, and on your diagram mark the points P and Q which correspond to $t = -1$ and $t = 2$ respectively. **1**

(ii) Show that the tangents to the parabola at P and Q intersect at $R\left(\frac{1}{2}, -2\right)$. **2**

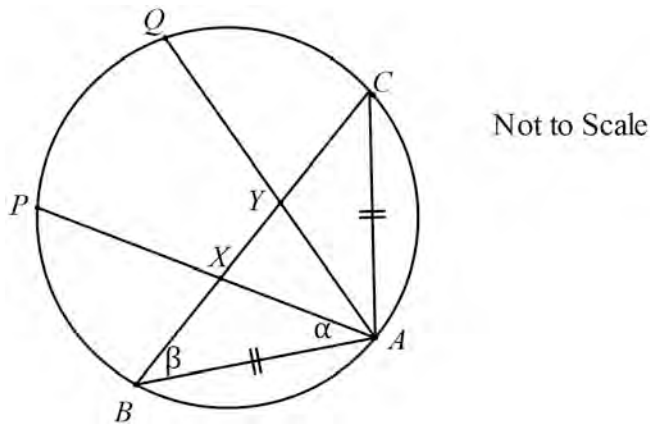
(iii) Let $T\left(t, t^2\right)$ be the point on the parabola between P and Q such that the tangent at T meets QR at the midpoint of QR. Show that the tangent at T is parallel to PQ. **3**

End of Question 12

Question 13. (15 marks) Use a **Separate** Booklet. **Marks**

(a) If $\int_0^{\frac{2}{3}} \frac{dx}{\sqrt{4-9x^2}} = k\pi,$ find the value of k. **3**

(b) In the circle below, $AB = AC$. Let $\angle PAB = \alpha$ and $\angle ABC = \beta$.



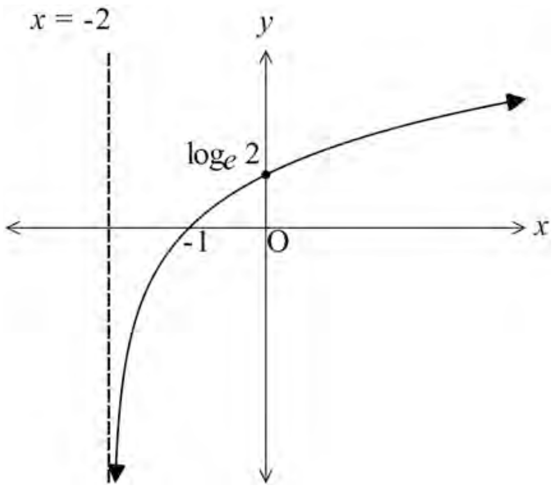
(i) Copy the diagram into your answer booklet and give a reason why $\angle PQB = \alpha$. **1**

(ii) Prove $\angle AQB = \beta$. **1**

(iii) Prove $XYQP$ is a cyclic quadrilateral. **2**

Question 13 continues on page 9

(c) The diagram below shows the graph of the function $y = f(x)$ where $f(x) = \ln(x+2)$.



(i) Copy the diagram and on it draw the graph of the inverse function $y = f^{-1}(x)$ showing the intercepts on the axes and the equation of the asymptote. **2**

(ii) Show that the x coordinates of the points of intersection of the curves $y = f(x)$ and $y = f^{-1}(x)$ satisfy the equation $e^x - x - 2 = 0$. **2**

(iii) Show that the equation $e^x - x - 2 = 0$ has a root α such that $1 < \alpha < 2$. **2**

(iv) Use one application of Newton's method with an initial approximation $\alpha_0 = 1.2$ to find the next approximation for the value of α, giving your answer correct to one decimal place. **2**

End of Question 13

Question 14. (15 marks) Use a **Separate** Booklet. **Marks**

(a) Use mathematical induction to prove that for all integers $n \geq 1$, **3**

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)....\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+2}{2n+2}.$$

(b) In an experiment recording the number, N, of daisies in a given area of garden, it was found that the rate of change of N is given by

$$\frac{dN}{dt}=kN\left(1-\frac{N}{2000}\right),$$ where k is a constant and t is number of days.

