QUESTION 1

SCEGGS REDLANDS 1998 3U

(a) (i) *Find* $\int \frac{dx}{x^2+4}$

(ii) *Find* $\int \frac{x^2dx}{x^3-8}$

(b) *Evaluate:* (i) $\int_2^7 \frac{xdx}{\sqrt{x+2}}$ using the substitution $u=x+2$

(ii) $\int_2^7 \frac{xdx}{\sqrt{x+2}}$ using the substitution $u=\sqrt{x+2}$

(c) (i) *Show* that $\tan x \equiv \frac{\sin 2x}{1+\cos 2x}$

(ii) *Hence* evaluate $\tan\frac{\pi}{12}$.

QUESTION 2:

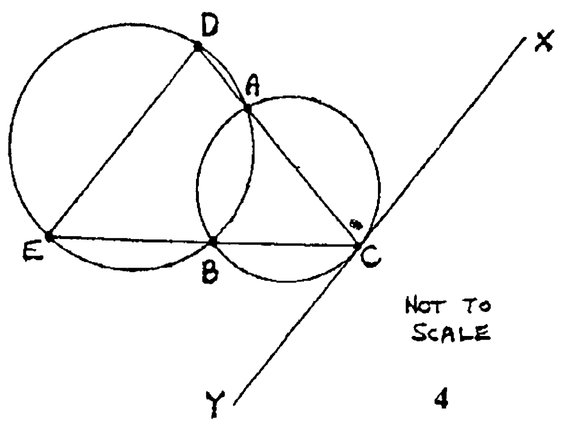
(a) A is the point $(3,2)$ and B is the point (x_B, y_B) .
The point P $(-4,2)$ divides AB internally in the ratio 2 : 5
(i.e. AP : PB = 2 : 5). *Find* the values of x_B and y_B .

(b) (i) If $f(x)=\sin^{-1}\frac{x}{2}$ *find* $f^{-1}(x)$.

(ii) *State* the domain and range of $f^{-1}(x)$.

(iii) *Sketch* the graph of $3y=\sin^{-1}\frac{x}{2}$ stating clearly its domain and range.

(c) Two circles intersect at A and B.
From any point C on the smaller circle lines CAD and CBE are drawn cutting the larger circle at D and E respectively.
XY is the tangent at C .
Prove formally that DE is parallel to XY.



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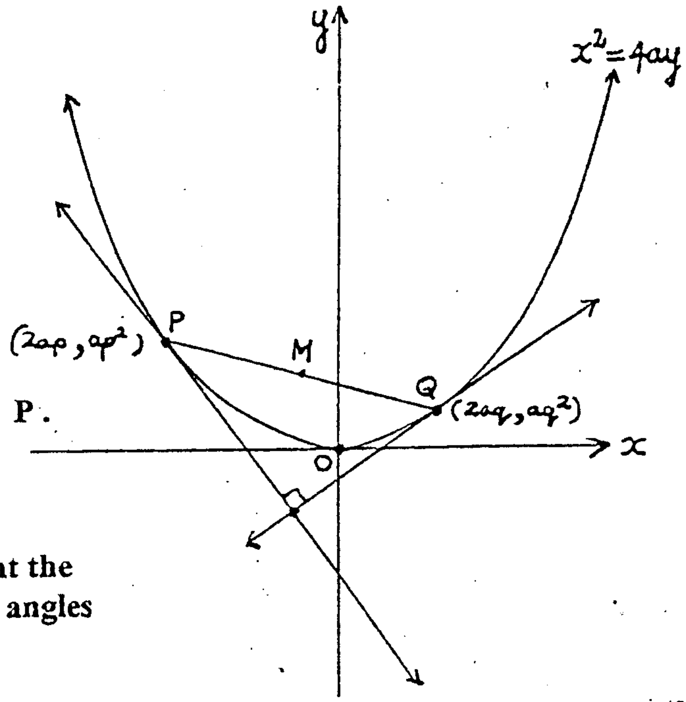
QUESTION 3: 12 marks (Start a new answer booklet) **MARKS**

(a) In the diagram P and Q are two points on the parabola $x^2 = 4ay$ having coordinates respectively of $(2ap, ap^2)$ and $(2aq, aq^2)$. **9**



(i) *Show* that the equation of PQ is $y = (\frac{p+q}{2})x - apq$.

