Student Name:_______________

Saint Ignatius' College
Riverview

2003

YEAR 12
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Time allowed – 2 hours, + 5 minutes reading time
- Write using blue or black pen
- Board-approved calculators and mathaids may be used
- Show all necessary working
- Answer each question in a separate booklet with your name and teacher's name

Total Marks (84)

- Attempt Questions 1 – 7

Total marks (84)
Attempt Questions 1 – 7
All questions are of equal value

Answer each question in a SEPARATE writing booklet.

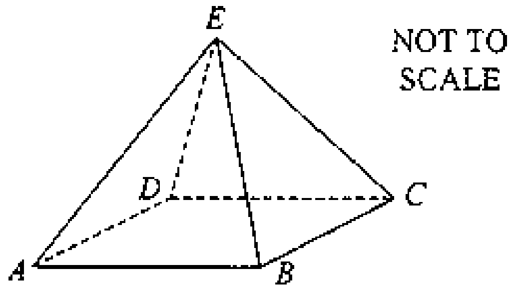
QUESTION 1 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the acute angle between the lines $y = 2x - 5$ and $y = 6 - 3x$. **2**

(b) Solve $\dfrac{x+4}{x} < 3$. **3**

(c) Find the general solutions of the equation $\sin 2\theta = \sin^2\theta$. **4**
Give your answer in terms of π.

(d) **3**



A right square pyramid $ABCDE$ has a base of length 6cm and a perpendicular height of 8cm.

Find the angle which the slant edge AE makes with the base $ABCD$.

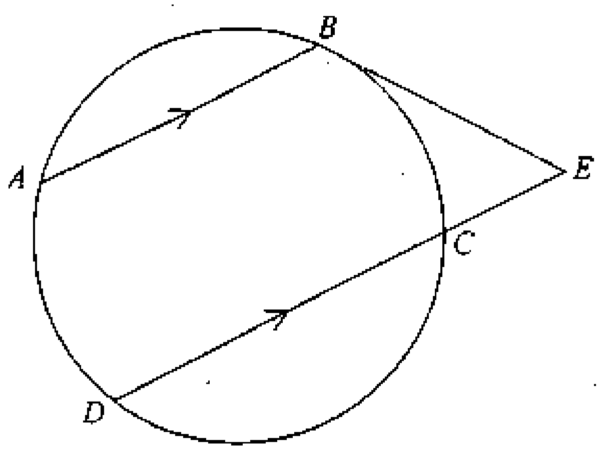
QUESTION 2 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) The point (2, 2) divides the join of (-2, 5) to (a, b) in the ratio 3:2. **2**
Find the values of a and b.

(b) If α, β, γ are the roots of the equation $2x^3 - 6x^2 + 5x - 1 = 0$, **3**
find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

(c) The polynomial equation $x^3 - 11x^2 + px + q = 0$ has a double root **4**
at $x = \alpha$ and a single root at $x = \alpha + 2$.
Using the formula for the sum of the roots, or otherwise, find the values of α, p and q.

(d) **3**



In the diagram, A, B, C and D lie on a circle.
AB is parallel to DC and the tangent at B meets DC produced at E.

Copy or trace the diagram onto your writing page, and join BC and AC.

Prove that ΔABC is similar to ΔBCE.

QUESTION 3 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the inverse function of the function $f(x) = \dfrac{5-2x}{3}$, expressing your answer in the form $f^{-1}(x) = ...$ **2**

(b) Evaluate $\cos^{-1}\left(\dfrac{1}{2}\tan\dfrac{2\pi}{3}\right)$. **2**

(c) Find the exact value of $\sin\left(2\cos^{-1}\dfrac{2}{3}\right)$. **3**

(d) Prove $\dfrac{\cos A - \sin A}{\cos A + \sin A} = \dfrac{\cos 2A}{1+\sin 2A}$. **2**

(e) Show that there is only one stationary point on the curve $y = x + \cos^{-1} x$, and determine its nature. **3**

QUESTION 4 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) From a standard pack of 52 cards, a hand of 4 cards is dealt.

(i) How many different hands can be selected? 1

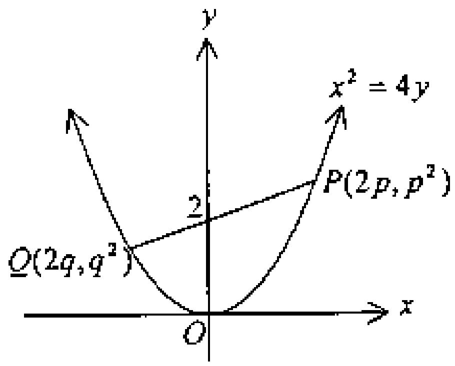
(ii) What is the probability I will be dealt exactly two aces? 2

(b) The letters of the word C A L C U L U S are arranged in a row.