At the beginning of the experiment there were 1000 daises.

(i) Verify that $N=\dfrac{2000e^{kt}}{1+e^{kt}}$ is a solution of the equation. **3**

(ii) If $N = 1500$ when t = 10 days, determine the value of t when N = 1800. **3**

Question 14 continues on page 11

(c) The depth of water y metres in a tidal creek is given by

$$4\frac{d^2y}{dt^2}=5-y,$$ the time t being measured in hours.

(i) Prove that the vertical motion of water is simple harmonic. **2**
Find the centre of motion.

(ii) Write down the period T of the motion. **1**

(iii) Given that y = 1 m at low tide and y = 9 m at high tide, and **1**

that $y=a-b\cos nt$ is a solution of the equation $4\frac{d^2y}{dt^2}=5-y$,

write down the values of a, b and n.

(iv) If the low tide one day is at 1.00 pm, when is the earliest time that **2**
a boat requiring 3 m of water can enter the creek?

Give your answer correct to the nearest minute.

End of Paper

Standard Integrals

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\, dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

2015 Extension 1 Trial Solutions

1. $8x^3 + 125$

$= (2x)^3 + (5)^3$

$= (2x+5)((2x)^2 - 10x + 5^2)$

$(2x+5)(4x^2 - 10x + 25)$ $\therefore$ B

2. $6^2 = (x+5).x$

$x^2 + 5x - 36 = 0$

$(x+9)(x-4) = 0$

$x = -9, 4$

But $x > 0$ $\therefore$ $x = 4$ $\therefore$ A

3. When $x = 2$ $-2\sin^{-1}\frac{2}{2} = -2 \times \pi/2 = -\pi$ $\therefore$ D

4. General term $\binom{8}{r}(2x)^{8-r}5^r$

For x^5 $r = 3$

$\therefore$ coeff is $\binom{8}{3}2^5.5^3$

$= 224000$ $\therefore$ B

5. $\sin\theta = \cos 2\theta$

$= 1 - 2\sin^2\theta$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\therefore \sin\theta = 1/2$ or -1 $0 \le \theta \le 2\pi$

$\theta = \pi/6, 5\pi/6, 3\pi/2$ $\therefore$ B

6. $m:n$ $A(\overset{x_1}{-5}, \overset{y_1}{6})$

$(3:-2)$ $B(\overset{x_2}{4}, \overset{y_2}{3})$

$$x = \frac{nx_1 + mx_2}{m+n} = \frac{(-2 \times -5) + (3 \times 4)}{3-2}$$

$= 22$ $\therefore$ C

7. $\int \cos^2 x \, dx$ $\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = 1/2(\cos 2x + 1)$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2x + x\right) + C$

$= \frac{\sin 2x + 2x}{4} + C$ $\therefore$ A

8. Gradient fn is $\frac{dy}{dx} = \frac{2}{2x+1}$

When $x = 0$ $M_1 = 2$

$x = 1/2$ $M_2 = 1$

$\therefore$ $\tan\alpha = \left|\frac{M_1 - M_2}{1 + M_1 M_2}\right| = \left|\frac{2-1}{1+2\times1}\right| = \frac{1}{3}$

$\alpha \doteqdot 18^\circ$ $\therefore$ B

9. $x = (V\cos\alpha)t$ $\therefore$ $V\cos\alpha = 36$.

