ST IGNATIUS COLLEGE RIVERVIEW

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2000

MATHEMATICS

3/4 UNIT COMMON

Time allowed: Two hours
(plus 5 minutes reading time)

Instructions to Candidates

- Attempt all questions
- All questions are of equal value.
- Show all necessary working. Marks may be deducted for missing or poorly arranged work.
- Standard integrals are provided
- Board approved calculators may be used.
- Each question attempted must be returned in a *separate* writing booklet clearly marked Question 1, Question 2 etc, on the cover
- **Each booklet must have your student number and the name of your Class Teacher.**

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2000 3/4 unit Mathematics Higher School Certificate Examination

Question 1 (12 marks) **Start a new booklet**

(a) Solve $|x-3|>5$ *2 marks*

(b) Find the exact value of $\cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$ *1 mark*

(c) Differentiate with respect to x: $e^{-\ln x}$ *2 marks*

(d) Show that $\displaystyle\int_{2}^{2\sqrt{3}}\frac{dx}{\sqrt{16-x^2}}=\frac{\pi}{2}$ *2marks*

(e) Find the coefficient of x^5 in the expansion of $\left(x+\dfrac{1}{x}\right)^{13}$ *2 marks*

(f) (i) Sketch $y=\dfrac{1}{x}$. *1 mark*

(ii) Hence or otherwise find the values of x for which $\dfrac{1}{x}>x$ *2 marks*

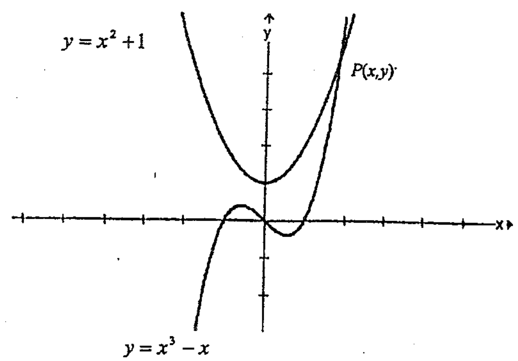
Question 2 (12 marks) **Start a new booklet**

(a) Find $\displaystyle\int_{e}^{e^2}\frac{\ln x}{x}dx$ using the substitution $u=\ln x$ *3marks*

(b) (i) Prove that $\dfrac{1-\cos x}{\sin x}=\tan\dfrac{x}{2}$ *2 marks*

(ii) Hence sketch $y=\dfrac{1-\cos x}{\sin x}$ for $-\pi<x<\pi$ *2 mark*

(c) The graphs of $y=x^3-x$ and $y=x^2+1$ intersect at $P(x,y)$ as shown in the diagram.



(i) Show that $1<x<2$. *2marks*

(ii) Taking $x=1.8$ as a first approximation to the x-value of P, use one application of Newton's method to find a closer value for x. *3marks*

Question 3 **(12 marks)** **Start a new booklet**

(a)

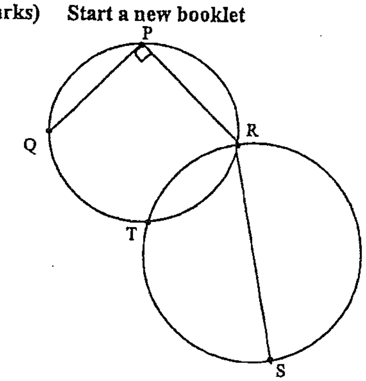


Diagram not to scale

RS is a diameter. PQ is perpendicular to PR.
Prove that Q, T and S are collinear.

3marks

(b)

(i) State the domain and range of

$$y=2\sin^{-1}3x.$$ *2marks*

(ii) Sketch $y=2\sin^{-1}3x$. *1mark*

(iii) The graph of $y=2\sin^{-1}3x$ is rotated about the y-axis. Show that the volume generated is $\frac{\pi^2}{9}$ *units*^{3}. *4marks*

(c)

Julian has 10 different pairs of socks where the left sock and right sock of each pair are indistinguishable.

Find the number of odd pairs of socks (ie a pair which do not match) that Julian can wear. Explain your reasoning.

2marks

Question 4 **(12 marks)** **Start a new booklet**

(a) Find all solutions to $\cos x=\frac{\sqrt{3}}{2}$. *2marks*

(b) (i) Show that $\frac{1}{(x+1)(x+2)}=\frac{1}{x+1}-\frac{1}{x+2}$ *1mark*

(ii) Hence evaluate $\int_0^1 \frac{dx}{(x+1)(x+2)}$ *2marks*

(c) Tangents from the point $T(x_0,y_0)$ touch the parabola $x^2=4y$ at $P(x_1,y_1)$ and $Q(x_2,y_2)$.

