SAINT IGNATIUS' COLLEGE

HSC Trial

2001

MATHEMATICS

Extension 1

2:00 – 4:05pm
Wednesday 5th September 2001

Directions to Students

- Reading Time : 5 minutes
- Time Allowed : 2 hours
- Attempt ALL questions.
- Board approved calculators may be used.
- A standard integral table is provided
- Answer each question in a separate writing booklet and clearly label your name and teacher's name.

Total Marks 84

Attempt Questions 1 – 7

All questions are of equal value

Students are reminded that this is a trial examination only and cannot in any way guarantee the content or the format of the 2001 Mathematics Extension 1 Higher School Certificate examination

Total marks (84) **Marks**
Attempt Questions 1 - 7
All questions are of equal value

Answer each question in a SEPARATE writing booklet.

QUESTION 1 (12 marks) Use a SEPARATE Writing Booklet.

(a) When $x^3 - 3x^2 - 4x + k$ is divided by $(x+2)$, the remainder is 3. Find the value of k. **2**

(b) The interval PQ has end points $P(5,-6)$ and $Q(-7,10)$. **2**
Find the coordinates of the point R which divides PQ internally in the ratio 5 : 3.

(c) Evaluate $\int_1^2 \frac{dx}{\sqrt{4-x^2}}$. **2**

(d) Solve $\frac{3}{2-x} > 1$. **3**

(e) If α, β, γ are the roots of the equation $x^3 - 5x^2 + 3x - 2 = 0$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **3**

QUESTION 2 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

(a) Using the substitution $u = x + 3$, find $\int_{-3}^{-2} x(x+3)^4\,dx$. **3**

(b) Consider the function $f(x) = \sin^{-1}(x-1)$.

(i) What is the domain of $y = f(x)$? **1**

(ii) Sketch the graph of $y = f(x)$. **1**

(c) Mr and Mrs Jones belong to a bush-walking club, which has a total of 20 members. A committee of 4 is chosen at random to plan the next bush-walk.

What is the probability that:

(i) both Mr and Mrs Jones will be on the committee? **2**

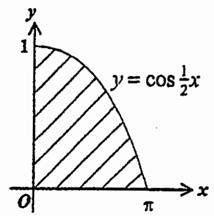
(ii) neither of Mr and Mrs Jones will be on the committee? **2**

(d) (i) Express $\tan 2\theta$ in terms of $\tan\theta$. **1**

(ii) By letting $\theta = \tan^{-1}2$, prove that $2\tan^{-1}2 = \tan^{-1}\left(-\frac{4}{3}\right)$ **2**

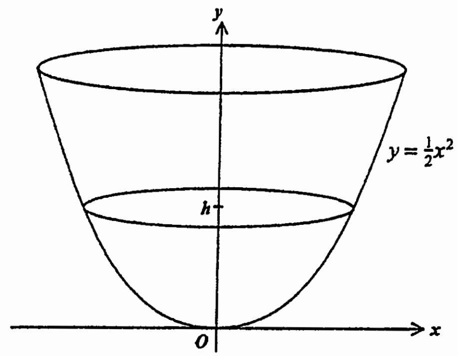
QUESTION 3 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

(a) **3**



The diagram shows the graph of $y = \cos\frac{1}{2}x$, for $0 \le x \le \pi$.
The shaded area is rotated about the x-axis.
Find the volume of the solid formed.

(b)



A large industrial container is in the shape of a paraboloid, which is formed by rotating the parabola $y = \frac{1}{2}x^2$ around the y-axis.
Liquid is poured into the container at a rate of 2 m^3 per minute.

