SAINT IGNATIUS' COLLEGE, RIVERVIEW

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1999

MATHEMATICS

3/4 UNIT COMMON

Time Allowed: 2 Hours
(plus 5 minutes reading time)

Directions to Candidates:

Attempt **ALL** questions.

All questions are of equal value.

All necessary working should be shown in every question. Marks may be deducted for areless or badly arranged work.

tandard integrals are provided.

oard-approved calculators may be used.

ach question attempted is to be returned in a ***separate*** Writing Booklet clearly marked uestion 1, Question 2, etc. on the cover.

ach booklet must show your Student Number and the name of your Class Teacher.

ou may ask for extra Writing Booklets if you need them.

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 1999 3/4 Unit Mathematics Higher School Certificate Examination.

Question 1

(a) Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$, giving your answer in terms of π. **(1 mark)**

(b) Differentiate $\tan^{-1}(4x)$ **(2 marks)**

(c) Find $\int \frac{x}{\sqrt{x^2-3}}\,dx$. Use the substitution $u = x^2 - 3$. **(2 marks)**

(d) The polynomial $x^3 + 2x^2 - kx + 3$ has a factor $(x+1)$.
Find the value of k.
(2 marks)

(e) Solve the equation $100e^{-3t} = 20e^{2t}$. **(2 marks)**

(f) (i) Sketch the graph of $y = f(x)$ where $f(x) = \log_e(x+2)$, showing features. **(1 mark)**

(ii) Sketch, on the same axes, the graph of the inverse function $y = f^{-1}(x)$ and state the equation of $f^{-1}(x)$. **(2 marks)**

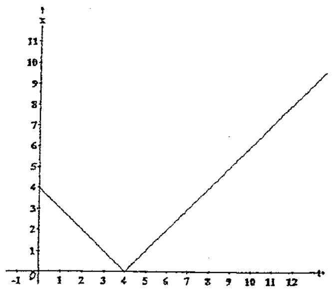
Question 2

(a) Use the table of standard integrals and the substitution $u = x - 1$ to evaluate

$$\int_1^2 \frac{dx}{\sqrt{(x-1)^2+4}}$$

(3 marks)

(b)



A particle moves along the *x-axis* such that displacement, x cm from 0 after t seconds, is give by $x = |t-4|, \quad t \geq 0$, as shown in the diagram.

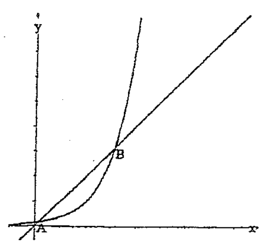
(i) Describe the motion of the particle when $t = 4$. **(1 mark)**

(ii) Find the velocity of the particle for $0 < t < 4$. **(1 mark)**

(iii) Find the total distance travelled in the first 6 seconds. **(2 marks)**

(iv) Sketch the graph of velocity versus time. **(1 mark)**

(c)



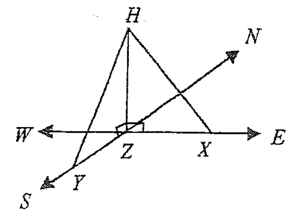
The diagram shows the graphs of $y = e^{x-2}$ and $y = x$ with points of intersection at A and B.

(i) How many roots has the equation $e^{x-2} - x = 0$? **(1 mark)**

(ii) Taking $x = 3.3$ as the first approximation, use one application of Newton's Method to find a better approximation to the x-coordinate of B. **(3 marks)**

Question 3

(a)



A helicopter at an altitude of 500 metres is observing two targets at X and Y on ground level. X is due East and Y is due South of the helicopter. The angles of depression from the helicopter to X and Y are 45° and 60° respectively.

(i) Find the length ZY in surd form. **(2 marks)**

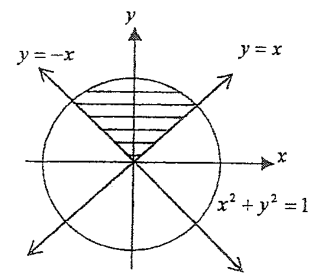
(ii) Find the distance between the two targets. **(2 marks)**

(b) Consider the function $f(x) = 3\cos^{-1} x$.

(i) Evaluate $f(-1)$. **(1 mark)**

(ii) Sketch the graph of $y = f(x)$, showing features. **(2 marks)**

(c)



The shaded area is bounded by the circle $x^2 + y^2 = 1$ and the lines $y = x$ and $y = -x$.