(i) State the equation of the chord of contact. *1 mark*

(ii) Show that the x-values of P and Q are given by the roots of the equation $x^2-2x_0x+4y_0=0$ *2marks*

(iii) Hence or otherwise prove that the midpoint M of QP is $\left(x_0, \frac{1}{2}x_0^{\,2} - y_0\right)$. *2marks*

(iv) If T moves on the line $y=x-1$ find the equation of the locus of M. *2marks*

Question 5 **(12 marks)** **Start a new booklet**

(a) Prove by Mathematical Induction that the expression 5^n-1 is divisible by 4 for all positive integers n.

4marks

(b)

Metal Fatigue is a phenomenon where a piece of steel will fail when repeatedly subjected to a force F. The endurance limit is the force below which the steel will not break even if subjected to an infinite number of applications of that force. Let the number of applications be n.

The force and the number of applications are related by the differential equation

$$\frac{dF}{dn} = -k(F - F_0) \quad \text{where } k \text{ and } F_0 \text{ are constants.}$$

(i) Show that $F = 275e^{-k(n-1)} + F_0$ is a solution to $\frac{dF}{dn} = -k(F - F_0)$ *1 mark*

(ii) If F=350 when n=1, find the value of F_0

1mark

(iii) Find the endurance limit. *1 mark*

(iv) Find the value of k if F=80 when n=200.

2marks

(c)

In today's society, statistics show that 28% of Australian women will never have children. Three women are selected at random. Find the probability that

(i) they will all have children *1marks*

(ii) at least one of them will have children *2marks*

Question 6 **(12 marks)** **Start a new booklet**

(a) Consider the function $f(x) = e^{-x^2}$

(i) Show that the function is even. *1mark*

(ii) Find the stationary point of $y=f(x)$. *1mark*

(iii) Show that $\frac{d^2y}{dx^2} = -2e^{-x^2}(1-2x^2)$ and hence find any points of inflexion.

2marks

(iv) Sketch the curve of $y=f(x)$, $x \geq 0$. *1 mark*

(v) Sketch the inverse function $f^{-1}(x)$ of $f(x) = e^{-x^2}$ $x \geq 0$. *1mark*

(vi) Find the equation of $f^{-1}(x)$ and state its domain. *3marks*

(b)

Malaysia has invented a minirail system which is completely automated, running to precision timing (ignoring passenger boarding and alightment). The journey between two stations A and B, where A is to the west of B, can be modelled by the equation $v^2 = 2(8x - x^2 - 7)$ where v is velocity in km/h and x is displacement in km from the central automated control office.

(i) Show that the motion is simple harmonic. *1mark*

(ii) Find the distance between the two stations. *1mark*

(iii) Where is the control office in relation to A and B? *1mark*

Question 7 (12 marks) **Start a new booklet**

(a)

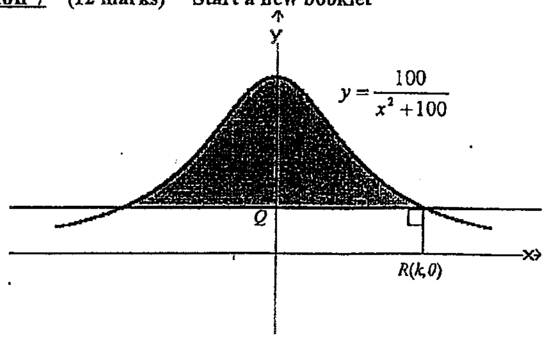


Diagram not to scale

The cross-section of a light fabric structure for a stadium roof is described by the equation $y=\dfrac{100}{x^2+100}$.

Dimensions are in metres.

(i) If Q is the point $\left(0,\dfrac{1}{4}\right)$ find the value of k. *1 mark*

(ii) Show that the shaded area is $\dfrac{5\left(4\pi-3\sqrt{3}\right)}{3}$ square metres. *2marks*

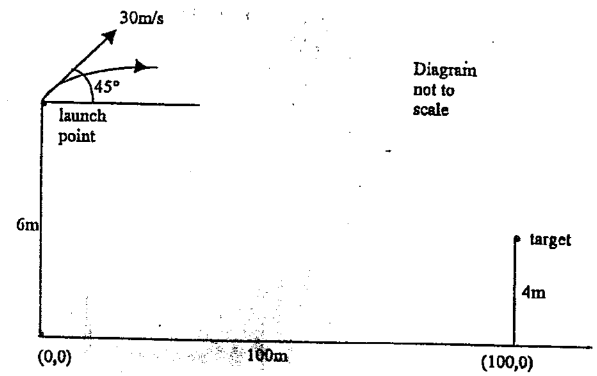
(iii) By considering the integral $\int_{-k}^{k}\dfrac{100}{x^2+100}dx$ or otherwise show that the area of the cross-section will never exceed 10π square metres. *2marks*

(b)