(ii) *Find* the gradient of the tangent at P.

(iii) Hence *write down* the condition that the tangents at P and Q are at right angles to each other.

(iv) What are the coordinates of the midpoint M of PQ ?

(v) *Show* that the locus of M, as the points P and Q move around the parabola with the tangents at P and Q being perpendicular to each other, is another parabola with equation $x^2 = 2a(y-a)$. *Write down* the coordinates of the vertex and focus of this locus parabola.

(b) Air is being pumped into a spherical balloon at the rate of 450 cm^3/sec. **3**

Given that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, *calculate* the rate at which the radius of the ballon is increasing at the instant when its radius reaches 15 cm.

QUESTION 4: 12 marks (Start a new answer booklet) **MARKS**

(a) If $f(x) = x^3 + 3x^2 - 10x - 24$ **3**

Calculate $f(-2)$ and *express* $f(x)$ as the product of three linear factors.

(b) If α, β, γ are the roots of the equation $x^3 + 2x^2 - 3x + 5 = 0$ **3**
state the values of :

(i) $\alpha + \beta + \gamma$

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$

(iii) $\alpha\beta\gamma$

(iv) *Hence calculate* the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$.

(c) *Solve* for x given that $\dfrac{2x+3}{x-4} > 1$. **3**
Sketch your solution on a number line.

(d) *Differentiate* with respect to x : **3**

(i) $y = x \sin^{-1}\dfrac{x}{2}$

(ii) $y = \tan(x^3)$

(iii) $y = \dfrac{e^{2x}}{1 + \cos x}$

QUESTION 5: 12 marks (Start a new answer booklet) **MARKS**

(a) (i) *Show* that $\dfrac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}=\dfrac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}$. 6

(ii) *Find* the most general solution for θ satisfying the equation

$$4\sin^2\theta-1=0 \quad .$$

(b) A body is heated to a temperature of 120 °C and left to cool in a room whose room temperature is 20 °C . After 10 minutes the temperature of the body cools to 80 °C . 6

You may assume that the rate of cooling can be expressed in the differential equation

$$\frac{dT}{dt}=-k(T-20) \quad .$$

(i) *Show* by integration that the temperature T can be expressed in the form

$$T=20+100e^{-kt} \text{ where } k=-\frac{1}{10}\ln\frac{3}{5} \ .$$

(ii) What will be the temperature to the nearest degree of the body after a further 25 minutes?

QUESTION 6: 12 marks (Start a new answer booklet) MARI

(a) The speed $|v|$ of a particle moving along the x-axis is given by the equation

$$v^2 = 12 + 8x - 4x^2$$

where x is the displacement of the point from the origin. 5

(i) *Prove* that the motion is simple harmonic.

(ii) *Find* its centre of motion.

(iii) *Calculate* its period.

(iv) *Show* that its amplitude is 2 units.

(b) (i) *Write down* an expression for $\sin^2\theta$ in terms of $\cos 2\theta$. 3

(ii) *Hence* evaluate $\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta$.

(c) (i) *Sketch* the curve $y = 1 + \sin x$ for the domain $-\pi \le x \le \pi$. 4

(ii) *Hence* sketch the shape of the solid of revolution formed by rotation of this curve about the x-axis .

(iii) *Show* that the volume of this solid formed by rotation about the x-axis is $3\pi^2$ units^2.

QUESTION 7: 12 marks (Start a new answer booklet) **MARKS**

(a) A projectile P is projected with initial velocity U at angle α to the horizontal. **8**

Show by using $\ddot{x} = 0$ and $\ddot{y} = -g$ and *without* assuming a numerical value for g that :

(i) The time taken to reach maximum height is given by

$$t = \frac{U \sin\alpha}{g} \quad .$$

(ii) *Find* this maximum height reached by the projectile.

