SAINT IGNATIUS' COLLEGE RIVERVIEW

1995 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time allowed: two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

Attempt all questions.

There are SEVEN questions. Each question is worth 12 marks.

The mark for each main part of a question is shown in square brackets on the right hand side of the page, eg [2].

All necessary working should be shown. Full marks may not be awarded if work is careless or badly arranged.

Approved calculators may be used. A table of standard integrals is provided.

EACH QUESTION is to be returned in a separate Writing Booklet with the question numbers clearly marked on the cover.
Your examination number must be written on each booklet.

Note: this is Trial Paper only and does not necessarily reflect the content or format of the final Higher School Certificate Examination Paper for this subject.

QUESTION 2. Use a Separate Writing Booklet

(a) Evaluate $\int_0^2 \sin\theta\cos\theta \ d\theta$ giving your answer correct to 3 significant figures.

[3]

(b) A particle undergoes simple harmonic motion about the origin 0.
The displacement, x cm, from 0 at time t seconds, is given by

$$x = 2\sin\left(t - \frac{\pi}{3}\right).$$

(i) Write down the amplitude of the motion.

(ii) Find the acceleration as a function of time.

(iii) Express this acceleration as a function of displacement.

(iv) Find the positive value of t for which the speed is a maximum and determine this speed.

[6]

(c) The equation $x^3 + 2x - 8 = 0$ has a root close to $x = 1 \cdot 6$. Use one application of Newton's method to find a better approximation to the root.

[3]

QUESTION 1. Use a Separate Writing Booklet

(a) Solve: $\dfrac{2}{x-2} < 1$. [2]

(b) Differentiate with respect to x

(i) $\log\sqrt{2x}$.

(ii) $f(x) = \sec x^3$. [4]

(c) Evaluate exactly $\displaystyle\int_1^{\sqrt{3}} \frac{dt}{\sqrt{4-t^2}}$. [3]

(d) (i) Sketch the graph of $y = \cos^{-1} x$.

(ii) Hence or otherwise state the number of solutions to the equation

$\cos^{-1} x = x$. [3]

QUESTION 3: Use a Separate Writing Booklet

(a)

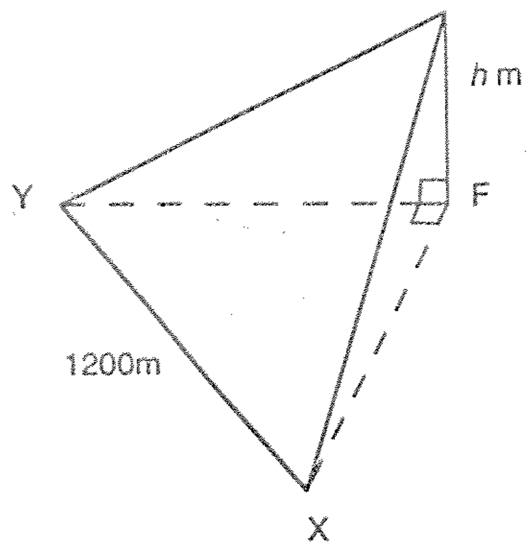


diagram not to scale

Point X is due south and point Y is due west of the foot, F, of a mountain. From X and Y, the angles of elevation of the top of the mountain M are 35° and 43° respectively. If X and Y are 1200 metres apart, show that the height, h metres, of the mountain is given by

$$h = 1200\left(\tan^2 55° + \tan^2 47°\right)^{-\frac{1}{2}}$$

Hence evaluate h correct to 2 significant figures.

[4]

(b) A cube of ice is melting in such a way that it retains the shape of a cube. The surface area of the cube is decreasing at the rate of 10 cm^2 /min. Find the rate at which the volume is decreasing when the edge is 4 cm.

[4]

(c) Use the Principle of Mathematical Induction to prove that $13 \times 6^n + 2$ is divisible by 5 for all n, where n is a positive integer.

[4]

QUESTION 4. Use a Separate Writing Booklet

(a) Evaluate $\int_3^4 t\sqrt{4-t}\ dt$ by using the substitution $t = 4 - u$.

[4]

(b) Evaluate $\int_0^{\frac{\pi}{4}} \cos x \sin^2 x\ dx$.