By considering the coefficient of x^{n+1} on both sides of the identity $(x+1)^n(x+1)^n=(x+1)^{2n}$ prove that

$${}^nC_0{}^nC_1+{}^nC_1{}^nC_2+{}^nC_2{}^nC_3+\dots\dots+{}^nC_{n-1}{}^nC_n=\frac{(2n)!}{(n-1)!(n+1)!}$$ *3marks*

(c)



One of the great historic problems which prompted the development of calculus was whether a cannonball would reach a target. Using the origin as shown and assuming $\ddot{x}=0$ and $\ddot{y}=-10$, if a cannonball is fired at an angle of 45 degrees at a velocity of 30m/s,

(i) show that

$$x=15t\sqrt{2}$$
$$y=-5t^2+15t\sqrt{2}+6$$ *2marks*

(ii) Hence determine whether or not the ball will reach its target. *2marks*

Saint Ignatius' College
3/4 Unit Common Trial HSC Exam 2000.

Question 1

a) $|x - 3| \geq 5$

$x - 3 > 5$ or $x - 3 < -5$

$x > 8$ or $x < -2$

b) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$

c) $\frac{d}{dx} e^{-\ln x}$

$= -\frac{1}{x} e^{-\ln x}$

$= -\frac{1}{x} e^{\ln \frac{1}{x}}$ $\left(e^{\ln \frac{1}{x}} = \frac{1}{x}\right)$

$= -\frac{1}{x} \cdot \frac{1}{x}$

$= -\frac{1}{x^2}$

Most students didn't get past this step to simplify answer ∴ second mark was not awarded to them.

d) $\int_2^{2\sqrt{3}} \frac{dx}{\sqrt{16 - x^2}} = \left[\sin^{-1}\frac{x}{4}\right]_{-2}^{2\sqrt{3}}$

$= \left(\sin^{-1}\frac{\sqrt{3}}{2}\right) - \left(\sin^{-1} -\frac{1}{2}\right)$

$= \frac{\pi}{3} + \frac{\pi}{6}$

$= \frac{\pi}{2}$

e) $\left(x + \frac{1}{x}\right)^{13}$

$T_{r+1} = {}^{13}C_r \left(x^r\right)\left(x^{-1}\right)^{13-r}$

$= {}^{13}C_r \left(x^r\right)\left(x^{r-13}\right)$

$= {}^{13}C_r \, x^{2r-13}$

Since we are finding the coefficient of x^5, let

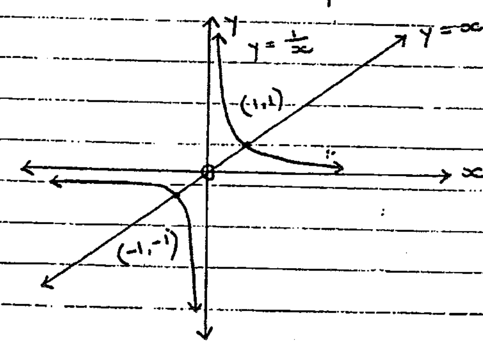
$2r - 13 = 5$

$2r = 18$

$r = 9$

∴ coefficient is ${}^{13}C_9 = 715$.

f) i)



ii) Easier to use graph to solve $\frac{1}{x} > x$ than t[o] solve algebraically.

∴ $\frac{1}{x} > x$ for $0 < x < 1$ and $x < -$

estion 2

$$\int_e^{e^2} \frac{\log_e x}{x}\,dx \quad \text{using the substitution } u = \log_e x$$

$$\int \frac{u}{\cancel{x}} \cdot \cancel{x}\, du$$

$$= \left[\frac{u^2}{2}\right]_1^2$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$dx = x\,du$$

when $x = e^2$, $u = 2$

when $x = e$, $u = 1$

get to limits in ...ms of u.

$$= 2 - \frac{1}{2}$$

$$= \frac{3}{2}.$$

i) Prove that $\dfrac{1-\cos x}{\sin x} = \tan\dfrac{x}{2}$

$\boxed{\begin{aligned} \text{let } \tan\tfrac{x}{2} &= t \\ \cos x &= \frac{1-t^2}{1+t^2} \\ \sin x &= \frac{2t}{1+t^2} \end{aligned}}$