(iii) *Show* that to obtain a maximum range, the angle of projection must be 45^o .

(b) A missile is projected with a speed of *100 m/s* at an elevation of 45^o aimed at a tall building which is a horizontal distance of *400 m* from the point of projection. **4**

(i) *Find* the time of flight until the missile strikes the building.

(ii) *Find* how high on the building the missile strikes. (You may use the approximation $g \approx 10m/s^2$ for this part, ie part (ii) .)

END OF EXAMINATION

Question No. 1

(i) $\int \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$ (1)

(ii) $\int \frac{x^2\,dx}{x^3-8} = \frac{1}{3}\log_e(x^3+8) + C$ (1)

(b). (i) $I = \int_2^7 \frac{x\,dx}{\sqrt{x+2}}$ $u = x+2$, $x = u-2$ $\therefore du = dx$ when $x=2$, $u=4$; when $x=7$, $u=9$

$\therefore I = \int_4^9 \frac{(u-2).du}{u^{1/2}}$

$= \int_4^9 (u^{1/2} - 2u^{-1/2})\,du$ (3)

$= \left[\frac{2}{3}u^{3/2} - 4u^{1/2}\right]_4^9$

$= (\frac{2}{3}\times 27 - 4\times 3) - (\frac{2}{3}\times 8 - 4\times 2)$

$= 18 - 12 - 5\frac{1}{3} + 8$

$= 8\frac{2}{3}$ Unit.

(ii) $I = \int_2^7 \frac{x\,dx}{\sqrt{x+2}}$ $u = \sqrt{x+2}$ $\therefore u^2 = x+2$ $\therefore 2u\,du = dx$ when $x=2$, $u=2$; when $x=7$, $u=3$

$= \int_2^3 \frac{(u^2-2).2u\,du}{u}$

$= 2\int_2^3 (u^2-2)\,du$ (3)

$= 2\left[\frac{u^3}{3} - 2u\right]_2^3$

$= 2[(9-6) - (\frac{8}{3} - 4)]$

$= 8\frac{2}{3}$ units

c). To show $\tan x = \frac{\sin 2x}{1+\cos 2x}$

$RHS = \frac{\sin 2x}{1+\cos 2x}$

$= \frac{2\sin x\cos x}{1+2\cos^2 x - 1}$ (2)

$= \frac{2\sin x\cos x}{2\cos^2 x}$

$= \frac{\sin x}{\cos x}.\frac{\cos x}{\cos x}$

$= \tan x$

$= LHS$

Using the above: $\tan\frac{\pi}{12} = \frac{\sin 2\times\frac{\pi}{12}}{1+\cos 2\times\frac{\pi}{12}}$
(Replace x by $\frac{\pi}{12}$)

$= \frac{\sin\frac{\pi}{6}}{1+\cos\frac{\pi}{6}}$ (2)

$= \frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}}$

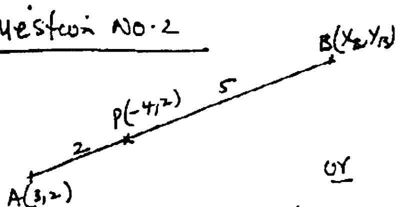
$= \frac{1}{2+\sqrt{3}}$

$(\text{or} = 2-\sqrt{3})$

Total 12 Marks

Question No. 2

(a).

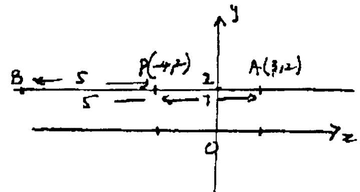


$\frac{2x_B + 15}{7} = -4$
$\therefore 2x_B + 15 = -28$
$\therefore 2x_B = -43$
$\therefore x_B = -21\frac{1}{2}$ (1)

$\frac{2y_B + 10}{7} = 2$
$\therefore 2y_B + 10 = 14$
$\therefore 2y_B = 4$
$\therefore y_B = 2$ (1)

$\therefore$ B is the point $(-21\frac{1}{2}, 2)$

or



AP = 7 units which is equivalent to

1 part = $3\frac{1}{2}$ Units.