(i) How many different arrangements are possible? 2

(ii) In how many of the arrangements will the letters U be at each end? 1

(c)



Points $P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola $x^2 = 4y$.

(i) Show that the equation of the chord PQ is $(p+q)x - 2y - 2pq = 0$. 2

(ii) Find the coordinates of M, the midpoint of PQ. 1

(iii) Hence find the equation of the locus of M if the chord PQ crosses the y axis at $(0, 2)$. 3

QUESTION 5 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the following indefinite integrals:

(i) $\int \frac{1}{4+x^2}\,dx$ **1**

(ii) $\int \frac{x}{4+x^2}\,dx$ **1**

(b) Find $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$. **3**

(c) Find $\int \frac{e^{2x}}{e^x - 2}\,dx$ using the substitution $u = e^x - 2$. **3**

(d) The acceleration of a particle moving in a straight line at position x is given by $\ddot{x} = -\frac{6}{(x+1)^2}$. Initially it has velocity 4 units when it is at the origin. **4**

Show that the velocity v at position x is given by $v = \pm 2\sqrt{\frac{x+4}{x+1}}$.

QUESTION 6 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Use the table of standard integrals to find $\int \frac{1}{\sqrt{x^2+16}}\,dx$. **1**

(b) Prove by mathematical induction, for positive integers n, that **4**

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}.$$

(c) Use one application of Newton's method to find an approximation to the root of $2x-4\sin 3x=0$ near $x=1$. **3**
Write your answer to two decimal places.

(d) (i) On the same set of axes, sketch the graphs of the equations $y=|2x|$ and $y=x^2-3$. **2**

(ii) Hence or otherwise solve the inequality **2**

$$|2x|>x^2-3.$$

QUESTION 7 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) An object is projected from level ground at an angle θ to the horizontal, with a velocity of V m/s. The object returns to the ground after 4 seconds and 100 metres from its point of projection.
Assume acceleration due to gravity is 10 m/s², and neglect air resistance.

(i) From the equations for acceleration in the x and y directions, find expressions for x and y in terms of time t ($t \leq 4$). **2**

(ii) Hence find the values of V and θ. **2**

(iii) What is the maximum height reached by the object? **2**

(b) Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference between the temperature of the body and that of the surrounding medium.

$$\text{i.e. } \frac{dT}{dt} = -k(T - T_0)$$

where T is the temperature of the body at time t and T_0 is the temperature of the surrounding medium, assumed constant.

(i) Show that $T = T_0 + Ae^{-kt}$ is a solution to this equation. **1**

(ii) A body whose temperature is 150°C is cooled by placing it in a liquid at 25°C. In one minute, the temperature of the body had cooled to 100°C.
How long will it take for the body to cool to 50°C? **5**

End of paper

SIC Trials – 2003

Mathematics Extension 1: Question

Suggested Solutions	Marks Awarded	Marker's Comments

1(a) $y = 2x - 5 \qquad y = 6 - 3x$

$$\tan\theta = \left|\frac{2-(-3)}{1+2\times(-3)}\right| \qquad \left(\frac{m_1 - m_2}{1+m_1 m_2}\right)$$

$$= 1$$

$$\theta = 45^\circ$$

(2)

(b) $$\frac{x+4}{x} < 3$$

$$x(x+4) < 3x^2$$

$$2x^2 - 4x > 0$$

$$2x(x-2) > 0$$

$$x < 0,\ x > 2$$

(3)

(c) $$\sin 2\theta = \sin^2\theta$$

$$2\sin\theta\cos\theta - \sin^2\theta = 0$$

$$\sin\theta(2\cos\theta - \sin\theta) = 0$$

$$\sin\theta = 0 \quad \text{or} \quad \sin\theta = 2\cos\theta$$

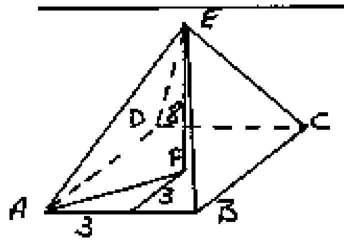
$$\tan\theta = 2.$$

$$\theta = n\pi, \quad \theta = n\pi + \tan^{-1}2$$

or $n\pi + 1{\cdot}11$ (2dp)

(4)