[2]

(c) The acceleration of a particle moving in a straight line is given by

$$\frac{d^2x}{dt^2} = -\frac{72}{x^2}$$

where x metres is the displacement from the origin after t seconds. When $t = 0$ the particle is 9 metres to the right of the origin with a velocity of 4 metres per second.

You may use the result $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

(i) Show that the velocity, v, of the particle, in terms of x, is

$$v = \frac{12}{\sqrt{x}}.$$

(ii) Find an expression for t in terms of x.

(iii) How many seconds does it take for the particle to reach a point 35 metres to the right of the origin?

[6]

QUESTION 5. Use a Separate Writing Booklet

(a) Consider the circle $x^2 + y^2 - 2x - 14y + 25 = 0$.

Show that if the line $y = mx$ intersects the circle in two distinct points then

$$(1 + 7m)^2 - 25(1 + m^2) > 0 .$$

[3]

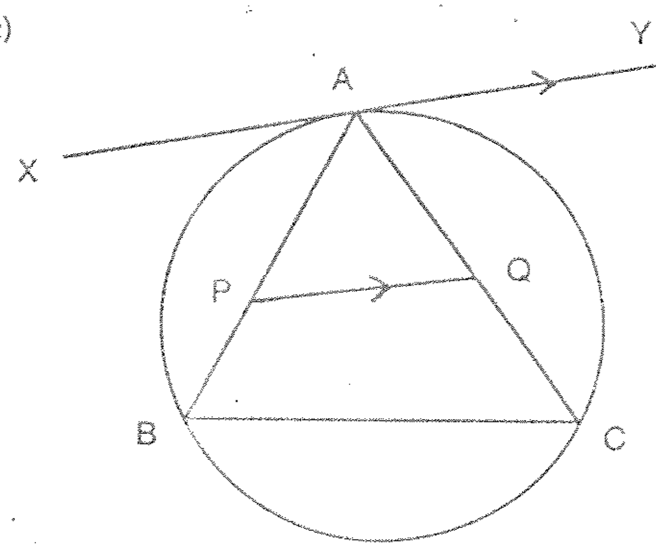
(b) (i) Show that $x = -4 + Qe^{-3t}$ where Q is a constant, satisfies the equation

$$\frac{dx}{dt} = -3(x + 4).$$

(ii) Describe the rate of change of x with respect to t as t increases without bound.

[3]

(c)



Given $AB = AC$.
The tangent at A is parallel to PQ.

diagram not to scale

Prove

(i) $AP = AQ$.

(ii) BC is parallel to the tangent at A.

(iii) $PQCB$ is a cyclic quadrilateral.

[6]

QUESTION 6. Use a Separate Writing Booklet

(a) (i) Write down the expansion of $\tan(A + B)$.

(ii) Find the value of $\tan\left(\frac{7\pi}{12}\right)$ in simplest surd form.

[4]

(b) Find $\frac{d}{dx} 2\cos^{-1}\left(\frac{x}{4}\right)$.

[2]

(c) During the medieval wars, the enemy wanted to attack a fortress with a 5 metre opening along the front wall. The strategy was to stand at the point P, x metres away from the wall, thus giving an angle of vision α, through which to fire arrows from a cross-bow.

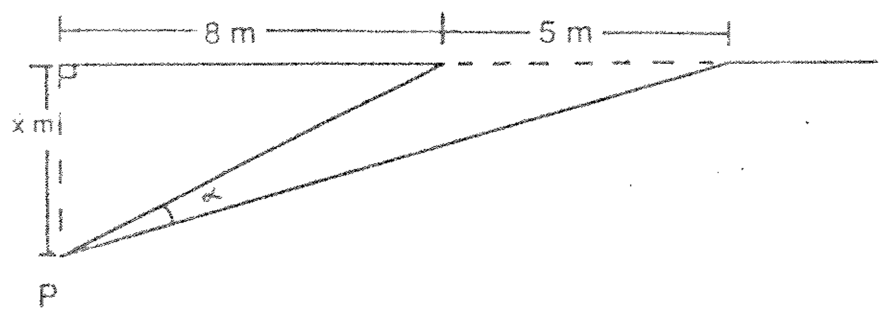


diagram not to scale

(i) Show that the angle of vision α is given by

$$\alpha = \tan^{-1}\left(\frac{13}{x}\right) - \tan^{-1}\left(\frac{8}{x}\right).$$

(ii) Find the value of x in order to give a maximum angle of vision and hence find the maximum angle of vision in radians.