PB corresponds to 5 parts $\therefore$ length PB $= 5\times 2\frac{1}{2}$ Unit $= 17\frac{1}{2}$ Units

$\therefore$ Co-ords of B are: $(-21\frac{1}{2}, 2)$

(b). (i) $f(x) = \sin^{-1}\frac{x}{2}$

$f: \quad y = \sin^{-1}\frac{x}{2}$ (2)
$f^{-1}: \quad x = \sin^{-1}\frac{y}{2}$

$\therefore \sin x = \frac{y}{2}$

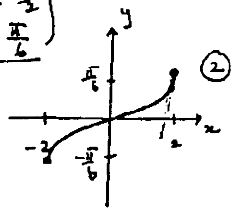
$\therefore y = 2\sin x$

(ii) $\boxed{D: -\frac{\pi}{2} \le x \le \frac{\pi}{2}}$ $R: -1 \le \frac{y}{2} \le 1$ (1)

(iii) $3y = \sin^{-1}\frac{x}{2}$

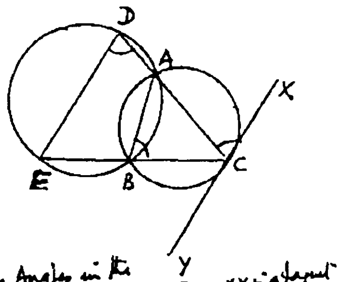
$D: -1 \le \frac{x}{2} \le 1$ ie $-2 \le x \le 2$ (1)

$R: -\frac{\pi}{2} \le 3y \le \frac{\pi}{2}$ ie $-\frac{\pi}{6} \le y \le \frac{\pi}{6}$



(2)

(c) (4)



Data: XY is a tangent at C. CAD, CBE and st lines.
To Prove: DE $\parallel$ XY
Constr: Join AB

Proof:
$\angle ACX = \angle ABC$ — Angles in the Alternate Segment - XY is a tangent

But $\angle ABC = \angle ADE$ — Exterior $\angle$ of Cyclic Quad ABED

$\therefore \angle ACX = \angle ADE$

But these are alternate angles to lines XY and DE

$\therefore$ XY $\parallel$ DE

Total 12 Marks

Question 2 (continued)

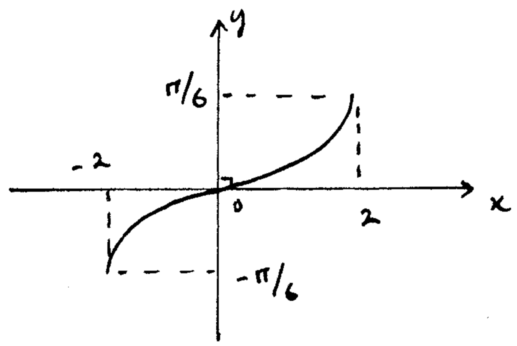
iii) $3y = \sin^{-1} \frac{x}{2}$

$\therefore \quad y = \frac{1}{3} \sin^{-1} (x/2)$

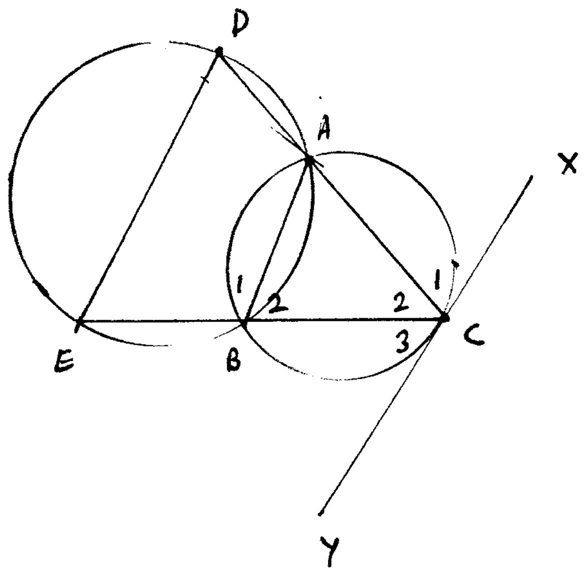
Domain : $-1 \le \frac{x}{2} \le 1$

$\therefore \quad -2 \le x \le 2$

Range : $-\pi/6 \le y \le \pi/6$



c)