(d).



$$AF = \sqrt{3^2+3^2} = \sqrt{18}$$

$$\tan\theta = \frac{EF}{AF} = \frac{8}{\sqrt{18}}$$

$$\theta = 62^\circ 04'$$

or 62° (nearest degree)

(3)

Mathematics Extension 1: Question 2

Suggested Solutions	Marks Awarded	Marker's Comments

2(a) $\begin{matrix} x_1\ y_1 & x_2\ y_2 & m_1\ m_2 \\ (-2,5) & (a,b) & 3:2 \end{matrix}$

$$\frac{3a+2(-2)}{3+2}=2\ ;\quad \frac{3\times b+2\times 5}{3+2}=2$$

$$a=4\tfrac{2}{3},\quad b=0 \qquad (2)$$

(b) $2x^3-6x^2+5x-1=0$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$$

$$=\frac{\frac{5}{2}}{\frac{1}{2}}$$

$$=5 \qquad (3)$$

(c) $x^3-11x^2+px+q=0$

Roots are $\alpha, \alpha, \alpha+2$.

Sum of roots: $\alpha+\alpha+(\alpha+2)=11$

$\alpha=3$.

$\therefore$ Roots are $3, 3, 5$.

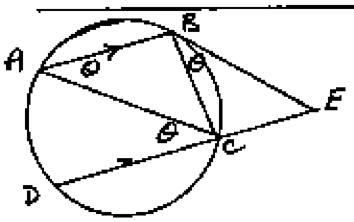
$\alpha\beta+\alpha\gamma+\beta\gamma$: $3\times3+3\times5+3\times5=p$

$p=39$.

$\alpha\beta\gamma$: $3\times3\times5=-q$

$q=-45$. (4)

(d).



$\angle CBE=\angle BAC=\theta$ (alt. segment thm) (A)

$\angle BAC=\angle ACD=\theta$ (alt. angles $AB \parallel DC$)

$\angle BCD=\angle CBE+\angle BEC$ (ext. angle of $\triangle BCE$)

$\theta+\angle ACB=\theta+\angle BEC$

$\therefore \angle ACB=\angle BEC$ (B)

From (A), (B), two pairs of angles are equal

$\therefore \triangle ABC \,|||\, \triangle BCE$ (3)

Marker's Comments:

Alternative:

$\angle ABC=\angle BCE$ (alternate $\angle$s in parallel lines)

Mathematics Extension 1: Question 3

Suggested Solutions	Marks Awarded	Marker's Comments

3.(a) Let $y = \frac{5-2x}{3}$

Inverse is: $x = \frac{5-2y}{3}$

$3x = 5 - 2y$

$y = \frac{5-3x}{2}$

$\therefore f^{-1}(x) = \frac{5-3x}{2}$ (2)

(b) $\cos^{-1}\left(\frac{1}{2}\tan\frac{2\pi}{3}\right) = \cos^{-1}\left(\frac{1}{2} \times -\sqrt{3}\right)$

$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{5\pi}{6}$ (2)

(c). $\sin\left(2\cos^{-1}\frac{2}{3}\right)$ Let $\theta = \cos^{-1}\frac{2}{3}$

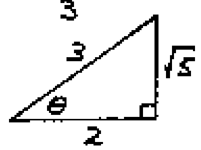
$\cos\theta = \frac{2}{3}$

$= \sin 2\theta$

$= 2\sin\theta\cos\theta$

$= 2 \times \frac{\sqrt{5}}{3} \times \frac{2}{3}$

$= \frac{4\sqrt{5}}{9}$ (3)



(d) $\frac{\cos A - \sin A}{\cos A + \sin A} = \frac{\cos A - \sin A}{\cos A + \sin A} \times \frac{\cos A + \sin A}{\cos A + \sin A}$

$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A + 2\sin A\cos A}$

$= \frac{\cos 2A}{1 + \sin 2A}$ (2)

(e) $y = x + \cos^{-1}x$

$\frac{dy}{dx} = 1 - \frac{1}{\sqrt{1-x^2}} = 1 - (1-x^2)^{-\frac{1}{2}}$

For stat. point: $1 = \frac{1}{\sqrt{1-x^2}}$ $\therefore \sqrt{(1-x^2)} = 1$

$\therefore x = 0$

$\frac{d^2y}{dx^2} = \frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x) = \frac{-x}{(1-x^2)^{3/2}}$

When $x=0$, $\frac{d^2y}{dx^2} = 0$.