$V\sin\alpha = 15$

$\therefore$ $\tan\alpha = 15/36 = 5/12$

$V^2(\sin^2\alpha + \cos^2\alpha) = 15^2 + 36^2$

$\therefore$ $V = \sqrt{15^2+36^2} = 39$ $\therefore$ D

10. $r\sin(x-\alpha) = r\sin x\cos\alpha - r\cos x\sin\alpha$

$r\cos\alpha = 8$ $\tan\alpha = 15/8$

$r\sin\alpha = 15$ $\alpha = 61^\circ 56'$

$r = \sqrt{8^2+15^2}$

$= 17$ $\therefore$ C

Question 11

$a = -2 \qquad \alpha+\beta+\gamma = -b/a = 0/-2 = 0$

$b = 0 \qquad \alpha\beta+\alpha\gamma+\beta\gamma = c/a = 5/-2$

$c = 5$

$d = -1 \qquad \alpha\beta\gamma = -d/a = -(-1)/-2 = -1/2$

a)

i) $\beta\alpha + \beta\gamma + \alpha\gamma = -5/2$ // 1 ✓/w

ii) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$

$= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ 1 for correct amalgamation

$= \frac{-5/2}{-1/2}$

$= 5$ // 1 correct answer

b) $\frac{5-x}{x} < 3$

$\times x^2 \qquad \times x^2$

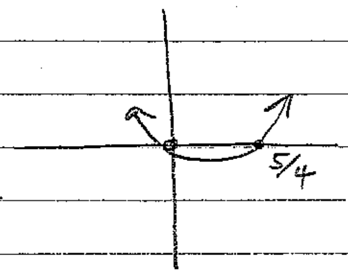
$x(5-x) < 3x^2$

$5x - x^2 < 3x^2$

$4x^2 - 5x > 0$

$x(4x-5) > 0$

$x < 0$ or $x > 5/4$ //



③ – Completely correct

② – 1 error

① – 2 errors

c) $v = x - 5$ m/s

$a_x = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\,\frac{1}{2}(x-5)^2$

$= 2\left(\frac{1}{2}\right)(x-5)^1 \cdot 1$

$= x - 5$

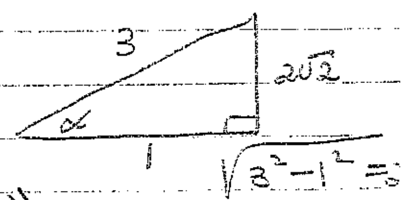
$= v$

$\therefore$ acceleration is velocity for all x.

② – correct method and execution

① – 1 error.

d) $\tan\left(\cos^{-1}\left(-\frac{1}{3}\right)\right) = -2\sqrt{2}$ //

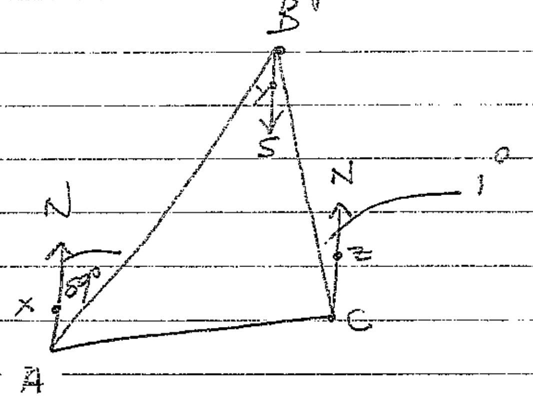


$\cos^{-1}(-1/3)$ is obtuse $\therefore \tan(\cos^{-1}(-1/3)) < 0$

② – correct answer; accept $\sqrt{8}$

① – one error

11e) Birds eye view



i) $\angle ABC = \angle ABY + \angle CBY$ (adjacent angles)

$\angle ABY = \angle XAB = 59°$ (alternate angles $AX \parallel BY \parallel NC$

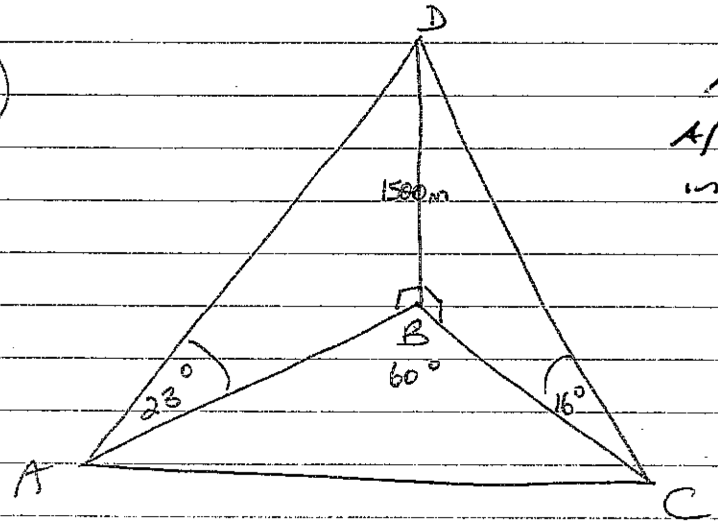
$\angle CBY = \angle NCB = 1°$ (" " "