DE is // to XY if

$\angle D = \angle C_1$ (alternate $\angle$'s)

Here, $\angle D + \angle B_1 \overset{=180^\circ}{}$ (opp. $\angle$'s in cyclic quad.)

$\therefore \quad \angle D = 180^\circ - \angle B_1$

$= \angle B_2$

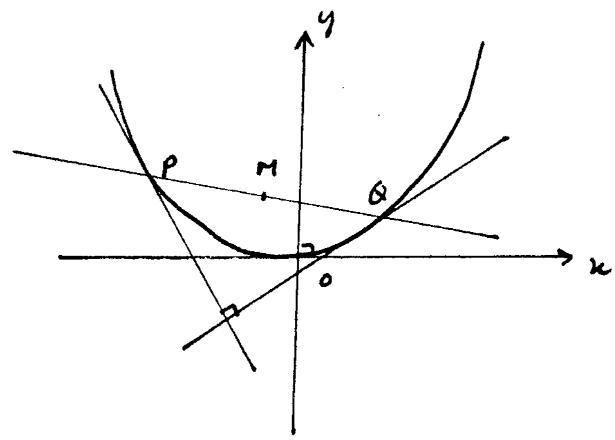
Also, $\angle B_2 = \angle C_1$

($\angle$'s between tangent & chord = $\angle$ opposite to that chord)

$\Rightarrow \quad \angle D = \angle C_1$

$\Rightarrow \quad ED \parallel XY$ as required.

Question 3



i)

Equation of PQ is

$$\frac{y - ap^2}{x - 2ap} = \frac{aq^2 - ap^2}{2aq - 2ap}$$

$$\frac{y - ap^2}{x - 2ap} = \frac{a(\cancel{q-p})(q+p)}{2a(\cancel{q-p})}$$

$$y - ap^2 = \left(\frac{q+p}{2}\right) \cdot (x - 2ap)$$

$$\therefore y = \left(\frac{p+q}{2}\right) x - apq$$

ii) $x^2 = 4ay \Rightarrow y = \frac{x^2}{4a}$

$\therefore \quad \frac{dy}{dx} = \frac{x}{2a} \Rightarrow m_1 = \frac{2ap}{2a}$

$\therefore \quad m_1 = p$ (gradient of tange at P)

Similarly, gradient of tangent at Q is $m_2 = q$

Question 3 (continued)

iii) The tangents are $\perp$ if

$m_1 \times m_2 = -1$

$\therefore \quad p \times q = -1 \Rightarrow pq = -1$

iv) $M\left(a(p+q), \dfrac{a(p^2+q^2)}{2} \right)$

$x_M = a(p+q)$

$y_M = \frac{a}{2}(p^2+q^2)$

and $pq = -1$

$\therefore y = \frac{a}{2}\left[(p+q)^2 - 2pq \right]$

$= \frac{a}{2}\left[\frac{x^2}{a^2} - 2pq \right]$

$y = \dfrac{x^2}{2a} - \frac{a}{2} \times 2 \times -1$

$y = \dfrac{x^2}{2a} + a$

$\therefore \quad x^2 = 2a(y-a)$

$x^2 = 4 \times \frac{a}{2}(y-a)$

$\therefore$ vertex at $(0, a)$

Focus at $\left(0, \frac{3a}{2}\right)$

b) $\dfrac{dv}{dt} = 450 \text{ cm}^3/\text{sec}.$

$V = \frac{4}{3}\pi R^3$

$\therefore \dfrac{dv}{dR} = 4\pi R^2$

$\therefore \dfrac{dR}{dt} = \dfrac{dR}{dv} \cdot \dfrac{dv}{dt}$

$\therefore \dfrac{dR}{dt} = \dfrac{1}{4\pi \times 15^2} \times 450 \quad (R = 15)$

$\dfrac{dR}{dt} = \dfrac{1}{2\pi}$ cm/sec.