[6]

QUESTION 7. Use a Separate Writing Booklet

(a) The roots, α, β and γ of the equation $2x^3 + 9x^2 - 27x - 54 = 0$ are in geometric sequence.

(i) Show that $\beta^2 = \alpha\gamma$.

(ii) Write down the value of $\alpha\beta\gamma$.

(iii) Find α, β and γ.

[5]

(b)

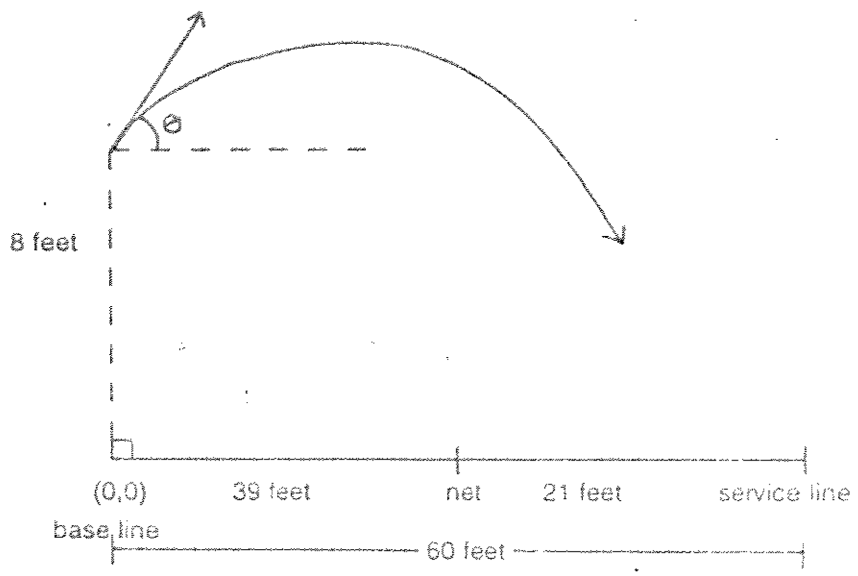


diagram not to scale

In the 1995 Wimbledon Men's Final, "Boom-boom" Becker's serve was measured to have an initial velocity of 184.8 feet/second, or 126 miles/ hour. Becker served the ball at the base line from a height of 8 feet at an angle of inclination of θ. In order not to fault, the ball must land within a range of 60 feet.

Taking acceleration due to gravity as 32 feet/second^2 and the origin as in the diagram,

(i) derive the equations of motion and show that the position of the ball (x, y) after t seconds is given by

$$x = 184 \cdot 8t\cos\theta$$

$$y = -16t^2 + 184 \cdot 8t\sin\theta + 8$$

(ii) Hence show that $y = \frac{-16x^2 \sec^2 \theta}{184 \cdot 8^2} + x \tan \theta + 8$

(iii) Show that if Becker serves the ball horizontally, he will fault.

[7]

END OF PAPER

3 Unit Trial: Riverview: 1995.

Q1. (a). $\frac{2}{x-2} < 1.$

$2(x-2) < (x-2)^2.$

$x^2 - 4x + 4 - 2x + 4 > 0.$

$x^2 - 6x + 8 > 0.$

$(x-2)(x-4) > 0$

$x > 4$ or $x < 2.$

(b). (i) $y = \log(2x)^{1/2}$

$= \frac{1}{2}\log 2x.$

$y' = \frac{1}{2}\cdot\frac{1}{2x}\cdot 2$

$= \frac{1}{2x}.$

(ii) $f(x) = \sec x^3.$

$= (\cos x^3)^{-1}$

$f'(x) = -1.(\cos x^3)^{-2}.\sin x^3.3x^2$

$= \frac{-3x^2.\sin x^3}{(\cos x^3)^2}$

$= -3x^2.\tan x^3.\sec x^3.$

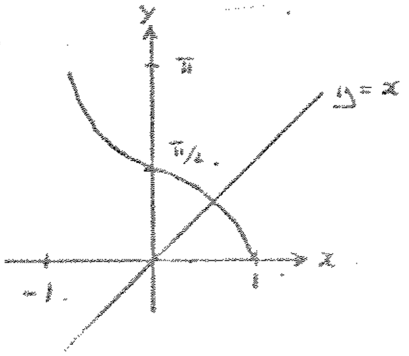
(c) $\int_1^{\sqrt{3}} \frac{dt}{\sqrt{2^2 - t^2}}$