$\therefore \angle ABC = 59° + 1°$

$= 60°$

1 – Correctly demonstrating

ii)



Any sol^{n}s which assumed A/C were East/west were incorrect

1 — ✓/D

Ignored any diagrams which used error in i)

iii) In $\triangle ABD$ $\tan 23° = \frac{1500}{AB}$ $\therefore AB = \frac{1500}{\tan 23°} = 1500\cot 23° = 1500\tan 67°$

In $\triangle DBC$ $\tan 16° = \frac{1500}{BC}$ $\therefore BC = \frac{1500}{\tan 16°} = 1500\cot 16° = 1500\tan 74°$

$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos 60°$

$= (1500\tan 67°)^2 + (1500\tan 74°)^2 - \cancel{2}(1500\tan 67°)(1500\tan 74°) \times$

$= 1500^2(\tan^2 67° + \tan^2 74° - \tan 67°\tan 74°)$

$= 21366599.19$

$AC = 4622.401\ldots$

$= 4622$ m to nearest m.

③ – Completely correct.

② – 1 error;

① – 2 errors, or just finding AB and BC

Question 12:

a) i) Let $P(x) = x^3 - 5x^2 + 8x - 4$

$P(1) = 1 - 5 + 8 - 4 = 0$ ①

$\therefore x - 1$ is a factor

$$\begin{array}{r|l} & x^2 - 4x + 4 \\ \hline x-1 & x^3 - 5x^2 + 8x - 4 \\ & -(x^3 - x^2) \\ \hline & -4x^2 + 8x \\ & -4x^2 + 4x \\ \hline & 4x - 4 \\ & 4x - 4 \\ \hline & 0 \end{array}$$

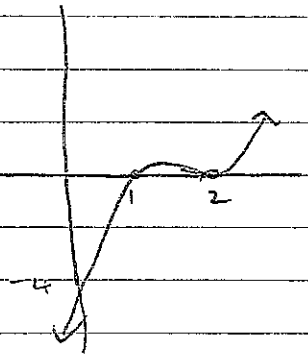
$P(x) = (x-1)(x-2)^2$ // ①

ii) $x^3 - 5x^2 + 8x - 4 > 0$

$(x-1)(x-2)^2 > 0$

$x > 1$ except $x = 2$ //

or $1 < x < 2,\ x > 2$ // ①



b) $u = x^2 + 1$

$\frac{du}{dx} = 2x \quad \therefore \quad du = 2x\,dx.$

$\frac{1}{2}du = x\,dx$

When $x = 0 \quad u = 1$ ①

$x = 2 \quad u = 5$

$$\int_0^2 \frac{x}{(x^2+1)^3}\,dx = \int_1^5 \frac{1}{u^3} \cdot \frac{1}{2} \cdot du \quad ①$$