$$\text{LHS} = \frac{1 - \dfrac{1-t^2}{1+t^2}}{\dfrac{2t}{1+t^2}}$$

$$= \frac{1+t^2-1+t^2}{\cancel{1+t^2}} \times \frac{\cancel{1+t^2}}{2t}$$

$$= \frac{2t^2}{2t}$$

$$= t$$

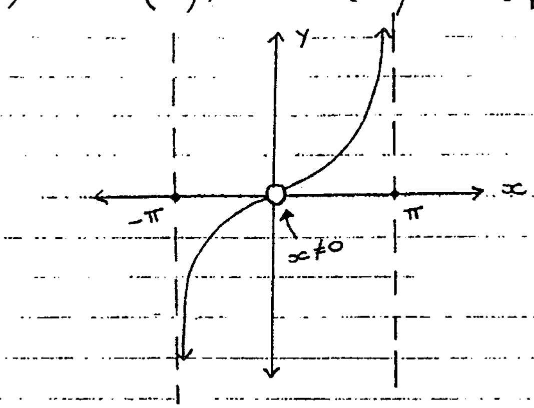
$$= \tan\frac{x}{2}.$$

ii) Sketch $y = \dfrac{1-\cos x}{\sin x}$ for $-\pi < x < \pi$ $(x \neq 0)$

Using the result in (i), sketch $y = \tan\frac{x}{2}$ $(x \neq 0$

$y = \tan\frac{x}{2}$

$$\text{Period} = \frac{\pi}{n} = \frac{\pi}{1/2} = 2\pi$$



c) i) Solve $y = x^2+1$ and $y = x^3 - x$ simultaneously to find the point of intersection.

$$x^2+1 = x^3 - x$$

$$x^3 - x^2 - x - 1 = 0, \quad \text{Let } f(x) = x^3 - x^2 - x - 1$$

$$f(1) = -2$$

$$f(2) = 1$$

Since $f(1)$ and $f(2)$ are opposite in sign, then the root of $x^3 - x^2 - x - 1 = 0$ lies between $x=1$ and $x=2$

$$\therefore 1 < x < 2.$$

ii) $f(x) = x^3 - x^2 - x - 1$

$f'(x) = 3x^2 - 2x$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

let $x_1 = 1.8$

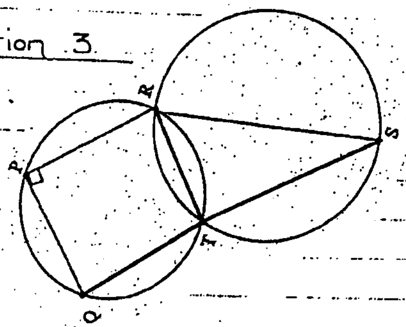
$$x_2 = 1.8 - \frac{f(1.8)}{f'(1.8)}$$

$$= 1.8 - \frac{-0.208}{5.12}$$

$$= 1.840625$$

Question 3.

(a)



Construction

Join Q to T,

T to S,

T to R.

$\angle RTS = 90^\circ$ (angle standing on a diameter)

$\angle QTR = 90^\circ$ (opposite angles of a cyclic quadrilateral are supplementary)

$\angle QTS = \angle RTS + \angle QTR$ (adjacent angles)

$= 90^\circ + 90^\circ$

$= 180^\circ$

$\therefore$ Q, T and S are collinear

(b) i) $y = 2\sin^{-1} 3x$

Domain: $-1 \le 3x \le 1$

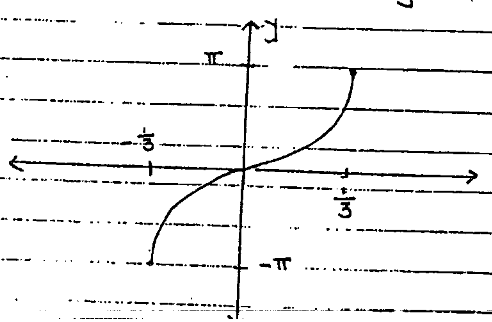
$-\frac{1}{3} \le x \le \frac{1}{3}$

Range: $-\frac{\pi}{2} \le \sin^{-1} 3x \le \frac{\pi}{2}$

$-\pi \le 2\sin^{-1} 3x \le \pi$

$-\pi \le y \le \pi$

ii)



(Many students had curve in the wrong quadrants or concavity of curve was incorrect).

iii) $V = \pi \int_a^b x^2 \, dy$

$= \pi \int_{-\pi}^{\pi} \frac{1}{9} \sin^2 \frac{y}{2} \, dy$

$= 2\pi \cdot \frac{1}{9} \int_0^{\pi} \sin^2 \frac{y}{2} \, dy$

$= \frac{\cancel{2}\pi}{9} \int_0^{\pi} \frac{1}{\cancel{2}} (1 - \cos y) \, dy$

$= \frac{\pi}{9} \left[y - \sin y \right]_0^{\pi}$

$= \frac{\pi}{9} \left[(\pi - \sin \pi) - (0 - 0) \right]$

$= \frac{\pi^2}{9}$

$y = 2\sin^{-1} 3x$

$\frac{y}{2} = \sin^{-1} 3x$

$3x = \sin \frac{y}{2}$

$x = \frac{1}{3} \sin \frac{y}{2}$

$x^2 = \frac{1}{9} \sin^2 \frac{y}{2}$

$\sin^2 \frac{y}{2} = \frac{1}{2}(1 - \cos$

This is from

$\sin^2 A = \frac{1}{2}(1 - \cos$

iv)

A	A
B	B
C	C
D	D
E	E
F	F
G	G
H	H
I	I
J	J

10 choices for 1st sock ($^{10}C_1$)

9 choices for 2nd sock ($^{9}C_1$)

$\therefore$ number of odd pairs $= 10 \times 9$

$= 90$

(Note: left sock and right sock are indistinguishable).