$= \left[\sin^{-1}\left(\frac{t}{2}\right)\right]_1^{\sqrt{3}}$

$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right).$

$= \pi/6.$

(d) (i)



(ii) one ...

Q2. (a). $\frac{1}{2}\int_0^2 \sin 2\theta . d\theta.$

$= \frac{1}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^2$

$= -\frac{1}{4}\left\{\cos 4 - \cos 0\right\}$

$= -0.413.$

(b). (i) 2

(ii) $x = 2\cos(t - \pi/3)$

$\dot{x} = -2\sin(t - \pi/3)$

(iii) $\ddot{x} = -x.$

(iv) When $\ddot{x} = 0.$

$\therefore -2\sin(t - \pi/3) = 0.$

$\sin(t - \pi/3) = 0.$

$t - \pi/3 = 0, \pi \ldots$

$t = \pi/3, (\pi + \pi/3) \ldots$

$\dot{x} = 2\cos(\pi/3 - \pi/3) = 2$

(c). $x = 1.6 - \frac{f(1.6)}{f'(1.6)}$

$= 1.6 - \frac{-0.704}{9.68}$

$\doteq 1.6 + 0.07272$

$\doteq 1.6727$

$[f'(x) = 3x^2 + 2].$

Q3 (a). $XF = h \tan 55$

$YF = h \tan 47$

$XY^2 = YF^2 + XF^2$.

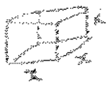
$1200^2 = h^2 \tan^2 47 + h^2 \tan^2 55$

$1200^2 = h^2(\tan^2 47 + \tan^2 55)$

$h = \dfrac{1200}{\sqrt{\tan^2 47 + \tan^2 55}}$

$= 1200(\tan^2 47 + \tan^2 55)^{-1/2}$.

$h \doteqdot 670$ m.

(b)  $S = 6x^2$, $\dfrac{dS}{dt} = -10$

(1) $\dfrac{dS}{dt} = \dfrac{dS}{dx} \cdot \dfrac{dx}{dt}$

$-10 = 12x \cdot \dfrac{dx}{dt}$

$\dfrac{dx}{dt} = -\dfrac{5}{6x}$

(2) $V = x^3$

$\dfrac{dV}{dt} = \dfrac{dV}{dx} \cdot \dfrac{dx}{dt}$

$= 3x^2 \cdot \dfrac{-5}{6x} = -\dfrac{5x}{2}$

(3) when $x = 4$

$\dfrac{dV}{dt} = \dfrac{-5(4)}{2}$

$= -10$

∴ Volume decreases at 10 cm³/min

(c)(1) When $n = 1$

LHS $= 13 \times 6 + 2$

$= 80$ which is divisible by 5

∴ result holds for $n = 1$

(2) Assumption: True for $n = k$ $k \in N$

∴ $13 \times 6^k + 2 = 5m$

(3) To prove true for $k+1$

ie prove $13 \times 6^{k+1} + 2 = 5P$

LHS $= 13 \times 6^{k+1} + 2$

$= 6(13 \times 6^k) + 2$

$= 6(13 \times 6^k) + 12 - 10$

$= 6[13 \times 6^k + 2] - 10$

$= 6[5m] - 10$

$= 5(6m - 2)$ = RHS.

∴ result is true for $n = k+1$

(4) Since the result holds for an initial value, $n = 1$ and if it holds for $n = k$ then it also holds for $n = k+1$, ∴ result holds for $n = 2, 3, 4$ etc for all values of $n \in N$

∴ Result is true.

Q4. (a).