$$= \frac{1}{2}\int_1^5 u^{-3}\,du$$

$$= \frac{1}{2}\left[\frac{u^{-2}}{-2}\right]_1^5$$

$$= -\frac{1}{4}\left[\frac{1}{u^2}\right]_1^5$$

$$= -\frac{1}{4}\left(\frac{1}{25} - \frac{1}{1}\right)$$

$$= \frac{6}{25} \quad // \quad ①$$

12e) $V = 2\cos\frac{3\pi}{x}$

Find dx/dt when $dV/dt = 12$ and $x = 1.2$

$$dx/dt = dx/dV \cdot dV/dt.$$

$$dV/dx = d/dx\ 2\cos(3\pi x^{-1})$$

$$= \frac{-3\pi}{x^2} \cdot 2 \cdot -\sin\left(\frac{3\pi}{x}\right)$$

$$= \frac{6\pi}{x^2} \cdot \sin\left(\frac{3\pi}{x}\right) \qquad 1$$

$$\frac{dx}{dV} = \frac{x^2}{6\pi \sin\left(\frac{3\pi}{x}\right)}$$

When $x = 1.2$ $\quad dx/dV = \frac{1.2^2}{6\pi \sin 3\pi/1.2} = \frac{36/25}{6\pi \sin 5\pi/2} \qquad 1$

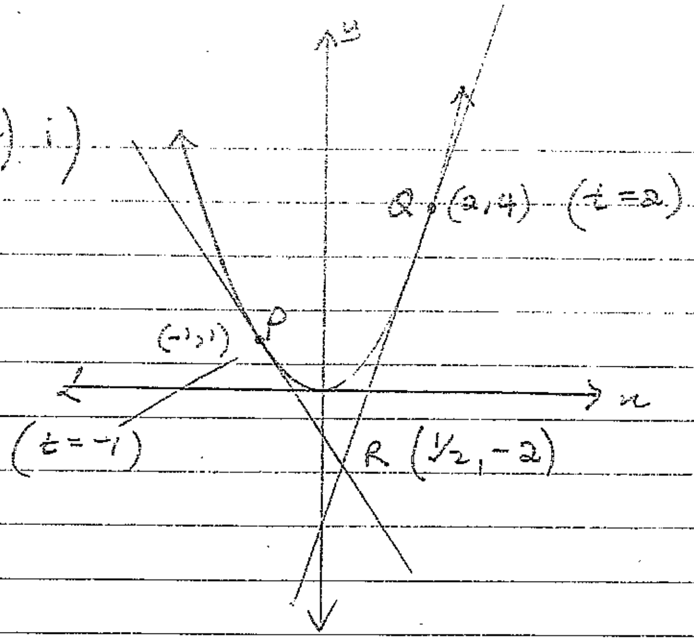
$$= \frac{36}{25} \times \frac{1}{6\pi}$$

$$= 6/25\pi$$

$$\therefore dx/dt = \frac{6}{25\pi} \times 12$$

$$= \frac{72}{25\pi}\ m/h \qquad 1$$

a) i)



1

ii) $x = t, \; y = t^2$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= 2t$

$\therefore$ At P, $\frac{dy}{dx} = -2$

Tangent at P is $y - 1 = -2(x - -1)$

$y = -2x - 1$ (1)

At Q, $\frac{dy}{dx} = 4$

Tangent at Q is $y - 4 = 4(x - 2)$

$y = 4x - 4$ (2) 1

Solve (1) + (2) simultaneously to find R

$-2x - 1 = 4x - 4$

$6x = 3$

$x = 1/2$

When $x = 1/2$ $y = 4(1/2) - 4$ using (2)

$y = 2 - 4 = -2$ 1

$\therefore$ R is $(1/2, -2)$ //

iii) Gradient at $T(t, t^2)$ is $\frac{dy}{dx} = 2t$

Tangent at T is $y - t^2 = 2t(x - t)$

Midpoint of QR is $\left(\frac{2 + 1/2}{2}, \frac{4 + -2}{2}\right) = (5/4, 1)$

$(5/4, 1)$ lies on tangent at T $\therefore$ $1 - (t)^2 = 2t(5/4 - t)$ 1

$1 - t^2 = \frac{10t}{4} - 2t^2$

$4 - 4t^2 = 10t - 8t^2$

$4t^2 - 10t + 4 = 0$

$2t^2 - 5t + 2 = 0$

$$2t^2 - 5t + 2 = 0$$
$$(2t - 1)(t - 2) = 0$$
$$t = \tfrac{1}{2} \text{ or } t = 2$$
but $t = 2$ is at Q $\therefore$ T is $\left(\frac{1}{2}, \frac{1}{4}\right)$ 1