(i) Show that $x^2 + y^2 = 1$ intersects $y = x$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ **(1 mark)**

(ii) Find the volume generated when this shaded area is rotated about the x – axis. **(4 marks)**

Question 4

(a) Consider the function $f(x)=\sqrt[3]{x^2}$

(i) Show that f is an even function. **(1 mark)**

(ii) Find $f(0)$. **(1 mark)**

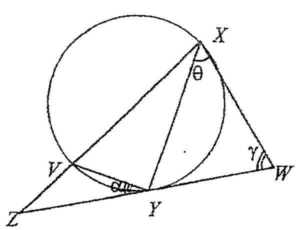
(iii) Find $f'(x)$ and $f''(x)$. **(2 marks)**

(iv) Show that $y=f(x)$ has no stationary points **(1 mark)**

(v) Show that $y=f(x)$ is concave down for all $x>0$. **(1 mark)**

(vi) Sketch the graph of $y=f(x)$. **(1 mark)**

(b)



XV is a diameter of a circle. WY and WX are tangents to the circle. ZYW is a straight line. $\angle ZYV=\alpha$, $\angle YWX=\gamma$ and $\angle WXY=\theta$

(i) Prove that $\theta+\alpha=90^\circ$ **(3 marks)**

(ii) Hence or otherwise prove that $\gamma=2\alpha$. **(2 marks)**

Question 5

(a) Australia is the leading producer of the famous South Sea Pearls. The probability of finding a natural, perfect pearl in an oyster is $\frac{1}{20000}$.

In parts (i) (ii) and (iii) give your answer unsimplified using a power of a fraction.

If three oysters are selected at random and opened, find the probability that

(i) no pearl will be found **(1 mark)**

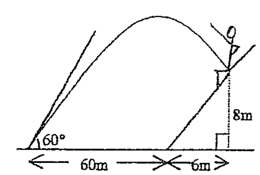
(ii) a perfect pearl will be found **(2 marks)**

If n oysters are opened,

(iii) write down an expression for the probability that a perfect pearl is found. **(1 mark)**

(iv) hence find the number of oysters required to have at least a 50% chance of finding a perfect pearl. **(2 marks)**

(b)



An ardent fan at an AFL match waits anxiously for his favourite player "Slogger" to kick the ball from the field into his lap, as shown in the diagram. Slogger kicks the ball at an angle of elevation of 60° with initial speed Vm/s. Acceleration due to gravity is $g=10\text{ m/s}^2$ and the equations of motion are $x=Vt\cos\theta$ and $y=Vt\sin\theta-\frac{1}{2}gt^2$, where x and y are the respective horizontal and vertical components of displacement in metres after t seconds,

(i) show that $y=x\tan\theta-\dfrac{gx^2}{2V^2\cos^2\theta}$. **(1 mark)**

(ii) Hence show that Slogger must kick the ball at an initial speed of approximately 28.63 m/s. **(2 marks)**

(c) In a co-educational class there are 4 girls and 7 boys. Their classroom has 5 rows of 5 desks neatly arranged. Each student occupies a desk with a chair. Find the number of seating arrangements possible if,

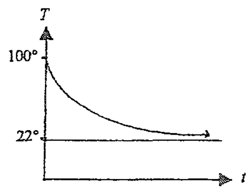
(i) students can sit anywhere, **(1 mark)**

(ii) all the girls want to occupy the first row. **(1 mark)**

(iii) Two particular girls and three particular boys fill the back row seated alternately. **(1 mark)**

Question 6

(a)



The graph represents the relationship between T, the temperature in $°C$ of cooling spa water, and time, t minutes.
The rate of cooling is given by
$$\frac{dT}{dt} = -k(T-A),$$
where k and A are constants, $k>0$.