$\int_3^4 t\sqrt{4-t}\, dt$.

$= \int_1^0 (4-v)\sqrt{v} \cdot -dv$. ✓

$t = 4 - v$

$\dfrac{dt}{dv} = -1$

$dt = -dv$

4 $= -\int_1^0 4v^{1/2} - v^{3/2}\, dv$. ✓

$= -\left[\dfrac{8v^{3/2}}{3} - \dfrac{2}{5} \cdot v^{5/2}\right]_1^0$

$= -\left(-\left(\dfrac{8}{3} - \dfrac{2}{5}\right)\right)$

$= \dfrac{34}{15} = 2\tfrac{4}{15}$ ✓

(b).

$\int_0^{\pi/4} \cos x (\sin x)^2\, dx$.

OR
$u = \sin x$
$\dfrac{du}{dx} = \cos x$
when $x = 0$, $u = 0$
$x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$

$= \frac{1}{3}\left[(\sin x)^3\right]_0^{\pi/4}$. ✓

∴ Integral becomes $\int_0^{1/\sqrt{2}} u^2 \dfrac{du}{dx} \cdot dx$

2 $= \frac{1}{3}\left\{(\sin \frac{\pi}{4})^3 - (0)\right\}$ $= \left[\frac{1}{3}u^3\right]_0^{1/\sqrt{2}}$

$= \dfrac{1}{6\sqrt{2}} = \dfrac{\sqrt{2}}{12}$ ✓

[illegible]

(c).

(i) $-\dfrac{72}{x^2} = \dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\int -72x^{-2}\, dx = \frac{1}{2}v^2$.

3 $c + 72x^{-1} = \frac{1}{2}v^2$. ✓

But $x = 9$ when $v = 4$
∴ $c = 0$ ✓

∴ $v^2 = \dfrac{144}{x}$.

$v = \dfrac{12}{\sqrt{x}}$. ✓ $v > 0$ only

(ii) $\dfrac{dx}{dt} = \dfrac{12}{x^{1/2}}$

$\dfrac{dt}{dx} = \dfrac{x^{1/2}}{12}$.

$t = \frac{1}{18} x^{3/2} + c$. ✓

2 When $t = 0$, $x = 9$.

$0 = \frac{1}{18} \cdot 9^{3/2} + c$

∴ $t = \frac{1}{18} x^{3/2} - \frac{3}{2}$. ✓

1 (iii) $t = \frac{1}{18} \cdot 35^{3/2} - \frac{3}{2}$

$\doteqdot 10$ sec. ✓

Question 5

a) $x^2 + y^2 - 2x - 14y + 25 = 0$

$y = mx$

Solve simultaneously for points of intersection

$x^2 + m^2x^2 - 2x - 14mx + 25 = 0$

$(1+m^2)x^2 + (-2-14m)x + 25 = 0$

Two distinct points will occur if there are two solutions to the equation. This occurs when $\Delta > 0$.

$b^2 - 4ac > 0$

$(-2-14m)^2 - 4(1+m^2)25 > 0$

$4(1+7m)^2 - 4.25(1+m^2) > 0$

Divide by 4

$(1+7m)^2 - 25(1+m^2) > 0.$

b) i) $x = -4 + Qe^{-3t}$

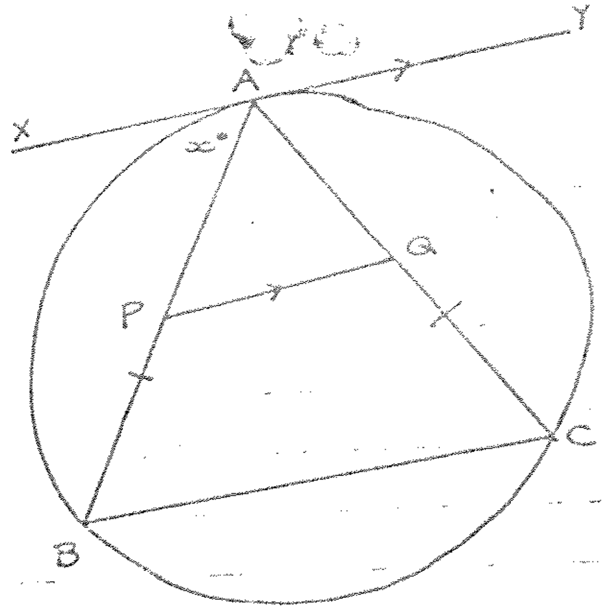
$\frac{dx}{dt} = -3Qe^{-3t}$

$Qe^{-3t} = x + 4$

$\therefore \frac{dx}{dt} = -3(x+4)$

ii) As $t \to \infty$ $e^{-3t} \to 0$

c)