$\doteqdot 0.16$ cm/sec.

Question No. 4

(a) $f(x) = x^3 + 3x^2 - 10x - 24$

$f(-2) = (-2)^3 + 3(-2)^2 - 10(-2) - 24$
$= -8 + 12 + 20 - 24$
$= 0$

$\therefore (x+2)$ is a factor

$$\begin{array}{r} x^2 + x - 12 \\ x+2 \,\big)\, \overline{x^3 + 3x^2 - 10x - 24} \\ \underline{x^3 + 2x^2} \\ x^2 - 10x \\ \underline{x^2 + 2x} \\ -12x - 24 \\ -12x - 24 \end{array}$$

$\therefore f(x) = (x+2)(x^2+x-12)$
$= (x+2)(x+4)(x-3)$ (3)

(b). $x^3 + 2x^2 - 3x + 5 = 0$ (α, β, γ)

(i) $\alpha + \beta + \gamma = -2$
(ii) $\alpha\beta + \beta\gamma + \alpha\gamma = -3$
(iii) $\alpha\beta\gamma = -5$
(1½)

(iv) $(\alpha-1)(\beta-1)(\gamma-1)$
$= (\alpha-1)[\gamma\beta - \beta - \gamma + 1]$
$= \alpha\beta\gamma - \alpha\beta - \alpha\gamma + \alpha - \gamma\beta + \beta + \gamma - 1$
$= \alpha\beta\gamma - (\alpha\beta + \beta\gamma + \alpha\gamma) + (\alpha + \beta + \gamma) - 1$
$= -5 - (-3) + (-2) - 1$
$= -5 + 3 - 2 - 1 = \underline{-5}$
(1½)

(c) $\dfrac{2x+3}{x-4} > 1$ $\quad x \neq 4$

$\therefore \dfrac{2x+3}{x-4} \cdot (x-4)^2 > (x-4)^2$
ie $(2x+3)(x-4) > (x-4)^2$
ie $(x-4)[2x+3-(x-4)] > 0$
ie $(x-4)(x+7) > 0$
ie $\underline{x > 4 \text{ or } x < -7}$ (3)

$$\longleftarrow\!\!\circ\text{———}\circ\!\!\longrightarrow \quad -7 \qquad 4$$

(d) (i) $y = x\sin^{-1}\left(\frac{x}{2}\right)$

$\frac{dy}{dx} = \sin^{-1}\left(\frac{x}{2}\right) \times 1 + x \times \frac{1}{2} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$ (1)
$= \sin^{-1}\frac{x}{2} + \dfrac{x}{\sqrt{4-x^2}}$

(ii) $y = \tan(x^3)$

$\frac{dy}{dx} = 3x^2 \cdot \sec^2(x^3)$ (1)

(iii) $y = \dfrac{e^{2x}}{1+\cos x}$

$\frac{dy}{dx} = \dfrac{(1+\cos x).2e^{2x} - e^{2x}(0 - \sin x)}{(1+\cos x)^2}$
$= \dfrac{e^{2x}(2 + 2\cos x + \sin x)}{(1+\cos x)^2}$ (1)

Total: 12 marks

Question No. 5

(a) (i) Show that $\dfrac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \dfrac{\tan\frac{\alpha+\beta}{2}}{\tan\left(\frac{\alpha-\beta}{2}\right)}$

LHS $= \dfrac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta}$
$= \dfrac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2} \cdot \sin\frac{\alpha-\beta}{2}}$ (3)
$= \tan\frac{\alpha+\beta}{2} \cdot \cot\frac{\alpha-\beta}{2}$
$= \dfrac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$ //