(i) Show that $T = A + Be^{-kt}$ is a solution to the differential equation $\frac{dT}{dt} = -k(T-A)$, where B is a constant. **(2 marks)**

(ii) By examining the graph when $t = 0$ and $t \to \infty$, find the values of A and B. **(2 marks)**

(iii) If the temperature is 50° C after 90 minutes, show that $k = -\frac{1}{90}\ln\left(\frac{14}{39}\right)$. **(2 marks)**

(iv) Find the rate at which the spa water is cooling after 90 minutes. **(1 mark)**

(b) The north West Australian town of Derby boasts the largest tides in the Southern hemisphere. On April 13, low tide occurred at 3:00am. At 5:14 am the tide had risen to a depth of 3.3m, then high tide occurred at 9:42 am. You may assume these tides rise and fall in simple harmonic motion according to the equation $D = -a\cos(nt) + 6$, where D is depth in metres, t is time after 3:00 am in hours, a, n are constants.

(i) Find the period in hours and minutes. **(1 mark)**

(ii) Hence show that $n = \frac{10\pi}{67}$. **(1 mark)**

(iii) Find the exact amplitude. **(2 marks)**

(iv) Show that the record-breaking depth at high tide is over 11 metres. **(1 mark)**

Question 7

(a) Show that $\frac{(n+9)^2(n+10)^2 - n^2(n-1)^2}{4} = 5(2n+9)(n^2+9n+45)$ **(2 marks)**

(b) (i) Prove by Mathematical Induction that
$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2}{4}(n+1)^2$ for $n \geq 1$, $n \in \mathbb{N}$ **(4 marks)**

(ii) Use (b)(i) to write a formula for $1^3 + 2^3 + 3^3 + \ldots + (n+k)^3, k \in \mathbb{N}$ **(1 mark)**

(iii) Hence use parts of (a) and (b) (ii) to show that
$n^3 + (n+1)^3 + (n+2)^3 + \ldots + (n+9)^3 = 5(2n+9)(n^2+9n+45)$. **(2 marks)**

(c)

Russian Petrouchka Dolls are a set of 10 dolls which are all similar, the smallest of which has a height of 4 cm. Each doll is 1cm taller than the previous one.
The picture (**Not to scale**) shows the smallest five dolls in the set.
The mass of each doll is $c \times h^3$ grams where h is the height of the doll in cm and c is a constant. Show that the total mass of the set is 8245c grams.

(3 marks)

TION 1

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ (1)

$\frac{d}{dx}\tan^{-1}(4x)$

$= \frac{\frac{1}{4}}{\frac{1}{16}+x^2} = \frac{4}{1+16x^2}$ (2)

$\int \frac{x}{\sqrt{x^2-3}}dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-3}}dx$

$= \frac{1}{2}\int 2x(x^2-3)^{-\frac{1}{2}}dx$ ✓

$= \frac{1}{2}\cdot\frac{(x^2-3)^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= \sqrt{x^2-3} + C$ ✓ (2)

$'(-1) = 0 = (-1)^3 + 2(-1)^2 - k(-1) + 3$ ✓

$\therefore 0 = -1 + 2 + k + 3$

$\therefore k = -4$ ✓ (2)

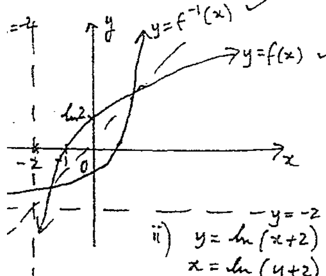
$10e^{-3t} = 2e^{2t}$

$5 = \frac{e^{2t}}{e^{-3t}}$

$5 = e^{5t}$ ✓ (2)

$\ln 5 = 5t$

$t = \frac{1}{5}\ln 5$ ✓



(1)

ii) $y = \ln(x+2)$

$x = \ln(y+2)$

$e^x = y+2$ ✓ (2)

$f^{-1}(x) = y = e^x - 2$

QUESTION 2

a) $\int_1^2 \frac{dx}{\sqrt{(x-1)^2+4}}$ let $u = x-1$, $du = 1dx$. If $x=2$ $u=1$, $x=1$ $u=0$ ✓

$= \int_0^1 \frac{du}{\sqrt{u^2+4}}$ ✓ (3)

$= \left[\ln\left(u+\sqrt{u^2+4}\right)\right]_0^1$ ✓

$= \ln(1+\sqrt{1+4}) - \ln(0+\sqrt{0+4})$

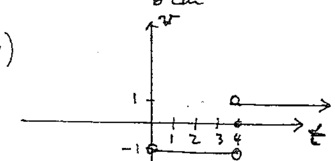
$= \ln\left(\frac{1+\sqrt{5}}{2}\right)$

b) i) P is at the origin O and changing direction, so velocity must be 0cm/s! ✓ (1)

ii) velocity = −1 cm/s ie 1cm/s to left. ✓ (1)

iii) Total d = 4 + 2 = 6 cm ✓✓ (2)

iv)



