SAINT IGNATIUS' COLLEGE

Trial Higher School Certificate

2002

MATHEMATICS

EXTENSION 1

2:00am – 4:05 pm
Friday 30th August 2002

General Instructions

- Reading time : 5 minutes
- Working time: 2 hours
- Write using blue or black pen
- ***Write your name and teacher's name on each answer booklet***
- Board approved calculators may be used
- A table of standard integrals is provided

- Total Marks **(84)**

- Attempt Questions 1 – 7
- All questions are of equal value

Students are reminded that this is a trial examination only and cannot in any way guarantee the content or the format of the 2002 Mathematics Extension 1 Higher School Certificate examination

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x>0$

Total marks (84)
Attempt Questions 1 – 7
All questions are of equal value

Answer each question in a SEPARATE writing booklet.

QUESTION 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Factorise $64 - y^3$. **1**

(b) Find the coordinates of the point that divides the join of A (1, -3) and B (-5, 7) externally in the ratio 5 : 2. **2**

(c) Solve $\dfrac{2x}{x-2} > 1$. **3**

(d) Find the acute angle, to the nearest degree, between the lines $3x - y + 2 = 0$ and $x + 2y - 5 = 0$. **3**

(e) Using the expansion of $\cos(A+B)$, find the exact value of $\cos 75°$ in surd form with a rational denominator. **3**

QUESTION 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the general solution of the equation $\sin^2 x - 2\cos x + 2 = 0$, where x is in radians. **3**

(b) Find $\int \cos^2 2x \, dx$. **2**

(c) Use the substitution $x = u^2 - 1$ for $u > 0$, to evaluate $\int_3^8 \dfrac{x-1}{\sqrt{x+1}} dx$. **4**

(d) The probability that a student will attend university after doing the HSC is 0.7. In a class of 25 students, what is the probability that at least 23 students will proceed to university after the HSC? **3**

QUESTION 3 (12 marks) Use a SEPARATE writing booklet. Marks

(a) Find the indefinite integral $\int \frac{dx}{9+16x^2}$. 2

(b) Consider the function $f(x)=\cos^{-1}(x+2)-\frac{\pi}{2}$.

(i) What is the domain of $y=f(x)$? 1

(ii) Sketch the graph of $y=f(x)$. 2

(c) Consider the function $f(x)=\tan^{-1}x+\tan^{-1}\frac{1}{x}$.

(i) Prove the derivative of $f(x)$ is zero. 2

(ii) What is the domain of $y=f(x)$? 1

(iii) Find the value of $f(x)$ over the domain of $y=f(x)$. 2

(d) The velocity v of a particle moving in a straight line at position x is given by: 2

$$v=1+2x.$$

Find the acceleration of the particle at position x.

QUESTION 4 (12 marks) Use a SEPARATE writing booklet. Marks

(a) Consider the letters of the word MILLER.

(i) How many arrangements of these letters are possible if the letters are arranged in a straight line? 1

(ii) What is the probability that the L's will be separated when the letters are arranged in a straight line? 2

(iii) If the letters are arranged in a circle, how many arrangements are possible? 1

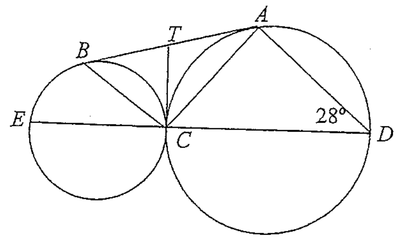
(iv) If the letters are arranged in a circle, what is the probability the L's will be opposite each other? 2

(b) Find the term independent of x in the expansion of $\left(x-\frac{2}{x^2}\right)^9$, expressing your answer as an integer. 3

(c) Prove that $\binom{n+1}{2}+\binom{n+2}{2}$ is the square of an integer. 3

QUESTION 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

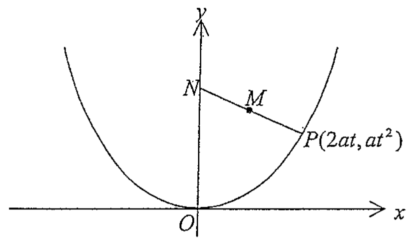
(a) 4



AB is a common tangent to the circles, which touch at C. CT is the tangent at C. A line through C meets the circles at D and E. $\angle ADC = 28°$.