Alternate method:

Total combinations $= {}^{10}C_2$

$= 190$

subtract 10 pairs of matching socks

$190 - 10 = 180$

Question 4 (12marks)

$$\cos x = \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{6} \text{ (acute)} \qquad (2)$$

gen sol^n $$x = 2n\pi \pm \frac{\pi}{6}$$

$$\frac{1}{(x+1)(x+2)} \equiv \frac{1}{x+1} - \frac{1}{x+2}$$

$$\begin{aligned} RHS &= \frac{1}{x+1} - \frac{1}{x+2} \\ &= \frac{x+2-(x+1)}{(x+1)(x+2)} \qquad (1) \\ &= \frac{1}{(x+1)(x+2)} \\ &= LHS \end{aligned}$$

$$\int_0^1 \frac{dx}{(x+1)(x+2)}$$

$$= \int_0^1 \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx \qquad (2)$$

$$= \Big[\ln(x+1) - \ln(x+2)\Big]_0^1$$

$$\left[\ln\left(\frac{x+1}{x+2}\right)\right]_0^1$$

$$\ln\frac{2}{3} - \ln\frac{1}{2}$$

$$\ln\frac{4}{3}$$

Comments

Most students did not know the general solution formula for $\cos^{-1}\alpha$.

part (i) was generally done well

most students were able to find the indefinite integral as a log function BUT

many made errors in evaluating the definite integral.

Question 4 (continued)

(c) (i) Chord of Contact: $x x_0 = 2(y + y_0)$ (1)

(ii)
$$x x_0 = 2(y+y_0) \quad \text{— ①}$$
$$x^2 = 4y$$
$$y = \frac{x^2}{4} \quad \text{— ②}$$

Solve simultaneously for x

$$x x_0 = 2\left(\frac{x^2}{4} + y_0\right)$$

$$x x_0 = \frac{x^2}{2} + 2y_0 \qquad (2)$$

$$2x x_0 = x^2 + 4y_0$$

$$x^2 - 2x_0 x + 4y_0 = 0 \quad \text{—(A)}$$

(iii) Use the quadratic formula to solve (A)

$$x = \frac{2x_0 \pm \sqrt{4x_0^2 - 4(4y_0)}}{2}$$

$$= \frac{2x_0 \pm 2\sqrt{x_0^2 - 4y_0}}{2}$$

$$= x_0 \pm \sqrt{x_0^2 - 4y_0}$$

Midpoint of PQ = average of roots

$$x = \frac{x_0 + \sqrt{x_0^2 - 4y_0} + x_0 - \sqrt{x_0^2 - 4y_0}}{2}$$

$$x = x_0$$

Sub into ① to find y.

Comments

(i) very few students could remember this formula.

(ii) Not knowing the formula made pa[rt] virtually impossibl[e]

(iii) Some students use[d] the sum of roots method.

$$\text{sum of roots} = -\frac{b}{a}$$
$$= 2x_0 \quad \text{from}$$

$$\text{Average of roots} = \frac{\text{sum of roots}}{2}$$
$$= x_0$$

etc to find y_M.

3U Trial SIC 2000

QUESTION 5 | COMMENTS

a) When $n=1$, $5^n-1=4$ which is divisible by 4. ∴ statement true when $n=1$

Assume result true when $n=k$

ie Assume $5^k-1=4M$, $M\in I$

Now prove that the result holds when $n=k+1$

ie Prove that $5^{k+1}-1=4N$, $N\in I$ ✓

you do not assume $n=k$

you do not prove that $n=k+1$

Now $5^{k+1}-1$

$=5\cdot5^k-1$ ✓

$=(4+1)5^k-1$

$=4\cdot5^k+5^k-1$ ✓ but $5^k-1=4M$

$=4\cdot5^k+4M$

$=4(5^k+1)$ ✓ but 5^k+1 is an integer

$=4N$, $N\in I$

$5\cdot5^k-1\neq5(5^k-1)$

or $5\cdot5^k-1$

$=5(4M+1)-1$

$=20M+4=4(5M+1)$

$=4N$

[4]

∴ result holds when $n=k+1$

∴ since the result is true for $n=1$, it is also true for $n=2$ and hence for $n=3$ and so on for all $n\in I^+$.

b) $\dfrac{dF}{dn}=-k(F-F_0)$ if $F=275e^{-k(n-1)}+F_0$

(i) LHS $=\dfrac{dF}{dn}=\dfrac{d(275e^{-k(n-1)}+F_0)}{dn}$

$=-k\cdot275e^{-k(n-1)}$

RHS $=-k(F-F_0)=-k(275e^{-k(n-1)}+F_0-F_0)$

$=-k\cdot275e^{-k(n-1)}$ ✓

LHS = RHS

∴ $F=275e^{-k(n-1)}+F_0$ is a solution to $\dfrac{dF}{dn}=-k(F-F_0)$

similar to showing that a point is on a line. It is best set up as an identity

Poorly done. You got away with very poor solutions because it was worth only 1 mark.