✓ (1)

c) i) $e^{x-2} - x = 0$ has 2 roots. ✓ (1)

ii) Let $f(x) = e^{x-2} - x$

$f'(x) = e^{x-2} - 1$ ✓ (3)

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\therefore x_2 = 3\cdot3 - \left(\frac{e^{3\cdot3-2} - 3\cdot3}{e^{3\cdot3-2} - 1}\right)$ ✓

$= 3\cdot3 - \frac{0\cdot3692966\ldots}{2\cdot6692966\ldots}$

$= 3\cdot3 - 0\cdot13834\ldots$

$= 3\cdot161650\ldots$

$= 3\cdot2$ to 2d.p. ✓

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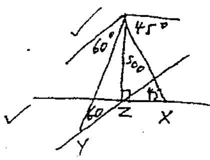
7Q × 12 marks, ✓ = 1 mark

QUESTION 3



a) i) $\tan 60° = \frac{500}{ZY}$ (2)

$ZY = \frac{500}{\sqrt{3}}$ m ✓

ii) $ZX = 500$ m

$\therefore XY^2 = ZX^2 + ZY^2$

$XY^2 = 500^2 + \left(\frac{500}{\sqrt{3}}\right)^2$ ✓ (2)

$= \frac{4}{3}\cdot 500^2$

$\therefore XY = \frac{2\times500}{\sqrt{3}}$ ✓

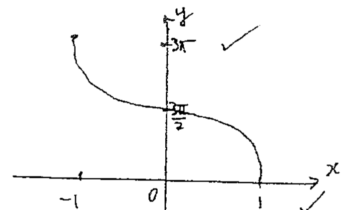
$XY = \frac{1000}{\sqrt{3}}$ m or $\frac{1000\sqrt{3}}{3}$ m

b) $f(x) = 3\cos^{-1}x$

i) $f(-1) = 3\cos^{-1}(-1)$

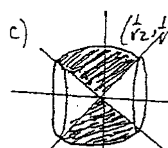
$= 3\pi$ ✓ (1)

ii)



✓ (2)

c)



i) $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}+\frac{1}{2} = 1$ (1)

and $y = \frac{1}{\sqrt{2}}$ if $x = \frac{1}{\sqrt{2}}$

$\therefore \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ satisfies. ✓

ii) Volume $= 2\pi \times \left[\int_0^{\frac{1}{\sqrt{2}}}(1-x^2)dx - \int_0^{\frac{1}{\sqrt{2}}} x^2 dx\right]$ ✓

$= 2\pi\left[\int_0^{\frac{1}{\sqrt{2}}} 1 - 2x^2\, dx\right]$ (4)

$= 2\pi\left[x - \frac{2x^3}{3}\right]_0^{\frac{1}{\sqrt{2}}}$ ✓

$= 2\pi\left[\frac{1}{\sqrt{2}} - \frac{2\left(\frac{1}{\sqrt{2}}\right)^3}{3}\right] - 0$ ✓

$= 2\pi\left[\frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}}\right] = \frac{2\sqrt{2}\pi}{3}u^3$.

QUESTION 5

a) i) $P(\text{no pearls}) = \left(\frac{19999}{20000}\right)^3$ ✓ (1)

ii) P(a pearl) = P(at least 1 pearl)

= 1 − P(no pearls) ✓

$= 1 - \left(\frac{19999}{20000}\right)^3$ ✓ (2)

iii) $P(\text{a pearl}) = 1 - \left(\frac{19999}{20000}\right)^n$ ✓ (1)

iv) $\therefore P(\text{a pearl}) \geq 50\%$

$\therefore 1 - \left(\frac{19999}{20000}\right)^n \geq \frac{1}{2}$ ✓

$\therefore \left(\frac{19999}{20000}\right)^n \leq \frac{1}{2}$ (2)

$n\log\left(\frac{19999}{20000}\right) \leq \log\left(\frac{1}{2}\right)$

$\therefore n \geq \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{19999}{20000}\right)}$