(ii) $4\sin^2\theta = 1$
$\therefore \sin^2\theta = \frac{1}{4}$
$\therefore \sin\theta = \pm\frac{1}{2}$ (3)
$\therefore \theta = \underline{\underline{n\pi \pm \frac{\pi}{6}}}$

(b) (i) $\dfrac{dT}{dt} = -k(T-20)$

$\therefore \dfrac{dT}{T-20} = -k\,dt$
$\therefore \displaystyle\int \frac{dT}{T-20} = -k\int dt + A$
$\therefore \log_e(T-20) = -kt + A$
$\therefore T-20 = e^{-kt+A} = Be^{-kt}$

$T = 20 + Be^{-kt}$ ——— (1)

when $t=0$, $T = 120°C$
$\therefore 120 = 20 + B$
$\therefore B = 100$
$\therefore \underline{T = 20 + 100e^{-kt}}$

when $t = 10$ min., $T = 80°C$
$\therefore$ Sub in (1):
$80 = 20 + 100e^{-10k}$
$\therefore 60 = 100e^{-10k}$
$\therefore e^{-10k} = \frac{60}{100} = \frac{3}{5}$ (5)
$\therefore -10k = \log_e\frac{3}{5}$
$\therefore k = -\frac{1}{10}\log_e\frac{3}{5}$

$\therefore T = 20 + 100e^{\frac{1}{10}\left(\log_e\frac{3}{5}\right)t}$

After a further 25 mts: total time taken to cool will be 35 mts.
$\therefore T = 20 + 100e^{\frac{1}{10}\left(\ln\frac{3}{5}\right)35}$ (1)
$= 36.73°\ (37°)$.

Total: 12 marks

Question 6

a) i) $v^2 = 12 + 8x - 4x^2$

$\frac{v^2}{2} = 6 + 4x - 2x^2$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4 - 4x$

$\ddot{x} = -4(x-1)$

This is in the form $\ddot{X} = -4X$

$(X = x - 1)$

$\therefore$ The motion is S.H.M.

ii) centre of motion $x = 1$

iii) $T = \frac{2\pi}{n} = \frac{2\pi}{2}$

$T = \pi$ sec.

iv) max. displacement when $v = 0$,

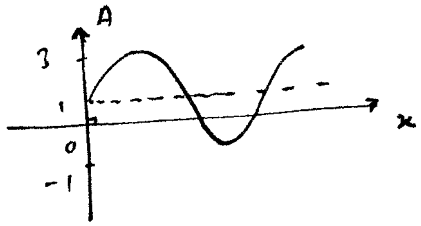
$\therefore 12 + 8x - 4x^2 = 0$

$\therefore 4(-x^2 + 2x + 3) = 0$

$\therefore x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$\therefore x = -1, \quad x = 3$



Amplitude is $A = 2$

b) i) $\cos 2\theta = 1 - 2\sin^2\theta$

$\therefore 2\sin^2\theta = 1 - \cos 2\theta$

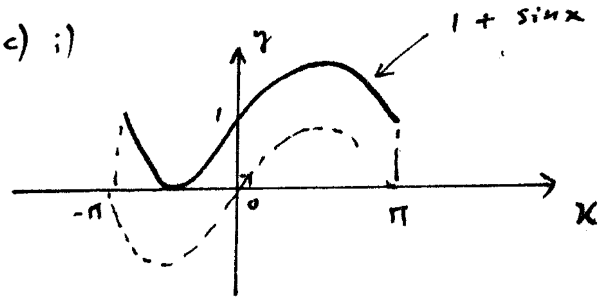
$\therefore \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

ii) $\int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2\theta)\, d\theta$

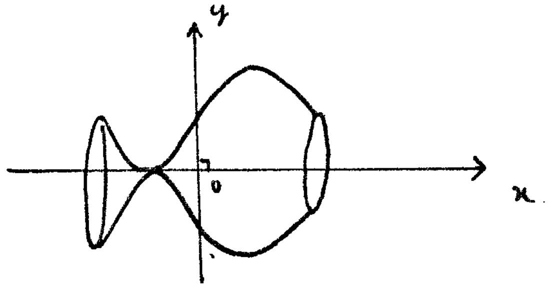
$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$