Gradient of tangent at T is $2t = 2 \times \tfrac{1}{2} = 1$
Gradient of PQ is $\frac{4-1}{2--1} = 1$

$\therefore$ Tangent at T is parallel to PQ // 1

Q13

a) $$\int_0^{2/3} \frac{dx}{\sqrt{4-9x^2}} = k\pi$$

$$\int_0^{2/3} \frac{dx}{\sqrt{9\left(\frac{4}{9} - x^2\right)}} = k\pi$$

$$\frac{1}{3}\int_0^{2/3} \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2 - x^2}} = k\pi$$

$$\frac{1}{3}\left[\sin^{-1}\frac{x}{(2/3)}\right]_0^{2/3} = k\pi$$ (1) coefficient $\frac{1}{3}$

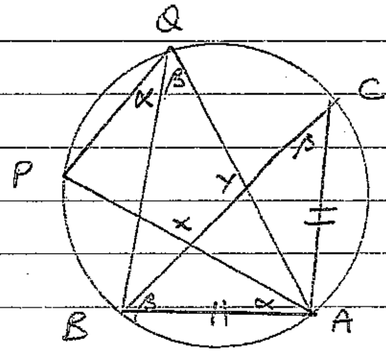
$$\sin^{-1}\frac{2/3}{2/3} - \sin^{-1}\frac{0}{2/3} = 3k\pi$$

$$\pi/2 - 0 = 3k\pi$$

$$3k = 1/2$$

$$k = 1/6 \quad //$$ (1)

b) i) $\angle PQB = \angle PAB = \alpha$
(Angles at the circumference subtended by arc PB are equal)



(1)

ii) $\angle ACB = \angle CBA = \beta$ (angles opposite equal sides in isosceles $\triangle ABC$)

$\angle AQB = \angle ACB$ (angles at the circumference subtended by arc AB are equal)

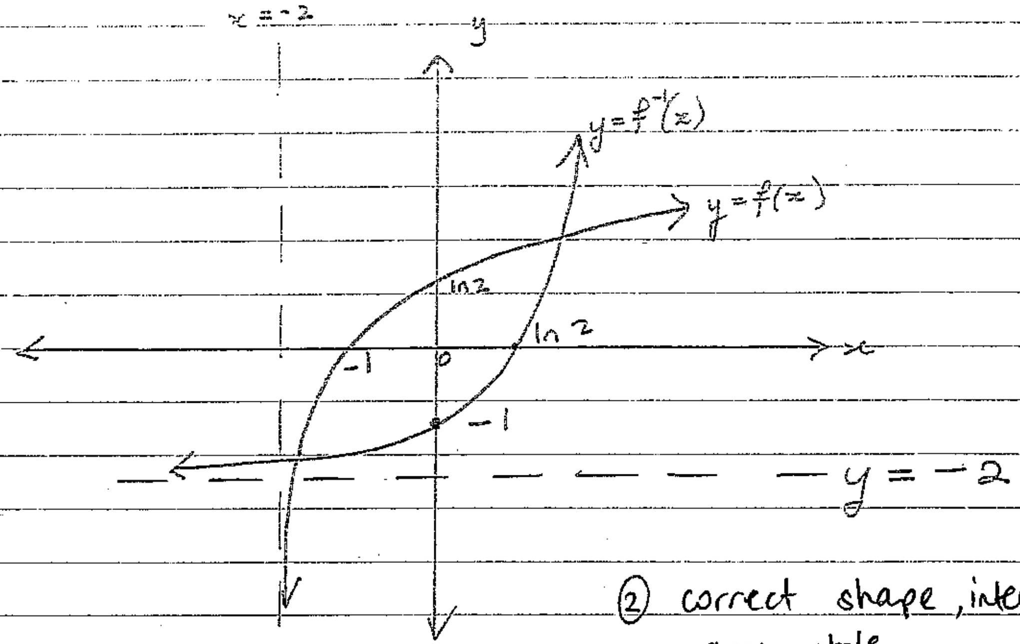
$\therefore \angle AQB = \beta$ (1)

iii) $\angle YXA = \angle XBA + \angle XAB = \beta + \gamma$ (exterior angle of $\triangle XAB$) (1)