i)
Let $\angle XAB = x°$

$\angle ACB = x°$ (angle in the alternate segment = angle made by tangent and chord)

$\angle ABC = x°$ (base angle of isosceles $\triangle ABC$)

$\angle YAC = x°$ (angle in the alternate segment = angle made by tangent & AC.

$\angle APQ = x$ (alternate angles equal $XY \parallel PQ$)

$\angle AQP = x$ (alternate angles)

$\therefore AP = AQ$ (sides opposite equal angles are equal)

ii) $\angle XAB = x° = \angle ABC$ (from i)

$\therefore$ BC is parallel to the tangent at A.

(alternate angles equal on parallel lines).

(iii) $\angle BPQ = 180° - x°$
(angles on straight line AB)

$\angle BPQ + \angle QCB = 180° - x° + x°$
$= 180°$.

$\therefore$ PQCB is a cyclic quadrilateral (opposite angles are supplementary)

Question 6

a) i) $\tan(A+B)$
$= \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

ii) $\tan\left(\frac{7\pi}{12}\right)$

$= \tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$= \dfrac{\tan \pi/3 + \tan \frac{\pi}{4}}{1 - \tan\frac{\pi}{3}\tan\frac{\pi}{4}}$

$= \dfrac{\sqrt{3} + 1}{1 - \sqrt{3}} \times \dfrac{1 + \sqrt{3}}{1 + \sqrt{3}}$

$= \dfrac{4 + 2\sqrt{3}}{-2}$

$= -2 - \sqrt{3}$.

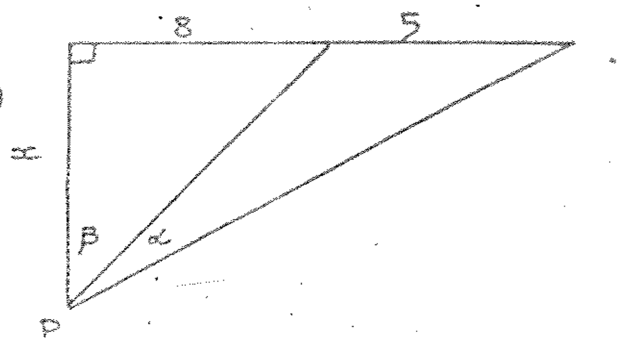
b) $\dfrac{d}{dx}\, 2\cos^{-1}\left(\frac{x}{4}\right)$

$= 2 \times \dfrac{-1}{\sqrt{1 - \left(\frac{x}{4}\right)^2}} \times \dfrac{1}{4}$

$= -\dfrac{1}{2} \times \dfrac{1}{\frac{1}{4}\sqrt{16 - x^2}}$

$= \dfrac{-2}{\sqrt{16 - x^2}}$

c)



i) $\tan\beta = \dfrac{8}{x}$

$\beta = \tan^{-1}\dfrac{8}{x}$

$\tan(\alpha + \beta) = \dfrac{13}{x}$

$\alpha + \beta = \tan^{-1}\dfrac{13}{x}$

$\alpha + \beta - \beta = \tan^{-1}\dfrac{13}{x} - \tan^{-1}\dfrac{8}{x}$

$\alpha = \tan^{-1}\dfrac{13}{x} - \tan^{-1}\dfrac{8}{x}$

ii) Maximum angle occurs when

$\frac{d\alpha}{\ldots}$

$$\frac{d\alpha}{dx} = \frac{1}{1+\left(\frac{13}{x}\right)^2} \times -13x^{-2} - \frac{1}{1+\left(\frac{8}{x}\right)^2} \times -8x^{-2}$$