$= \pi/4$

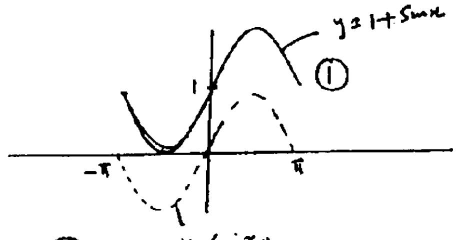
c) i)



ii)



(c). $y = 1 + \sin x$ for $-\pi \leq x \leq \pi$



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iii)

$V = \pi\int_{-\pi}^{\pi} y^2\,dx$

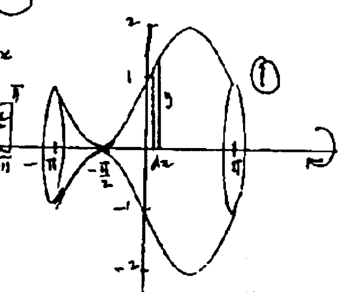
$= \pi\int_{-\pi}^{\pi} (1 + \sin x)^2\,dx$ 2

$= \pi\int_{-\pi}^{\pi} (1 + 2\sin x + \sin^2 x)\,dx$

$= \pi\left[x - 2\cos x + \frac{x}{2} - \frac{\sin 2x}{4}\right]_{-\pi}^{\pi}$

$= \pi\left[\left(\frac{3\pi}{2} + 2\right) - \left(-\frac{3\pi}{2} + 2\right)\right]$

$= 3\pi^2$ unit^3



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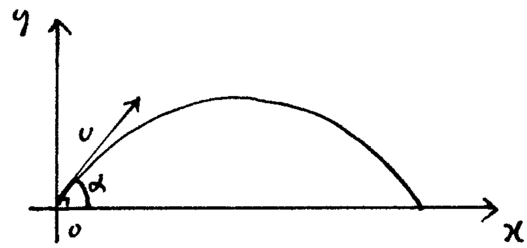
Total: 12 marks

Question 7

a)



Vertical component

$\ddot{y} = -g \Rightarrow \dot{y} = -gt + c_1$

$t = 0 \quad \dot{y} = U\sin\alpha$

$\therefore \dot{y} = -gt + U\sin\alpha$

$\therefore y = -\frac{1}{2}gt^2 + (U\sin\alpha)t + c_2$

$t = 0, \; y = 0 \Rightarrow c_2 = 0$

$\therefore y = -\frac{1}{2}gt^2 + (U\sin\alpha)t$

Time to reach max. height as

$\dot{y} = 0$

$\Rightarrow -gt + U\sin\alpha \Rightarrow t = \frac{U\sin\alpha}{g}$

$\therefore y_{max} = -\frac{1}{2}g \times \frac{(U\sin\alpha)^2}{g^2} +$

$U\sin\alpha \times \frac{U\sin\alpha}{g}$

$y_{max} = \frac{1}{2g}(U\sin\alpha)^2$

Horizontal Component

$\ddot{x} = 0$

$\therefore \dot{x} = c_3 = U\cos\alpha$

$\therefore x = U\cos\alpha \, t$

Time to complete the range

$T = \frac{2U\sin\alpha}{g}$

$\therefore$ Range $= U\cos\alpha \cdot T$

$= U\cos\alpha \times \frac{2U\sin\alpha}{g}$

Range $= \frac{U^2 \sin 2\alpha}{g}$

Range Max if $2\alpha = 90°$

$\therefore \alpha = 45°$

b) $x = U\cos\alpha \, t$

$\therefore 400 = 100\cos 45° \, t$

$\therefore t = \frac{400}{100\cos 45°} = 4\sqrt{2}$ se

$y = -5t^2 + 100\sin 45° \, t$

$t = 4\sqrt{2}$ sec.

$\therefore y = -5 \times (4\sqrt{2})^2 + 50\sqrt{2} \times 4\sqrt{2}$

$y = 240\,m$ (the height of the building)