If $x<0$, $\frac{d^2y}{dx^2} > 0$; If $x>0$, $\frac{d^2y}{dx^2} < 0$

Concavity changes $\therefore$ one stationary point is a horizontal point of inflexion. (3)

Mathematics Extension 1: Question 4

Suggested Solutions	Marks Awarded	Marker's Comments
4(a)(i) No. of different hands $= \binom{52}{4}$ $= 270\,725$ (1)		
(ii) $P(2\text{ aces}) = \dfrac{\binom{4}{2}\binom{48}{2}}{\binom{52}{4}}$ $= 0\cdot025$ (2)		
(b) (i) No. of arrangements $= \dfrac{8!}{2!\,2!\,2!}$ $= 5040$ (2)		
(ii) No. arrgts. with U at ends $= \dfrac{6!}{2!\,2!}$ $= 180$ (1)		
(c)(i) PQ: $\dfrac{y-q^2}{x-2q} = \dfrac{p^2-q^2}{2p-2q} = \dfrac{p+q}{2}$ $2y-2q^2 = (p+q)x-2q(p+q)$ $2y-2q^2 = (p+q)x-2pq-2q^2$ $(p+q)x-2y-2pq=0$ (2)		
(ii) M: $\left(\dfrac{2p+2q}{2}, \dfrac{p^2+q^2}{2}\right)$ i.e. $\left(p+q, \dfrac{p^2+q^2}{2}\right)$ (1)		
(iii) If PQ passes through $(0,2)$ Subst. $x=0, y=2$: $0-2\times2-2pq=0$ $pq=-2$ $\therefore x=p+q, \quad y=\dfrac{p^2+q^2}{2}$ $(p+q)^2 = p^2+q^2+2pq$ $x^2 = 2y+2\times(-2)$ $x^2 = 2y-4$ Locus of M is $x^2=2y-4$. (3)		

Mathematics Extension 1: Question 5

Suggested Solutions	Marks Awarded	Marker's Comments
5. (a)(i) $\int \frac{1}{4+x^2}\,dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$ (1)		
(ii) $\int \frac{x}{4+x^2}\,dx = \frac{1}{2}\int \frac{2x}{4+x^2}\,dx = \frac{1}{2}\log_e(4+x^2) + C$ (1)		
(b) $\int_0^{\frac{\pi}{6}} \sin^2 x\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}} = \frac{1}{2}\left[\left(\frac{\pi}{6} - \frac{1}{2}\times\frac{\sqrt{3}}{2}\right) - (0-0)\right] = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$ or $\frac{2\pi - 3\sqrt{3}}{24}$ (3)		
(c) $\int \frac{e^{2x}}{e^x - 2}\,dx$; $u = e^x - 2$, $\frac{du}{dx} = e^x$, $du = e^x\,dx$; $= \int \frac{e^x e^x\,dx}{e^x - 2} = \int \frac{(u+2)\,du}{u} = \int \left(1 + \frac{2}{u}\right)du = u + 2\ln u + c = e^x - 2 + 2\ln(e^x - 2) + C$ (3)		
(d) $\ddot{x} = \frac{-6}{(x+1)^2}$; $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -6(x+1)^{-2}$; $\frac{1}{2}v^2 = 6(x+1)^{-1} + c$; When $x=0, v=4$: $8 = 6 + c$, $c = 2$; $\therefore \frac{1}{2}v^2 = \frac{6}{x+1} + 2 = \frac{6 + 2(x+1)}{x+1} = \frac{2x+8}{x+1} = 2\left(\frac{x+4}{x+1}\right)$; $v^2 = 4\left(\frac{x+4}{x+1}\right)$; $v = \pm 2\sqrt{\frac{x+4}{x+1}}$ (4)		

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Mathematics Extension 1: Question 6

Suggested Solutions	Marks Awarded	Marker's Comments

6(a) $\displaystyle\int \frac{1}{\sqrt{x^2+16}}\,dx = \log_e\left(x+\sqrt{x^2+16}\right)+C$ (1)

(b) Prove $\dfrac{1}{1.3}+\dfrac{1}{3.5}+\dots+\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{2n+1}$

When $n=1$, $LHS=\frac{1}{1.3}=\frac{1}{3}$, $RHS=\frac{1}{2+1}=\frac{1}{3}$

$\therefore$ it is true for $n=1$.