Find the size of $\angle ACB$, giving reasons.

(b)



A normal is drawn to the parabola $x^2 = 4ay$ at a variable point $P(2at, at^2)$ on the parabola. The normal meets the y axis at N. M is the midpoint of PN. The coordinates of M are $(at, at^2 + a)$ and O is the origin.

(i) Show that the equation of the locus of M is the parabola $ay = x^2 + a^2$. 1

(ii) What is the vertex of the parabola $ay = x^2 + a^2$? 1

(iii) What is the domain of this locus? Explain. 1

(c) The rate of change of the amount of pollutant (x units) in a lake after t days of rain is given by: $\frac{dx}{dt} = \frac{1}{4} - \frac{x}{16}$.

(i) Show that $x = 4 + Ae^{-\frac{t}{16}}$ satisfies this equation. 2

(ii) If the initial amount of pollutant is 20 units, find the value of A. 1

(iii) In how many days will the amount of the pollutant drop to 8 units? 2

QUESTION 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Two of the roots of the equation $x^3 + 6x^2 - x + k = 0$ are $2, -3$, where k is a constant.

(i) By considering the sum of the roots of this equation, find the third root. 2

(ii) Hence or otherwise find the value of the constant k. 2

(b) Use one step of Newton's method to find an approximation to the root of the equation $x^4 - 10x + 7 = 0$ near $x = 2$. 3

(c) A particle moves in Simple Harmonic Motion about an origin O. The period of motion is 4 seconds and the amplitude is 6 cm.

(i) Write down an expression for x in terms of t, when the particle is x cm from O, t seconds after passing through O. 2

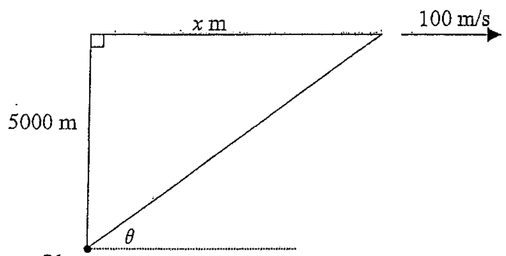
(ii) Find its speed as it passes through O. 2

(iii) What is the maximum acceleration of the particle? 1

QUESTION 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove by mathematical induction, where n is a positive integer and $n \geq 2$, that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$. **5**

(b)



At a certain instant, a plane flies overhead at a constant altitude of 5000 metres, at a constant speed of 100 metres per second.
When the plane has travelled x metres from the overhead position, its angle of elevation from the observer is θ radians.

(i) Show that $\frac{dx}{d\theta}=-\frac{5000}{\sin^2\theta}$. **2**

(ii) Hence show that $\frac{d\theta}{dt}=-\frac{1}{50}\sin^2\theta$. **2**

(iii) Find the rate at which the angle of elevation is changing:

(α) when the plane is overhead. **1**

(β) 50 seconds after the plane is overhead. **2**

End of Examination

SAINT IGNATIUS' COLLEGE

Trial Higher School Certificate

2002

MATHEMATICS

EXTENSION 1

Suggested Solutions

Mathematics Extension 1: Question 1

Suggested Solutions	Marks Awarded	Marker's Comments

(a) $64 - y^3 = (4-y)(16+4y+y^2)$ (1)

(b) $A(1,-3) \quad B(-5,7) \qquad 5:-2$

$$\left(\frac{5\times(-5)+(-2)\times 1}{5+(-2)}, \frac{5\times 7+(-2)\times(-3)}{5+(-2)}\right)$$

$$\left(-9, 13\tfrac{2}{3}\right)$$ (2)