(i) Prove that the volume V of liquid in the container when the depth of liquid is h, is given by $V = \pi h^2$. **1**

(ii) At what rate is the height of the liquid rising when the depth is 1.5 m? **3**

(iii) If the container is 3 m high, how long will it take to fill the container? **1**

(c) Prove, using mathematical induction, that $5^n + 11$ is divisible by 4, where n is a positive integer. **4**

QUESTION 4 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

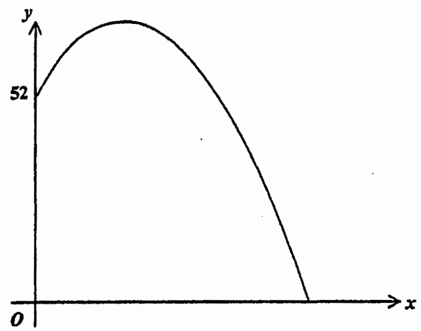
(a) Find the coefficient of x^2 in the expansion of $(3+2x)(2+x)^6$. **3**

(b) A squad of 18 boys is selected for rugby training, from which a team of 15 players is to be chosen for the Saturday game.
The probability that a player will be injured at training and not available for Saturday is 0.15.

(i) Find the probability that 3 players will be unavailable for the Saturday game. (Answer to 3 decimal places) **2**

(ii) Write the numerical expression for the probability that the team will not be able to field a team of 15 fit players on Saturday. **2**
Do not simplify the answer.

(c)



A ball is projected from the top of a 52 metre high tower.
Its position t seconds after it is thrown, is given by the equations

$$x = 12t, \qquad y = 52 + 16t - 5t^2$$

where the origin O is on the ground vertically below the point of projection.

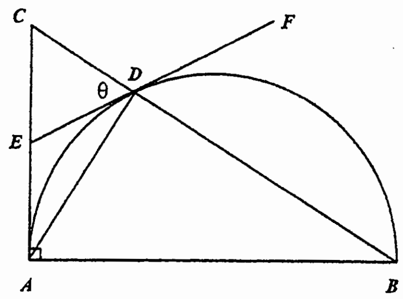
(i) Find the greatest height reached above ground level. **2**

(ii) For what length of time is the ball in flight? **2**

(iii) How far from O does the ball land? **1**

QUESTION 5 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

(a)



AB is the diameter of a semi-circle. ΔABC is right-angled at A, and BC cuts the semi-circle at D. EF is a tangent to the semi-circle at D. AD is joined. $\angle CDE = \theta$.

COPY OR TRACE THE DIAGRAM ONTO YOUR WRITING PAGE.

(i) Why is $\angle ADB = 90^0$? **1**

(ii) Why is $\angle ADE = \angle ABD$? **1**

(iii) Name two angles equal to $\angle CDE$. **1**

(iv) Prove ΔADE is isosceles. **2**

(v) Prove that E is the midpoint of AC. **2**

(b) Consider the function $f(x) = (x-2)^2 - 3$ for $x \le 2$.

(i) Sketch the function $y = f(x)$. **2**

(ii) Explain why $f(x)$ has an inverse function. **1**

(ii) Find the inverse function $y = f^{-1}(x)$. **2**

QUESTION 6 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

(a) The velocity of a particle moving in a straight line at position x is given by:

$$v = 2e^{-x}.$$

Initially the particle is at the origin.

(i) Show that the acceleration at position x is given by $a = -4e^{-2x}$. **2**

(ii) What is the initial acceleration ? **1**

(iii) The position of the particle at time t is given by $x = \log_e f(t)$. Find the function $f(t)$. **2**

(b) Consider the binomial expansion, where n is an even number:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots\dots\dots + \binom{n}{n}x^n.$$

(i) Prove that $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots\dots + \binom{n}{n-1} = 2^{n-1}$. **3**

(ii) Prove that $\sum_{r=1}^{n} r\binom{n}{r} = n \times 2^{n-1}$. **2**

(iii) Find an expression for $\sum_{r=0}^{n} (r+1)\binom{n}{r}$. **2**

QUESTION 7 (12 marks) Use a SEPARATE Writing Booklet. **Marks**

(a) (i) Express $2\sin t - 5\cos t$ in the form $A\sin(t - \alpha)$, where α is in radians, $A > 0$. **2**

(ii) What is the amplitude of the function $f(t) = 2\sin t - 5\cos t$? **1**

(b) Show that the derivative of $8t\tan^{-1}2t - 2\log_e(1+4t^2)$ is $8\tan^{-1}2t$. **2**