Assume it is true for $n=k$.

i.e. assume $\dfrac{1}{1.3}+\dfrac{1}{3.5}+\dots+\dfrac{1}{(2k-1)(2k+1)}=\dfrac{k}{2k+1}$

When $n=k+1$,

$$LHS=\frac{1}{1.3}+\frac{1}{3.5}+\dots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \quad \text{by assumption}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(k+1)(2k+1)}{(2k+1)(2k+3}$$

$$=\frac{k+1}{2(k+1)+1}=\frac{n}{2n+1} \quad \text{where } n=k+1.$$

$\therefore$ if it is true for $n=k$, it is true for $n=k+1$.

Since it is true for $n=1$, it is true for $n=2$, $n=3$, ... (4)

(c) Let $f(x)=2x-4\sin 3x$

$f'(x)=2-12\cos 3x$.

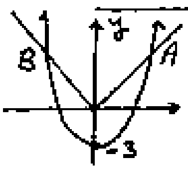
$f(1)=2-4\sin 3=1.4355$

$f'(1)=2-12\cos 3=13.880$

Approx'n $=1-\dfrac{f(1)}{f'(1)}=1-\dfrac{1.4355}{13.880}$

$=0.90$ (2dP) (3)

(d)(i)



(2)

(ii) $x^2-3=2x$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=3,-1$

$A(3,6)$ $B(-3,6)$

$\therefore |2x|>x^2-3$ for $-3<x<3$. (2)

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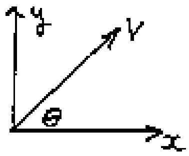
Mathematics Extension 1: Question 7(a)

Suggested Solutions	Marks Awarded	Marker's Comments

7(a) (i) $\ddot{x} = 0$

$\dot{x} = c$



When $t=0$, $\dot{x} = V\cos\theta$

$\therefore c = V\cos\theta$

$\therefore \dot{x} = V\cos\theta$

$x = V\cos\theta\, t + c'$

When $t=0$, $x=0$ $\therefore c'=0$

$\therefore x = V\cos\theta\, t$

$\ddot{y} = -10$

$\dot{y} = -10t + k$

When $t=0$, $\dot{y} = V\sin\theta$ $\therefore k = V\sin\theta$

$\therefore \dot{y} = V\sin\theta - 10t$

$y = V\sin\theta\, t - 5t^2 + k'$

When $t=0$, $y=0$ $\therefore k'=0$

$\therefore y = V\sin\theta t - 5t^2$ 2

(ii) When $t=4$, $y=0$, $x=100$

$100 = 4V\cos\theta$ $\quad 0 = 4V\sin\theta - 80$

$V\cos\theta = 25$ $\quad V\sin\theta = 20$

$\dfrac{V\sin\theta}{V\cos\theta} = \dfrac{20}{25}$ $\therefore \tan\theta = 0.8$

$\theta = 38°40'$.

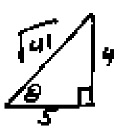
Also, $V^2\cos^2\theta + V^2\sin^2\theta = 25^2 + 20^2$

$V^2(\cos^2\theta + \sin^2\theta) = 1025$

$V = \sqrt{1025}$ or 32.0 m/s 2

$= 5\sqrt{41}$

(iii) Maximum height when $\dot{y} = 0$



$5\sqrt{41}\sin 38°40' - 10t = 0$

$t = 2$

When $t=2$, $y = 5\sqrt{41} \times \dfrac{4}{\sqrt{41}} \times 2 - 5 \times 2^2$

$= 20$

Maximum height is 20 m.

2

Mathematics Extension 1: Question 7(b)		
Suggested Solutions	Marks Awarded	Marker's Comments
7.(b) $\frac{dT}{dt} = -k(T - T_0)$ (i) $T = T_0 + Ae^{-kt}$ $\frac{dT}{dt} = -kAe^{-kt}$ $= -k(T - T_0)$ 1 (ii) When $t = 0$, $T = 150$, $T_0 = 25$ $150 = 25 + A$ $\therefore A = 125$ ✓ $\therefore T = 25 + 125e^{-kt}$ When $t = 1$, $T = 100$ $100 = 25 + 125e^{-k}$ $75 = 125e^{-k}$ $e^{k} = \frac{125}{75}$ $k = \ln\left(\frac{125}{75}\right)$ $= 0.5108$ (4dp) ✓✓ $\therefore T = 25 + 125e^{-0.5108t}$ When $T = 50$, $50 = 25 + 125e^{-0.5108t}$ $25 = 125e^{-0.5108t}$ $e^{0.5108t} = \frac{125}{25} = 5$ $0.5108t = \ln 5$ $t = \frac{\ln 5}{0.5108}$ $= 3.15$ (2dp) It takes 3.15 minutes to reach 50°. ✓✓ 5		