[1]

(ii) $F=275e^{-k(n-1)}+F_0$ if $F=350$ when $n=1$

$350=275e^{-k(1-1)}+F_0$

$F_0=75$ ✓

$350=275+F_0$ not division next $F_0=$

[1]

(iii) $F=275e^{-k(n-1)}+75$

$\lim\limits_{n\to\infty}(275e^{-k(n-1)}+75)=0+75=75$

∴ the endurance limit is 75 ✓

[1]

(iv) if $F=80$ when $n=200$

$80=275e^{-k(200-1)}+75$

$5=275e^{-199k}$

$e^{-199k}=\dfrac{5}{275}$ ✓

$-199k=\ln\left(\dfrac{5}{275}\right)$

$k=-\ln\left(\dfrac{5}{275}\right)\div199=0.02013735$ ✓ (7 s.f.)

[2]

this question was mostly well done

(c) P(no children) = .28

P(children) = .72

be careful of the "not"s

(i) P(CCC) = $.72^3=0.373248$ ✓

[1] 37%

(ii) P(at least one has children)

= 1 − P(none has children)

$=1-.28^3$ ✓

$=.978048$ ✓

if you add these up separately don't forget ${}^3C_1$

[2] 98%

(iv) $T(x_0, y_0)$ moves on the line $y = x - 1$

∴ $y_0 = x_0 - 1$ — (A)

$M(x_0, \frac{1}{2}x_0^2 - y_0)$ (2)

∴ $x = x_0$ — ①

$y = \frac{1}{2}x_0^2 - y_0$ — ②

(A): $y = \frac{1}{2}x_0^2 - (x_0 - 1)$

$= \frac{1}{2}x_0^2 - x_0 + 1$

∴ $y = \frac{1}{2}x^2 - x + 1$

It is important under examination conditions to realise that parts (iii) and (iv) of this question could have been attempted independently of parts (i) and (ii)

→ learn to <u>SCROUNGE FOR MARKS</u>

<u>Question 6</u> (12 marks)

$f(x) = e^{-x^2}$

$f(a) = e^{-a^2}$

$f(-a) = e^{-(-a)^2}$

$= e^{-a^2}$

$= f(a)$

∴ $f(x)$ is even since $f(a) = f(-a)$

(i) generally well done <u>HOWEVER</u>

it is not good enough to show that $f(1) = f(-1)$ ...etc

ie for only one value of x

→ you must show $f(x) = f(-x)$ for any value of x e.g. $x = a$.

(a) (ii) $f(x) = e^{-x^2}$

$f'(x) = -2x\, e^{-x^2}$

t.p. when $f'(x) = 0$ (1)

$-2xe^{-x^2} = 0$

$x = 0$

$y = 1$

x	-1	0	$+1$
$f'(x)$	$+$	0	$-$

Max at $(0,1)$

(iii) $f''(x) = -2x \cdot -2xe^{-x^2} + -2 \cdot e^{-x^2}$

$= -2e^{-x^2}(1 - 2x^2)$

infl. pt may occur when $f''(x) = 0$

ie $-2e^{-x^2}(1 - 2x^2) = 0$ (2)

$x^2 = \frac{1}{2}$

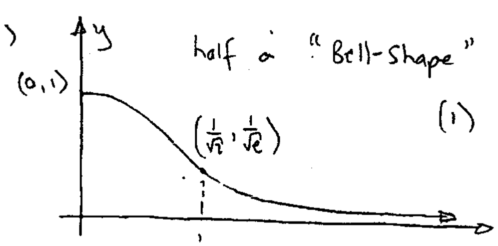
$x = \pm\frac{1}{\sqrt{2}}$

infl. pts $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$

x	-2	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	2
$f''(x)$	$+$	0	$-$	0	$+$

∴ change of concavity through I.P.'s.

(iv) half a "Bell-Shape" (1)



(ii) Many students could not differentiate e^{-x^2} → $-2e^{-x^2}$ (common e

→ should always classify the t.p's.

→ marks were not deducted since this w only worth 1 mark

(iii) When asked to find either t.p's or I. you should always find the y-co-ord

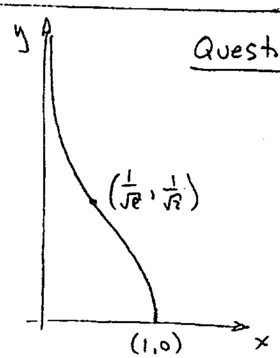
→ many students just found $x = 0$ part (ii), $x = \pm\frac{1}{\sqrt{2}}$ part (i

→ it is good practice to test the concavity. $f''(x) = 0$ <u>DOES NOT PROVE</u> change in concavity !!