$$= \frac{x^2}{x^2+169} \times \frac{-13}{x^2} - \frac{x^2}{x^2+64} \times \frac{-8}{x^2}$$

$$0 = \frac{-13}{x^2+169} + \frac{8}{x^2+64}$$

$$\frac{13}{x^2+169} = \frac{8}{x^2+64}$$

$$13(x^2+64) = 8(x^2+169)$$

$$13x^2+832 = 8x^2+1352$$

$$5x^2 = 520$$

$$x^2 = 104$$

$$x = 2\sqrt{26} \quad (x>0)$$

Test

x	10	$2\sqrt{26}$	11
$\frac{d\alpha}{dx}$	>0	0	<0

$x = 2\sqrt{26}$ gives a maximum value for α.

$$\text{Max. angle} = \tan^{-1}\frac{13}{2\sqrt{26}} - \tan^{-1}\frac{8}{2\sqrt{26}}$$

$\alpha \doteqdot 0{\cdot}24$ radians.

Question 7

a) $2x^3 + 9x^2 - 27x - 54 = 0$

i) $\frac{\beta}{\alpha} = \frac{\gamma}{\beta}$ geometric sequence.

$\beta^2 = \alpha\gamma$

ii) $\alpha\beta\gamma = \frac{-d}{a} = \frac{54}{2} = 27$

iii) $\alpha\gamma\beta = 27 \qquad \beta^2 = \alpha\gamma$

$\beta^3 = 27$

$\beta = 3$

$\alpha\gamma = \beta^2$

$\alpha\gamma = 9. \quad \left(\gamma = \frac{9}{\alpha}\right)$

$\alpha + \beta + \gamma = -\frac{9}{2}$

$\alpha + 3 + \frac{9}{\alpha} = -\frac{9}{2}$

$2\alpha^2 + 6\alpha + 18 = -9\alpha$

$2\alpha^2 + 15\alpha + 18 = 0$

$(2\alpha+3)(\alpha+6) = 0$

$\alpha = -\frac{3}{2}$ or $\alpha = -6$

when $\alpha = -\frac{3}{2}$, $\gamma = -6$

When $\alpha = -6$, $\gamma = -\frac{3}{2}$

So α, β, γ are $-6, 3, -\frac{3}{2}$ or $-\frac{3}{2}, 3, -6$

b) i) Initially

$\ddot{x}_0 = 0$, $\ddot{y}_0 = -32$ ft/s^2

$\dot{x}_0 = 184.8\cos\theta$ $\dot{y}_0 = 184.8\sin\theta$

$x_0 = 0$ $y_0 = 8$.

$\ddot{x} = 0$	$\ddot{y} = -32$
$\dot{x} = C_1$	$\dot{y} = -32t + C_3$
when $t=0$, $C_1 = 184.8\cos\theta$	when $t=0$ $\dot{y} = 184.8\sin\theta$
$\dot{x} = 184.8\cos\theta$	$\dot{y} = -32t + 184.8\sin\theta$
$x = 184.8t\cos\theta + C_2$	$y = -16t^2 + 184.8t\sin\theta + C_4$
when $t=0$ $x=0$ $\therefore C_2 = 0$	when $t=0$, $y=8$ $C_4 = 8$
$x = 184.8t\cos\theta$	$y = -16t^2 + 184.8t\sin\theta + 8$

ii) Eliminating t.

$$x = 184.8t\cos\theta$$

$$t = \frac{x}{184.8\cos\theta}$$

$$y = \frac{-16 \times x^2}{(184.8)^2\cos^2\theta} + \frac{184.8\sin\theta \times x}{184.8\cos\theta} + 8$$

$$y = \frac{-16x^2\sec^2\theta}{184.8^2} + x\tan\theta + 8$$

(iii) If ball is served horizontally then $\theta = 0$, $y = 0$.

$$0 = \frac{-16x^2}{184.8^2} + 8$$

$$\frac{16x^2}{184.8^2} = 8$$

$$16x^2 = 8 \times 184.8^2$$

$$x = \sqrt{\frac{8 \times 184.8^2}{16}}$$

$$x \doteqdot 130.7$$

$$x \doteqdot 131 \text{ feet.}$$

> 60 feet

$\therefore$ Becker will fault.