(c) $$\frac{2x}{x-2} > 1$$

$$2x(x-2) > (x-2)^2$$

$$2x^2 - 4x > x^2 - 4x + 4$$

$$x^2 > 4$$

$$x < -2, \quad x > 2$$ (3)

(d) $3x - y + 2 = 0: \quad m = 3$

$x + 2y - 5 = 0: \quad m = -\frac{1}{2}$

$$\tan\theta = \left|\frac{3-(-\frac{1}{2})}{1+3\times(-\frac{1}{2})}\right|$$

$$= 7$$

$$\theta = 82^\circ \text{ (nearest degree)}$$ (3)

(e) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$\cos 75^\circ = \cos(45^\circ + 30^\circ)$$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$ ([illegible])

Mathematics Extension 1: Question 2

Suggested Solutions	Marks Awarded	Marker's Comments

(a) $\sin^2 x - 2\cos x + 2 = 0$

$1 - \cos^2 x - 2\cos x + 2 = 0$

$\cos^2 x + 2\cos x - 3 = 0$

$(\cos x + 3)(\cos x - 1) = 0$

$\cos x = -3 \quad$: no solution

$\cos x = 1 \quad : \quad x = 2n\pi$

where n is an integer. (3)

(b) $$\int \cos^2 2x\, dx = \int \tfrac{1}{2}(1+\cos 4x)\, dx$$

$$= \tfrac{1}{2}\left(x + \tfrac{1}{4}\sin 4x\right) + C$$ (2)

(c) $$\int_3^8 \frac{x-1}{\sqrt{x+1}}\, dx$$

$x = u^2 - 1$

$dx = 2u\, du$

$x = 3 \Rightarrow u = 2$

$x = 8 \Rightarrow u = 3$

$$= \int_2^3 \frac{u^2-2}{u}\times 2u\, du$$

$$= \int_2^3 (2u^2 - 4)\, du$$

$$= \left[\tfrac{2}{3}u^3 - 4u\right]_2^3$$

$$= (18-12) - (5\tfrac{1}{3} - 8)$$

$$= 8\tfrac{2}{3}$$ (4)

(d) $P(\text{at least 23 students})$

$$= P(23) + P(24) + P(25)$$

$$= \binom{25}{23}(0\cdot 7)^{23}(0\cdot 3)^2 + \binom{25}{24}(0\cdot 7)^{24}(0\cdot 3) + (0\cdot 7)^{25}$$

$$= 0\cdot 00896$$ (3)

Mathematics Extension 1: Question 3

Suggested Solutions	Marks Awarded	Marker's Comments

(a) $\int \frac{dx}{9+16x^2} = \int \frac{dx}{16(\frac{9}{16}+x^2)}$

$= \frac{1}{16} \times \frac{1}{\frac{3}{4}} \tan^{-1} \frac{x}{\frac{3}{4}} + C$

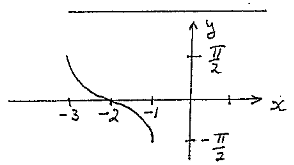
$= \frac{1}{12} \tan^{-1} \frac{4x}{3} + C$

OR Use substitution $u = 4x$.

(b) $f(x) = \cos^{-1}(x+2) - 1$

(i) Domain: $-1 \le x+2 \le 1$

i.e. $-3 \le x \le -1$ (1)



(2)

(c) $f(x) = \tan^{-1}x + \tan^{-1}\frac{1}{x}$

(i) $f'(x) = \frac{1}{1+x^2} + \frac{1}{1+(\frac{1}{x})^2} \times \left(\frac{-1}{x^2}\right)$

$= \frac{1}{1+x^2} - \frac{1}{x^2+1}$

$= 0$ (2)

(ii) Domain: all x, except $x=0$ (1)

(iii) For $x>0$: $f(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

For $x<0$: $f(-1) = -\frac{\pi}{4} + (-\frac{\pi}{4}) = -\frac{\pi}{2}$.