(iv) Students were not penalised for sketchi the full bell shape nor for omitting the inflection point.

Question 6 (continued)

i)



(1)

)

$$y = e^{-x^2}$$
$$x = e^{-y^2}$$
$$\ln x = -y^2$$
$$y^2 = -\ln x$$
$$= \ln \frac{1}{x}$$

(3)

ise ction $\cdot$ $y = +\sqrt{\ln\left(\frac{1}{x}\right)}$ take $+\sqrt{\ }$ since range is $y \geq 0$

$D_{f^{-1}}: 0 < x \leq 1$

(i) $v^2 = 2(8x - x^2 - 7)$

$$\tfrac{1}{2}v^2 = 8x - x^2 - 7$$
$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 8 - 2x$$
$$\ddot{x} = -2(x-4)$$

HM Since it is in the form $\ddot{x} = -n^2 x$ where $n = \sqrt{2}$,
nd centre of oscillation is
t $x = 4$ km from Control office.

(v) This was poorly done
→ most students had difficulty in finding the inverse shape
→ reflection in $y = x$
→ interchange $x \Leftrightarrow y$

(vi) students picked up marks here, even though they were unable to sketch the inverse they were able to perform the inverse algebraic operations.

(i) Many students forgot about the formula

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \ddot{x}$$

∴ They struggled to prove the motion was S.H.M.

→ ie. that acceleration is proportional to displacement.

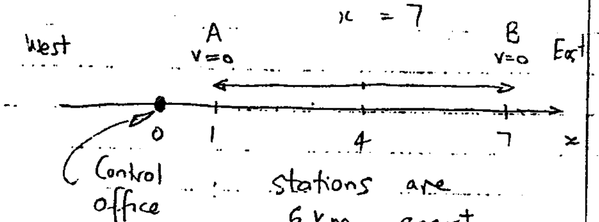
Markers Comments

(b) (ii) $v^2 = 2(8x - x^2 - 7)$

max. displacement when $v = 0$

ie $2(8x - x^2 - 7) = 0$
$$x^2 - 8x + 7 = 0$$
$$(x-7)(x-1) = 0$$
$$x = 1$$
$$x = 7$$

(1)



∴ stations are 6 km apart

(iii) Control office is 1 km West of Station (A) (1)

(ii) generally well d... by those who attempted this part

(iii) Students thought t... the control office would be at the centre of the moti...
→ question stated the distance was measur... FROM THE CONTROL O...
ie $(x = 0)$.

$y = \dfrac{100}{x^2+100}$

) when $y = \frac{1}{4}$, $\quad \frac{1}{4} = \dfrac{100}{x^2+100}$

$x^2+100 = 400$

$x^2 = 300 \qquad x = \pm 10\sqrt{3}$

From diagram, $k>0 \quad \therefore\ k = 10\sqrt{3}$ ✓ [1]

) $A = 2\int_0^{10\sqrt{3}} \dfrac{100}{x^2+100}\,dx - \frac{1}{4} \times 20\sqrt{3}$ ✓

$\frac{1}{4} \times 20\sqrt{3}$ is area of rectangle below shaded area from $(-k,0)$ to $(k,0)$

or $2\int_0^{10\sqrt{3}}\left(\dfrac{100}{x^2+100} - \frac{1}{4}\right)dx$

$= 2\times 100 \times \frac{1}{10}\left[\tan^{-1}\frac{x}{10}\right]_0^{10\sqrt{3}} - 5\sqrt{3}$ ✓

$= 20\left[\tan^{-1}\sqrt{3} - \tan^{-1}0\right] - 5\sqrt{3}$

$= 20 \cdot \frac{\pi}{3} - 5\sqrt{3} = 5\left(\frac{4\pi}{3} - \sqrt{3}\right) = \dfrac{5(4\pi - 3\sqrt{3})}{3}$

[2]

Area $= 2\int_0^k \dfrac{100}{100+x^2}\,dx - 2k \times \dfrac{100}{k^2+100}$ $\qquad$ or $2\int_0^k\left(\dfrac{100}{100+x^2} - \frac{1}{4}\right)dx$

$= 20\left[\tan^{-1}\left(\frac{x}{10}\right)\right]_0^k - \dfrac{200k}{k^2+100}$

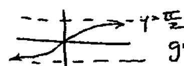
$= 20\tan^{-1}\left(\frac{k}{10}\right) - \dfrac{200k}{k^2+100}$ ✓

now as $k \to \infty$

$\tan^{-1}\left(\frac{k}{10}\right) \to \frac{\pi}{2}$ and $\dfrac{200k}{k^2+100} \to 0$ ✓

$\therefore$ limit of Area is $20 \times \frac{\pi}{2} = 10\pi$

$\therefore$ area never exceeds $10\pi\ m^2$.