$n \geq 13862\cdot59\ldots$ ✓

∴ Need to open at least 13863 pea[rls]

b) i) $x = Vt\cos\theta \Rightarrow t = \frac{x}{V\cos\theta}$

$\therefore y = V\left(\frac{x\sin\theta}{V\cos\theta}\right) - \frac{1}{2}g\left(\frac{x^2}{V^2\cos^2\theta}\right)$ ✓

$\therefore y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$. (1)

ii) Bell must reach point (66, 8) θ=6[0°]

$\therefore 8 = 66\sqrt{3} - \frac{10\times66^2}{2V^2\times0\cdot25}$ ✓

$\therefore \frac{87120}{V^2} = 66\sqrt{3} - 8$

$\therefore V^2 = \frac{87120}{66\sqrt{3}-8}$ (2)

$\therefore V = \sqrt{819\cdot44\ldots}$

$= 28\cdot626\ldots$ m/s ✓

$\doteq 29$ m/s ✓ (3)

c) i) No. of ways $= {}^{25}P_{11}$ ✓

ii) " " " $= {}^{5}P_4 \times {}^{21}P_7$ ✓

iii) " " " $= 3! \times 2! \times {}^{20}P_6$ ✓

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7Q × 12 marks ✓ = 1 mark

QUESTION 4

a) i) $f(-x) = \sqrt[3]{(-x)^2}$ ✓ (1)

$= \sqrt[3]{x^2}$

$= f(x)$ ∴ $f(x)$ is even

ii) $f(0) = \sqrt[3]{0^2}$

$= 0$ ✓ (1)

iii) $f(x) = x^{2/3}$

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$ ✓ , $f''(x) = -\frac{2}{9}x^{-\frac{4}{3}}$ ✓

$= \frac{2}{3\sqrt[3]{x}}$ $\qquad = \frac{-2}{9\sqrt[3]{x^4}}$ (2)

iv) $0 = \frac{2}{3\sqrt[3]{x}}$

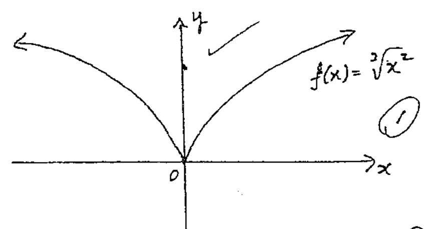
∴ $0 = 2$ No solution. $\frac{dy}{dx} \neq 0$ (1) ✓

∴ No stat. points. ✓

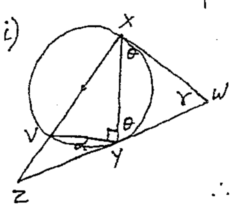
) $f'' = \frac{-2}{9\sqrt[3]{x^4}} < 0$ for all $x > 0$ since $x^4 > 0$ (1)

∴ $f(x)$ is concave down.

i) ✓ (1)



) i)



$\angle XYV = 90°$ ✓ (3)

(∠ in semicircle)

$\angle XYW = \theta$

(isos Δ XW = YW) ✓

tangents from ext. pt

∴ $\alpha + 90° + \theta = 180°$ ✓

(straight line ZYW)

∴ $\alpha + \theta = 90°$ QED

ii) $\gamma + 2\theta = 180°$ (∠ sum ΔXWY) ✓ (2)

∴ $\gamma + 2(90° - \alpha) = 180°$ from (i) ✓

$\gamma + 180 - 2\alpha = 180$

... QED.

QUESTION 6

a) i) $T = A + Be^{-kt} \Rightarrow Be^{-kt} = T - A$ ✓

$\frac{dT}{dt} = 0 - kBe^{-kt}$ ✓ (2)

∴ $\frac{dT}{dt} = -k(T - A)$ on substitution.

ii) $t \to \infty$ $e^{-kt} \to 0$ ∴ $T \to A$. ✓

From graph, asymptote = 22 ∴ A = 22.