$\therefore f(x) = \frac{\pi}{2}$ for $x>0$; $f(x) = -\frac{\pi}{2}$ for $x<0$. (2)

(d) $v = 1+2x$

$a = \frac{d}{dx}(\frac{1}{2}v^2)$

$= \frac{d}{dx} \frac{1}{2}(1+2x)^2$

$= \frac{1}{2} \times 2(1+2x) \times 2$

$= 2(1+2x)$.

Mathematics Extension 1: Question 4

Suggested Solutions	Marks Awarded	Marker's Comments

(a) MILLER.

(i) No. of arrange's $= \frac{6!}{2!} = 360$ (1)

(ii) Prob (L's separated)

$= 1 - P(\text{L's together})$

$= 1 - \frac{5!}{360}$

$= \frac{2}{3}$ (2)

(iii) No. of arrange's $= \frac{5!}{2!} = 60$ (1)

(iv) P(L's opposite) $= \frac{1}{5}$

or $\frac{4!}{5!} = \frac{1}{5}$ (2)

(b) $\left(x - \frac{2}{x^2}\right)^9$.

Term indep. of $x = \binom{9}{3} x^6 \left(-\frac{2}{x^2}\right)^3$

(by inspection)

$= 504 \times (-8) \times x^0$

$= -672$

OR $T_{r+1} = \binom{9}{r} x^{9-r} \left(\frac{-2}{x^2}\right)^r$

$= \binom{9}{r} (-2)^r x^{9-3r}$ (3)

(c) $\binom{n+1}{2} + \binom{n+2}{2} = \frac{(n+1)n}{2} + \frac{(n+2)(n+1)}{2}$

$= \frac{n+1}{2}[n + n+2]$

$= \frac{n+1}{2} \times 2(n+1)$

$= (n+1)^2$

which is the square of an integer. (3)

Mathematics Extension 1: Question 5

Suggested Solutions	Marks Awarded	Marker's Comments

(a) $x^3 + 6x^2 - x + k = 0$

(i) Let the third root be α:

$2 + (-3) + \alpha = -6$

$\alpha = -5$

(2)

(ii) By product of roots:

$2 \times (-3) \times (-5) = -k$

$k = -30$

(2)

(b) $f(x) = x^4 - 10x + 7$

$f'(x) = 4x^3 - 10$

$x_1 = 2 - \dfrac{f(2)}{f'(2)}$

$= 2 - \dfrac{3}{22}$

$= 1.86$ (2 d.p.)

(3)

(c) Period = 4 sec, Amplitude = 6 cm.

(i) $T = \frac{2\pi}{n} = 4 \quad \therefore n = \frac{\pi}{2}$.

$x = 6 \sin \frac{\pi}{2} t$.

(2)

(ii) $v = 6 \times \frac{\pi}{2} \cos \frac{\pi}{2} t$.

When $t = 0$, $v = 3\pi$.

Speed is 3π cm/s at O.

(2)

(iii) $a = -3\pi \times \frac{\pi}{2} \sin \frac{\pi}{2} t$

$= -\dfrac{3\pi^2}{2} \sin \frac{\pi}{2} t$

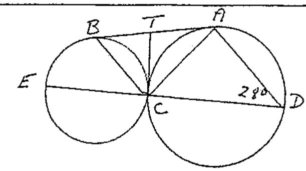
Maximum acceleration is $\frac{3\pi^2}{2}$ cm/sec^2

(1)

Mathematics Extension 1: Question 6

Suggested Solutions	Marks Awarded	Marker's Comments

(a)



$\angle TAC = 28°$ (alternate seg. theorem)

$\angle TCA = 28°$ (TA = TC : tangents from an external point are equal)

$\angle BTC = 56°$ (exterior angle of ΔTAC).

But TB = TC (tangents from ext. point)

$\therefore \angle TCB = \frac{1}{2}(180° - 56°)$ (angle sum of isos. ΔTBC)

$= 62°$

$\therefore \angle ACB = 28° + 62° = 90°$.

(4)

(b) (i) Coord of M. $x = at$, $y = at^2 + a$.