$\angle PQY = \angle PQB + \angle AQB = \alpha + \beta$ (adjacent angles, i + ii)

$\therefore \angle PQY = \angle YXA = \alpha + \beta$

$\Rightarrow$ XYQP is a cyclic quadrilateral (exterior angle equals the opposite interior angle) (1) i)



② correct shape, intercepts, asymptote
−① each error

ii) $y=f(x)$: $y=\ln(x+2)$ $f(x)$

$y=f^{-1}(x)$: $x=\ln(y+2)$

$e^x = y+2$

$y = e^x - 2$ $f^{-1}(x)$

∴ The intersection of the 2 curves is where :

$$\ln(x+2) = e^x - 2$$ ①

Since $f(x)$ and $f^{-1}(x)$ intersect on $y=x$ ∴ $x = e^x - 2$

$\Rightarrow e^x - x - 2 = 0$ gives the ①
x – co-ordinates of the points of intersection

iii) When $x=1$ $e^x - x - 2 = e^1 - 1 - 2 = e^1 - 3 < 0$ ①
When $x=2$ $e^x - x - 2 = e^2 - 2 - 2 = e^2 - 4 > 0$

Since there is a change in sign of $e^x - x - 2$ from $x=1$ to $x=2$ and $e^x - x - 2$ is continuous between these values, there must be a root α, of $e^x - x - 2 = 0$ such that $1 < \alpha < 2$. ①

iv) Let $f(x) = e^x - x - 2$ $f(1.2) = e^{1.2} - 1.2 - 2$
$f'(x) = e^x - 1$ $f'(1.2) = e^{1.2} - 1$

New approximation
$$a_1 = a_0 - \frac{f(a_0)}{f'(a_0)} = 1.2 - \frac{e^{1.2} - 3.2}{e^{1.2} - 1}$$ ①
$$= 1.14822\ldots = 1.1 \text{ to 1 dp}$$ ①

Q14

a) Let the statement be:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\cdots\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+2}{2n+2}$$

$n \geq$

When $n=1$ $\quad LHS = 1-\frac{1}{(1+1)^2} = 1-\frac{1}{2^2} = 3/4$

$RHS = \frac{1+2}{2(1)+2} = \frac{3}{4}$

Since LHS = RHS, statement is true for $n=1$ (1)

Assume the statement is true for $n=k$ $\quad k \geq 1$

ie $\left(1-1/2^2\right)\left(1-1/3^2\right)\cdots\cdots\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+2}{2k+2}$ •

Show the statement is true for $n=k+1$

ie $\left(1-1/2^2\right)\left(1-1/3^2\right)\cdots\cdots\left(1-\frac{1}{(k+1)^2}\right)\left(1-\frac{1}{(k+2)^2}\right)=\frac{k+3}{2k+4}$ •

$LHS = \left(\frac{k+2}{2k+2}\right)\left(1-\frac{1}{(k+2)^2}\right)$ • By assumption (1)

must have 3 dot points for mark.

not penalized if forgotten

$$= \frac{\cancel{k+2}}{2(k+1)} \times \frac{(k+2)^2-1}{(k+2)^{\cancel{2}}}$$

$$= \frac{(k+2)^2-1}{2(k+1)(k+2)}$$

$$= \frac{k^2+4k+3}{2(k+1)}$$

$$= \frac{(k+3)\cancel{(k+1)}}{2\cancel{(k+1)}(k+2)}$$

$$= \frac{k+3}{2k+4}$$

$= RHS$.

logical and shown to the end (1)

$\therefore$ The statement is true for $n=k+1$ if it is true for $n=k$ and since it is true for $n=1$, it is also true for $n=1+1=2$, $2+1=3$ and so on for all positive integer values of n.