(c) In the Olympic 100 metres running event, the speed v metres per second of a runner t seconds after the start is given by:

$$v = 8\tan^{-1}2t.$$

(i) Using the result of part (b), explain why the time taken, T seconds, to complete the 100 metres is given by the equation **2**

$$8T\tan^{-1}2T - 2\log_e(1+4T^2) - 100 = 0.$$

(ii) Show that a root of this equation lies between $T = 9$ and $T = 10$. **2**

(iii) Using $T = 9$ as a first approximation, use Newton's method to find a better approximation, to one decimal place. **2**

(iv) Using this value of T, what is the runner's speed at the end of the 100m race? **1**

End of paper

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3 Unit - Question 1.

(a) $P(x) = x^3 - 3x^2 - 4x + k$.

$P(-2) = -8 - 12 + 8 + k$.

$P(-2) = 3$: $\quad -12 + k = 3$

$k = 15$ $\quad \boxed{2}$

(b) $P(5, -6)$ $\quad Q(-7, 10)$ $\quad 5 : 3$

($x_1 \; y_1$, $x_2 \; y_2$, $m_1 \; m_2$)

$R\left(\dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}, \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}\right) = \left(\dfrac{5 \times (-7) + 3 \times 5}{5+3}, \dfrac{5 \times 10 + 3 \times (-6)}{5+3}\right)$

$= (-2\tfrac{1}{2}, 4)$ $\quad \boxed{2}$

(c) $\displaystyle\int_1^2 \frac{dx}{\sqrt{4 - x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_1^2$

$= \sin^{-1} 1 - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$ $\quad \boxed{2}$

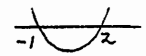
(d) $\dfrac{3}{2 - x} > 1$

$\times (2-x)^2$: $\quad 3(2 - x) > (2 - x)^2$

$6 - 3x > 4 - 4x + x^2$

$x^2 - x - 2 < 0$

$(x - 2)(x + 1) < 0$



$-1 < x < 2$ $\quad \boxed{3}$

(e) $x^3 - 5x^2 + 3x - 2 = 0$

$\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \dfrac{3}{2}$ $\quad \boxed{3}$

3 Unit – Question 2

(a) $\int_{-3}^{-2} x(x+3)^4\,dx = \int_0^1 (u-3)\,u^4\,du$ $u = x+3$

$= \int_0^1 (u^5 - 3u^4)\,du$ $du = dx$

$= \left[\frac{1}{6}u^6 - \frac{3}{5}u^5\right]_0^1$ When $x = -3$, $u = 0$

$= \left(\frac{1}{6} - \frac{3}{5}\right) - (0 - 0)$ When $x = -2$, $u = 1$

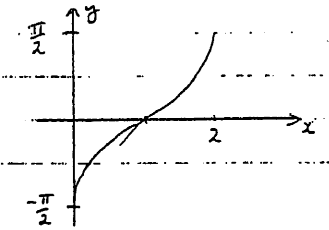
$= -\frac{13}{30}$ $\boxed{3}$

(b) $f(x) = \sin^{-1}(x-1)$

(i) $-1 \le x - 1 \le 1$ ∴ $0 \le x \le 2$

Domain is $0 \le x \le 2$ $\boxed{1}$

(ii)



$\boxed{1}$

(c) (i) Prob $= \dfrac{\binom{18}{2}}{\binom{20}{4}} = \dfrac{153}{4845} = \dfrac{3}{95}$ (or 0.032, 3dp) $\boxed{2}$

(ii) Prob $= \dfrac{\binom{16}{4}}{\binom{20}{4}} = \dfrac{1820}{4845} = \dfrac{364}{969}$ (or 0.376, 3dp) $\boxed{2}$

(d) (i) $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$ $\boxed{1}$

(ii) Let $\theta = \tan^{-1}2$, ∴ $\tan\theta = 2$

$\tan 2\theta = \dfrac{2 \times 2}{1 - 2^2}$

$= -\frac{4}{3}$

$2\theta = \tan^{-1}\left(-\frac{4}{3}\right)$

∴ $2\tan^{-1}2 = \tan^{-1}\left(-\frac{4}{3}\right)$ $\boxed{2}$

3 Unit – Question 3

Question 3.