$t = 0$ $100 = 22 + Be^0$ ✓ (2)

∴ $B = 78$.

iii) ∴ $T = 22 + 78e^{-kt}$

$50 = 22 + 78e^{-90t}$ ✓ (2)

∴ $\frac{28}{78} = e^{-90t}$

∴ $\ln\left(\frac{14}{39}\right) = -90t$

∴ $t = -\frac{1}{90}\ln\left(\frac{14}{39}\right)$. ✓

iv) $\frac{dT}{dt} = --\frac{1}{90}\ln\left(\frac{14}{39}\right)[50 - 22]$ ✓ (1)

$= -0.3187\ldots$

∴ Cooling ≈ 0.32°C per minute.

b) i) Period = 2 × (Time between low & high)

$= 2 \times (6h\ 42min)$ ✓ (1)

$= 13h\ 24min$

ii) Period $= \frac{2\pi}{n}$ ∴ $n = \frac{2\pi}{13h24min}$

∴ $n = \frac{2\pi}{13\frac{2}{5}} = \frac{10\pi}{67}$. ✓ (1)

iii) Sub. 5:14am and 3.3m (2) ✓

ie. $t = 2h\,14min$, $D = 3.3$

$3.3 = -a\cos\left(\frac{10\pi}{67} \times 2h\,14min\right) + 6$

$-2.7 = -a \times 0.5$ ✓ (in RADIAN MODE)

∴ $a = 5.4$

iv) ∴ 9:42am = 6h 42min after 3am

∴ $D = -5.4\cos\left(\frac{10\pi}{67} \times 6h\,42min\right) + 6$

High Tide = 11.4 m ✓ (1)

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7Q × 12 marks ✓ = 1 mark

QUESTION 7

a) $\frac{(n+9)^2(n+10)^2 - n^2(n-1)^2}{4}$ ✓

$= \frac{[(n+9)(n+10) - n(n-1)][(n+9)(n+10) + n(n-1)]}{4}$

$= \frac{(n^2 + 19n + 90 - n^2 + n)(n^2 + 19n + 90 + n^2 - n)}{4}$

$= \frac{(20n + 90)(2n^2 + 18n + 90)}{4}$ (2)

$= 5\frac{\cancel{10}(2n+9)(\cancel{2}(n^2 + 9n + 45))}{\cancel{4}}$ ✓

$= 5(2n+9)(n^2 + 9n + 45)$ QED.

b) i) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2}{4}(n+1)^2$ $n \geq 1$

Proof:

$n = 1$ LHS $= 1^3$, RHS $= \frac{1}{4}(1+1)^2$

$= 1$ $\qquad = 1$ ∴ True ✓

Assume True for n = k

ie $1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2}{4}(k+1)^2$ ✱

Required to Prove true for n = k + 1 ✓

ie $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^2}{4}(k+2)^2$

Proof: Consider the LHS of n = k + 1 statement

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{k^2}{4}(k+1)^2 + (k+1)^3$ by ✱

$= (k+1)^2\left[\frac{k^2}{4} + \frac{4}{4}(k+1)\right]$ (4)

$= (k+1)^2\left[\frac{k^2 + 4k + 4}{4}\right]$

$= \frac{(k+1)^2(k+2)^2}{4}$ ✓ ✓

= RHS

∴ Truth of n = k statement implies truth of n = k + 1 statement but n = 1 also true ∴ true for all $n \geq 1$, $n \in N$.

ii) $1^3 + 2^3 + 3^3 + \ldots + (n+k)^3 = \frac{(n+k)^2}{4}(n+k+1)^2$ (1) ✓

iii) $n^3 + (n+1)^3 + (n+2)^3 + \ldots + (n+9)^3$ (2)

$= \sum_{k=n}^{n+9} k^3$ ✓

$= \sum_{k=1}^{n+9} k^3 - \sum_{k=1}^{n-1} k^3$

b) iii) cont'd

$= \frac{(n+9)^2}{4}(n+10)^2 - \frac{(n-1)^2}{4}(n-1+1)^2$

$= \frac{(n+9)^2(n+10)^2 - n^2(n-1)^2}{4}$ ✓

$= 5(2n+9)(n^2 + 9n + 45)$ QED.

iv) Total mass of dolls

$= c \times 4^3 + c \times 5^3 + c \times 6^3 + \ldots + c \times 13^3$

$= c(4^3 + 5^3 + 6^3 + \ldots + 13^3)$

$= c(n^3 + (n+1)^3 + (n+2)^3 + \ldots + (n+9)^3)$, where $n = 4$

$= c[5(2n+9)(n^2 + 9n + 45)]$, $n = 4$

$= c[5(2 \times 4 + 9)(4^2 + 9 \times 4 + 45)]$

$= c \times 8245$

$= 8245c$ grams ✓

∴ Total mass is 8245c grams. (3)