Q14 continued

b) $\frac{dN}{dt} = kN\left(1 - \frac{N}{2000}\right)$ When $t=0$ $N=1000$

i) If $N = \frac{2000e^{kt}}{1+e^{kt}}$

then $\frac{dN}{dt} = \frac{(1+e^{kt})\,2000ke^{kt} - 2000e^{kt}\cdot ke^{kt}}{(1+e^{kt})^2}$

(1) correctly differentiated

$= \boxed{\frac{2000ke^{kt}}{(1+e^{kt})}}\frac{((1+e^{kt}) - e^{kt})}{(1+e^{kt})}$

$\frac{dN}{dt} = \frac{2000ke^{kt}}{(1+e^{kt})^2}$

(1) must show how

$= kN\left(\frac{1}{1+e^{kt}}\right)$

Now $1 - \frac{N}{2000} = 1 - \frac{\cancel{2000}e^{kt}}{\cancel{2000}(1+e^{kt})}$

$= 1 - \frac{e^{kt}}{1+e^{kt}}$

$= \frac{1+e^{kt}-e^{kt}}{1+e^{kt}}$ (1) must show how.

$= \frac{1}{1+e^{kt}}$

$\therefore \frac{dN}{dt} = kN\left(\frac{1}{1+e^{kt}}\right)$

$= kN\left(1 - \frac{N}{2000}\right)$

$\Rightarrow N = \frac{2000e^{kt}}{1+e^{kt}}$ is a solution of the equation

ii) When $t=10$ $N=1500$ $\therefore 1500 = \frac{2000e^{10k}}{1+e^{10k}}$

$1500 + 1500e^{10k} = 2000e^{10k}$

$500e^{10k} = 1500$

$e^{10k} = 3$ (1)

$10k = \ln 3$

$k = \frac{1}{10}\ln 3$ (1)

$\therefore$ When $N = 1800$ $1800 = \frac{2000e^{kt}}{1+e^{kt}}$

$$1800 + 1800e^{kt} = 2000e^{kt}$$ $\left(k = \frac{\ln 3}{10}\right)$

$$200e^{kt} = 1800$$
$$e^{kt} = 9$$
$$kt = \ln 9$$
$$t = \tfrac{1}{k}\ln 9$$
$$= \frac{10\ln 9}{\ln 3}$$
$$= \boxed{20} \text{ days} \quad (1)$$

as value of t asked for.

c) i) $4\frac{d^2y}{dx^2} = 5 - y$

For SHM $\ddot{x} = -n^2(x-b)$ where b is the centre of motion.

$$4\ddot{y} = 5 - y$$
$$4\ddot{y} = -(y-5)$$
$$\boxed{\ddot{y} = -\tfrac{1}{4}(y-5)} \quad (1)$$ which is SHM with $\boxed{\text{centre } 5}$. (1)

ii) $T = \frac{2\pi}{n} = \frac{2\pi}{(\frac{1}{2})} = \boxed{4\pi}$ (1)

iii) $y = a - b\cos nt$ $\quad y = 1 \leftarrow$ c of M. = 5
$= -b\cos nt + a$ $\quad y = 9$

$y = -b\cos nt$ begins at $-b + a$ $\quad \therefore$ amplitude is 4
$\therefore -4 + a = 1 \Rightarrow a = 5$. $\quad$ ie. $b = 4$

$$\therefore \boxed{a = 5 \quad b = 4 \quad n = \tfrac{1}{2}} \quad (1)$$

(IV) For $y = 3$

$$3 = 5 - 4\cos \tfrac{t}{2}$$
$$\tfrac{1}{2} = \cos \tfrac{t}{2}$$
$$t/2 = \pi/3, \ 2\pi/3 \text{ etc}$$
$$t = 2\pi/3, \ 4\pi/3 \text{ etc} \quad (1)$$

* needed to round up!!

$\therefore$ earliest time for boat to enter the creek is when $t = \frac{2\pi}{3}$ h $= 2h\, 5'\, 39.82''$
$= 2h\, 6'$ to nearest min.

ie: 3:06pm (1)