$\lim\limits_{k\to\infty} \dfrac{200k}{k^2+100} = \lim\limits_{k\to\infty} \dfrac{\frac{200}{k}}{1+\frac{100}{k^2}} = \dfrac{0}{1+0} = 0$

[2]

LHS $= (x+1)^n (x+1)^n$

$= \left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \dots + \binom{n}{n-2}x^2 + \binom{n}{n-1}x + \binom{n}{n}\right]$

$\times \left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \dots + \binom{n}{n-2}x^2 + \binom{n}{n-1}x + \binom{n}{n}\right]$

learn how to set t up.

In this product the coefficient of x^{n+1} will be

$\binom{n}{0}\binom{n}{n-1} + \binom{n}{1}\binom{n}{n-2} + \binom{n}{2}\binom{n}{n-3} + \dots + \binom{n}{n-2}\binom{n}{1} + \binom{n}{n-1}\binom{n}{0}$ ✓

But $\binom{n}{n-k} = \binom{n}{k} \quad \therefore \binom{n}{n-1} = \binom{n}{1}, \binom{n}{n-2} = \binom{n}{2}, \dots \binom{n}{0} = \binom{n}{n}$ ✓

$\therefore$ coefficient can be written as

$\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \binom{n}{2}\binom{n}{3} + \dots + \binom{n}{n-2}\binom{n}{n-1} + \binom{n}{n-1}\binom{n}{n}$

LHS $= (x+1)^{2n}$

$= \binom{2n}{0}x^{2n} + \binom{2n}{1}x^{2n-1} + \dots + \binom{2n}{n-1}x^{n+1} + \dots + \binom{2n}{2n}$

Here the coefficient of x^{n+1} is $\binom{2n}{n-1}$

but $\binom{2n}{n-1} = \dfrac{(2n)!}{(n-1)!\,(2n-(n-1))!}$

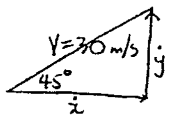
$= \dfrac{(2n)!}{(n-1)!\,(n+1)!}$ ✓

$\therefore \binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \dots + \binom{n}{n-1}\binom{n}{n} = \dfrac{(2n)!}{(n-1)!\,(n+1)!}$ [3]

QED

c)

Initially



$\dot{y} = V\sin\theta = 30\sin 45°$
$= 30 \times \frac{1}{\sqrt{2}} = 15\sqrt{2}$

$\dot{x} = V\cos\theta = 30\cos 45° = 15\sqrt{2}$

i) $\ddot{x} = 0$

$\dot{x} = c_1$ but $\dot{x} = 15\sqrt{2}$ initially

$\therefore \dot{x} = 15\sqrt{2}$

$x = 15\sqrt{2}\,t + c_2$ but $x=0$ when $t=0$ $\therefore c_2 = 0$

$\therefore x = 15\sqrt{2}\,t$

$\ddot{y} = -10$

$\dot{y} = -10t + c_3$ but $\dot{y} = 15\sqrt{2}$ when $t=0$ $\therefore c_3 = 15\sqrt{2}$

$\therefore \dot{y} = -10t + 15\sqrt{2}$

$y = -5t^2 + 15\sqrt{2}\,t + c_4$ but $y=6$ when $t=0$ $\therefore c_4 = 6$

$\therefore y = -5t^2 + 15\sqrt{2}\,t + 6$

Target is at (100, 4) $\therefore$ for the ball to reach target $y \geq 4$ when $x = 100$

when $x = 100$ $\quad 100 = 15\sqrt{2}\,t$

$t = \frac{100}{15\sqrt{2}} = \frac{100\sqrt{2}}{30} = \frac{10\sqrt{2}}{3}$ ✓

when $t = \frac{10\sqrt{2}}{3}$ $\quad y = -5\left(\frac{10\sqrt{2}}{3}\right)^2 + 15\sqrt{2}\left(\frac{100}{15\sqrt{2}}\right) + 6$

$= \frac{-5 \times 200}{9} + 100 + 6$

$= -5\frac{1}{9}$

$\therefore$ the ball will not reach the target. ✓

COMMENTS

set initial $\dot{x}$ & $\dot{y}$ up before you start

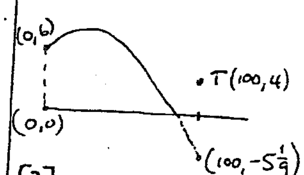
Integrating wrt t

[1] must set up & find constants all the way through. **

[1]

"hit" is not the same as "reach".
(would hit only if $y=4$ when $x=100$)

OR set $y=0$, find t, find x (=95.65)



[2]