(a) $V = \pi\int y^2\,dx$

$= \pi\int_0^\pi \cos^2\tfrac{1}{2}x\,dx$

$= \pi\int_0^\pi \tfrac{1}{2}(1 + \cos x)\,dx$

$= \frac{\pi}{2}\left[x + \sin x\right]_0^\pi$

$= \frac{\pi}{2}\left[(\pi + 0) - (0 + 0)\right]$

Volume $= \frac{\pi^2}{2}$ unit^3. $\boxed{3}$

(b) (i) $V = \pi\int x^2\,dy$

$= \pi\int_0^h 2y\,dy$

$= \pi\left[y^2\right]_0^h$

$= \pi h^2$ $\boxed{1}$

(ii) $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$

$\dfrac{dV}{dt} = 2\pi h \times \dfrac{dh}{dt}$

when $\frac{dV}{dt} = 2$, $h = 1.5$

$2 = 2 \times \pi \times (1.5) \times \dfrac{dh}{dt}$

$\dfrac{dh}{dt} = \dfrac{2}{3\pi}$

Liquid is rising at $\frac{2}{3\pi}$ m/min or 0.212 m/min (3dp) $\boxed{3}$

(iii) When $h = 3$, $V = 9\pi$

Time taken $= \dfrac{9\pi}{2}$ minutes

or 14.14 min (2dp) $\boxed{1}$

(c) Prove $5^n + 11$ is divisible by 4.

When $n = 1$, $5^n + 11 = 5 + 11$

$= 16$

∴ It is true when $n = 1$.

Assume it is true for $n = k$

i.e. assume $5^k + 11 = 4I$ where I is integ

When $n = k+1$,

$5^n + 11 = 5^{k+1} + 11$

$= 5^k \times 5 + 11$

$= 5(4I - 11) + 11$ by assumption

$= 20I - 44$

$= 4(5I - 11)$

which is divisible by 4

If it is true for $n = k$, then it is true for $n = k+1$.

Since it is true for $n = 1$, then it is true for $n = 2$ ∴ it is true for $n = 3$, ...

i.e. it is true for all positive integers n. $\boxed{4}$

30

Question 4.

(a) $(3+2x)(2+x)^6$

$= (3+2x)\left[\binom{6}{0}2^6 + \binom{6}{1}2^5 x + \binom{6}{2}2^4 x^2 + \dots\right]$

Term in $x^2 = 3 \times \binom{6}{2} 2^4 x^2$

$+ 2x \times \binom{6}{1} 2^5 x$

$= 3 \times 15 \times 16 x^2 + 2 \times 6 \times 32 \times x^2$

$= (720 + 384)x^2$.

Coefficient of $x^2 = 1104$. $\boxed{3}$

(b)(i) $P = \binom{18}{3}(0.15)^3 (0.85)^{15}$

$= 0.241$ (3 d p) $\boxed{2}$

(ii) Probability of not fielding a full team

$= P$(4 or more injured)

$= 1 - P$(0 or 1 or 2 or 3 injured)

$= 1 - \left[(0.85)^{18} + \binom{18}{1}(0.15)(0.85)^{17} + \binom{18}{2}(0.15)^2(0.85)^{16} + \binom{18}{3}(0.15)^3(0.85)^{15}\right]$ $\boxed{2}$

(c) $x = 12t$, $y = 52 + 16t - 5t^2$

$\dot{x} = 12$, $\dot{y} = 16 - 10t$

(i) greatest height when $\dot{y} = 0$

$16 - 10t = 0$

$t = 1.6$

$y = 52 + 16 \times 1.6 - 5 \times 1.6^2$

$= 64.8$

Greatest height = 64.8 metres. $\boxed{2}$

(ii) Hits ground when $y = 0$

$52 + 16t - 5t^2 = 0$

$5t^2 - 16t - 52 = 0$

$t = \dfrac{16 \pm \sqrt{256 - 4 \times 5 \times (-52)}}{10}$

$= \dfrac{-16 \mp 36}{10}$

$= 2$ or -5.2.