$t = \frac{x}{a} \quad \therefore y = a \times \frac{x^2}{a^2} + a$

$ay = x^2 + a^2$

(1)

(ii) $x^2 = a(y - a)$ Vertex is $(0, a)$.

(1)

(iii) Domain – all x except $x = 0$, because normal at $(0,0)$ does not cut the y-axis at one point – it is on the y-axis

(1)

(c) $\dfrac{dx}{dt} = \dfrac{1}{4} - \dfrac{x}{16}$.

(i) $x = 4 + Ae^{-\frac{t}{16}}$

$\dfrac{dx}{dt} = -\frac{1}{16} A e^{-\frac{t}{16}} = -\frac{1}{16}(x - 4) = \frac{1}{4} - \frac{x}{16}$.

(2)

(ii) $t = 0, x = 20$: $20 = 4 + Ae^0 \therefore A = 16$

(1)

(iii) $x = 4 + 16e^{-\frac{t}{16}}$

When $x = 8$, $8 = 4 + 16e^{-\frac{t}{16}}$

$e^{-\frac{t}{16}} = \frac{1}{4}$, $e^{\frac{t}{16}} = 4$

$t = 16 \ln 4 = 22.18$

Approx 22 days.

(2)

Mathematics Extension 1: Question 7(a)

Suggested Solutions	Marks Awarded	Marker's Comments

(a)

Prove $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for $n\geq 2$.

When $n=2$, $LHS = 1-\frac{1}{2^2} = \frac{3}{4}$

$RHS = \frac{2+1}{2\times 2} = \frac{3}{4}$

$\therefore$ it is true for $n=2$.

Assume it is true for $n=k$:

i.e. assume $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$.

When $n=k+1$,

$$LHS=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) \text{ by assumption}$$

$$= \frac{k+1}{2k}\times\frac{(k+1)^2-1}{(k+1)^2}$$

$$= \frac{k^2+2k+1-1}{2k(k+1)}$$

$$= \frac{k^2+2k}{2k(k+1)}$$

$$= \frac{k(k+2)}{2k(k+1)}$$

$$= \frac{k+2}{2(k+1)}$$

$$= \frac{n+1}{2n} \text{ when } n=k+1.$$

$\therefore$ if it is true for $n=k$, then it is true for $n=k+1$.

Since it is true for $n=2$, it is true for $n=3$, $n=4$, ...

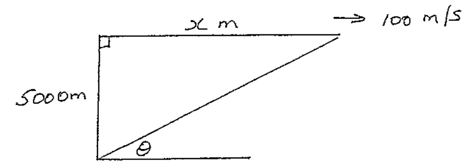
i.e. it is true for all positive integers $n\geq 2$

(5)

Mathematics Extension 1: Question 7(b)

Suggested Solutions	Marks Awarded	Marker's Comments

(b)



(i) $\frac{x}{5000} = \tan(90°-\theta)$

$x = 5000\cot\theta$

$\frac{dx}{d\theta} = -5000\,\text{cosec}^2\theta$

$= -\frac{5000}{\sin^2\theta}$

(2)

(ii) $\frac{dx}{dt} = \frac{dx}{d\theta}\times\frac{d\theta}{dt}$

$100 = -\frac{5000}{\sin^2\theta}\times\frac{d\theta}{dt}$

$\frac{d\theta}{dt} = -\frac{100\sin^2\theta}{5000}$

$= -\frac{1}{50}\sin^2\theta$

(2)

(iii) (α) When overhead, $\theta = 90°$,

$\frac{d\theta}{dt} = -\frac{1}{50}\sin^2 90°$

$= -\frac{1}{50}$

Angle is changing at $\frac{1}{50}$ rad/sec.

(1)

(β) When $t=50$, $x = 100\times 50$

$= 5000$

$\theta = \frac{\pi}{4}$.

$\frac{d\theta}{dt} = -\frac{1}{50}\sin^2\left(\frac{\pi}{4}\right)$

$= -\frac{1}{100}$

Angle is changing at $\frac{1}{100}$ rad/sec.

(2)