Ball is in flight for 2 seconds $\boxed{2}$

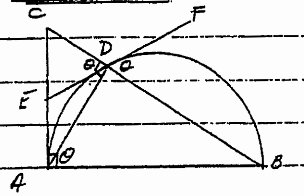
(iii) When $t = 2$, $x = 12 \times 2$

$= 24$.

Ball lands 24 m. from O. $\boxed{1}$

Question 5.

(a)



(i) $\angle ADB = 90^\circ$ because it is an angle in the semi-circle $\boxed{1}$

(ii) $\angle ADE = \angle ABD$ because the angle between a tangent and a chord drawn to the point of contact is equal to an angle in the alternate segment.

OR the alternate segment theorem. $\boxed{1}$

(iii) $\angle CDE = \angle FDB$ (vert. opp.)

$= \angle DAB$ (alt. seg. thm) $\boxed{1}$

(iv) $\angle EDA = 90^\circ - \theta$ from diagram

$\angle EAD = 90^\circ - \theta$ from diagram

$\therefore AE = ED$ (sides opposite equal angles)

$\therefore \triangle ADE$ is isosceles. $\boxed{2}$

(v) Since $\angle EAD = 90^\circ - \theta$

then $\angle ACD = \theta$ (angle sum of $\triangle ADC$)

$\therefore \angle EDC = \angle ECD$

$\therefore CE = ED$

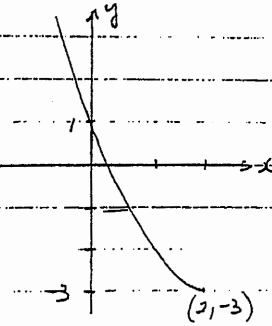
But $AE = ED$ in (iv)

$\therefore AE = CE$

$\therefore E$ is midpoint of AC. $\boxed{2}$

(b) $f(x) = (x-2)^2 - 3$ for $x \le 2$.

(i)



$\boxed{2}$

(ii) $f(x)$ has an inverse because it is a 1:1 relation

i.e. for each value of x there is one value of y and for each value of y there is one value of x.

OR It satisfies the horizontal line test. $\boxed{1}$

(iii) Inverse is

$x = (y-2)^2 - 3$ for $y \le 2$

$x + 3 = (y-2)^2$

$y - 2 = \pm\sqrt{x+3}$

$y = 2 \pm \sqrt{x+3}$

But $y \le 2$

$\therefore y = 2 - \sqrt{x+3}$

or $f^{-1}(x) = 2 - \sqrt{x+3}$. $\boxed{2}$

3 Unit - Question 6.

Question 6.

(a) $v = 2e^{-x}$

(i) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2} \times 4e^{-2x}\right)$

$= 2 \times (-2) e^{-2x}$

$= -4e^{-2x}$ $\boxed{2}$

(ii) When $t=0$, $x=0$

$\therefore a = -4e^0$

$= -4$.

Initial acceleration is -4 units. $\boxed{1}$

(iii) $\frac{dx}{dt} = 2e^{-x}$

$= \frac{2}{e^x}$

$\frac{dt}{dx} = \frac{e^x}{2}$

$t = \frac{1}{2}e^x + c$

When $t=0$, $x=0$

$\therefore 0 = \frac{1}{2}e^0 + c$

$c = -\frac{1}{2}$

$\therefore t = \frac{1}{2}e^x - \frac{1}{2}$

$2t = e^x - 1$

$e^x = 2t+1$

$x = \ln(2t+1)$

$\therefore f(t) = 2t+1$. $\boxed{2}$

(b) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n$

(n is even).

(i) Let $x = -1$.

$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots - \binom{n}{n-1} + \binom{n}{n}$

$\therefore \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n-1} = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n}$

Let $x = 1$:

$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

$\therefore \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n-1} = \frac{2^n}{2}$

$= 2^{n-1}$. $\boxed{3}$

(ii) Differentiate:

$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots$

$\cdots + n\binom{n}{n}x^{n-1}$

Let $x=1$:

$n \times 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}$

i.e. $\sum_{r=1}^{n} r\binom{n}{r} = n \times 2^{n-1}$ $\boxed{2}$

(iii) $x(1+x)^n = \binom{n}{0}x + \binom{n}{1}x^2 + \binom{n}{2}x^3 + \cdots$

$\cdots + \binom{n}{n}x^{n+1}$.

Differentiate:

$(1+x)^n + x\,n(1+x)^{n-1}$

$= \binom{n}{0} + 2\binom{n}{1}x + 3\binom{n}{2}x^2 + \cdots$

$\cdots + (n+1)\binom{n}{n}x^n$.

Let $x=1$:

$2^n + n \times 2^{n-1} = \binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} + \cdots$

$\cdots + (n+1)\binom{n}{n}$

i.e. $\sum_{r=0}^{n} (r+1)\binom{n}{r} = 2^{n-1}(2+n)$ $\boxed{2}$

3 Unit - Question 7.

Question 7.

(a) (i) $2\sin t - 5\cos t = A\sin(t-\alpha)$

$= A\sin t\cos\alpha - A\cos t\sin\alpha$.

Equating coefficients of $\sin t$, $\cos t$.

$A\cos\alpha = 2$ (1)

$A\sin\alpha = 5$ (2)

$A^2\cos^2\alpha + A^2\sin^2\alpha = 2^2 + 5^2$

$A^2(\cos^2\alpha + \sin^2\alpha) = 29$

$A = \sqrt{29}$ $(A>0)$

From (1) (2) since $A>0$, α is acute.

(2) ÷ (1) $\tan\alpha = \frac{5}{2}$

$\alpha = 1.19$

$\therefore 2\sin t - 5\cos t = \sqrt{29}\sin(t - 1.19)$ $\boxed{2}$

(ii) Amplitude of $f(t)$ is $\sqrt{29}$. $\boxed{1}$

(b) $\frac{d}{dt}\left[8t\tan^{-1}2t - 2\log_e(1+4t^2)\right]$

$= \tan^{-1}2t \times 8 + 8t \times \frac{2}{1+4t^2} - 2 \times \frac{8t}{1+4t^2}$

$= 8\tan^{-1}2t$ $\boxed{2}$

(c) (i) $v = 8\tan^{-1}2t$

$x = \int 8\tan^{-1}2t\,dt$

$= 8t\tan^{-1}2t - 2\log(1+4t^2) + C$.

When $t=0$, $x=0$

$0 = 0 - 2\log 1 + C$

$\therefore C = 0$

$\therefore x = 8t\tan^{-1}2t - 2\log(1+4t^2)$

When $x = 100$, $t = T$.

$\therefore 100 = 8T\tan^{-1}2T - 2\log(1+4T^2)$

i.e. $8T\tan^{-1}2T - 2\log(1+4T^2) - 100 = 0$ $\boxed{2}$

(ii)

$f(9) = 8 \times 9\tan^{-1}18 - 2\log_e(1+4\times 9^2) - 100$

$= -2.466$

$f(10) = 8\times 10\tan^{-1}20 - 2\log(1+4\times 10^2) - 100$

$= 9.679$

Since $f(9)$ and $f(10)$ have opposite signs, a root lies between 9, 10. $\boxed{2}$

(iii)

$T = 9 - \frac{f(9)}{f'(9)}$

$= 9 - \frac{(-2.466)}{8 \times \tan^{-1}18}$

$= 9 - \frac{(-2.466)}{12.122}$

$= 9.2$ (1 d.p.) $\boxed{2}$

(iv) When $T = 9.2$,

$v = 8\tan^{-1}(2\times 9.2)$

$= 12.13$

Runner's speed is 12.13 m/s. $\boxed{1}$