Number: ....

Roseville College

Year 12

Trial Higher School Certificate Examination

2001

EXTENSION 1 MATHEMATICS

Time Allowed: 2hours, plus 5 minutes reading time.

Instructions

- All questions are of equal value.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- **Start each question on a new page.** Write your number on each page.
- Staple each question separately

QUESTION 1. Start a new page (12 marks)

(a) Use the substitution $u = x^2 + 2$ to evaluate

$$\int_0^1 \frac{x}{x^2+2}\,dx$$ (3)

b) Solve for x if $\dfrac{4}{x-2} > 3$ (3)

(c) Find the exact value of $\tan\left(2\tan^{-1}\dfrac{3}{4}\right)$ (2)

(d) A box contains 12 jellybeans of which 5 are red, 4 are blue and 3 are white. If 3 jellybeans are picked up at once what is the probability that all three are different colours? (2)

(e) Sketch a continuous smooth curve which satisfies the following conditions

$f(0) = 1$

$f'(x) < 0$ and $f''(x) > 0$ for $0 < x < 2$

$f'(2) = 0$ (2)

$f(2) = -2$

$f'(x) < 0$ and $f''(x) < 0$ for $x > 2$

QUESTION 2. Start a new page (12 marks)

(a) State the domain and range

$f(x) = 4\sin^{-1}(\frac{x}{3})$ (3)

(b) (i) Show that the equation $x^3 + x - 3 = 0$ has 1 root between 1.2 and 1.3

(ii) Taking 1.2 as the first approximation to the root, use Newton's method once to find a second approximation. (3)

(c) A polynomial $P(x)$ of degree three, has zeros at $x = -2$, $x = -1$ and $x =$ [illegible] and a remainder of 36 when divided by $(x-2)$. Find $P(x)$, expressing it in the form

$$p_0x^3 + p_1x^2 + p_2x + p_3$$ (3)

(d) The tangent at $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ meets the directrix at K

(i) Show that the coordinates of K are $(\dfrac{ap^2 - a}{p}, -a)$ (1)

(ii) Prove that angle PSK is a right angle, where S is the focus (2)

QUESTION 3. Start a new page (12 marks)

(a) The acceleration of a particle is given by $4(1+x)$, where x is the displacement from the origin. If initially, the particle is at the origin with a velocity of $2ms^{-1}$,

(i) show that $v = 2(x+1)$ (2)

(ii) show that $x = e^{2t} - 1$ (2)

(iii) find its acceleration after 1 second (2)

(b) Express the solution to the equation
$\sin 2\theta = \sin\theta$ in general form, θ in radians (2)

(c) Find

(i) $\int \frac{dx}{\sqrt{9-4x^2}}$ (2)

(ii) $\int \sin^2 x\,dx$ (2)

QUESTION 4. Start a new page (12 marks)

(a) Show that

$$\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$$ (2)

(b) Kool has decided to invest in a superannuation fund. She calculates that she will need \$1 000 000 if she is to retire in 20 years time and maintain her present lifestyle. The superannuation fund pays 12% per annum interest on her investments.

(i) Kool invests \$P at the beginning of each year. Show that at the end of the first year her investment is worth \$P(1.12) (1)

(ii) Show that at the end of the third year the value of her investment is given by the expression \$P(1.12)(1.12^2 + 1.12 + 1) (2)

(iii) Find a similar expression for the value of her investment after 20 years and hence calculate the value of P needed to realise the total of \$1 000 000 required for his retirement. (3)

(c) The daily growth of the population of a colony of insects is 10% of the excess of the population over 1.2×10^6. At $t=0$ the population is 2.7×10^3 (Given $P = N + Ae^{0.1t}$)

(i) Determine the population after $3\frac{1}{2}$ days. (2)

(ii) If a scientist checks the population each day, which is the first day on which she should notice the original population has tripled? (2)

QUESTION 5. Start a new page (12 marks)

(a) A sphere is being heated so that its surface area is increasing at a constant rate of 15mm^2 per second. Find the rate of increase of the volume when the radius is 5mm. (3)

(b) Find the value of the constant m if e^{mx} satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0 \qquad (3)$$

(c) A javelin is thrown across level ground from a height of 2m at a speed of 20m/s at an angle of 60° to the horizontal. Taking acceleration due to gravity as 10m/s^2, find

(i) the height reached (2)
(ii) the time the javelin is in the air (2)
(iii) the length of the throw (2)

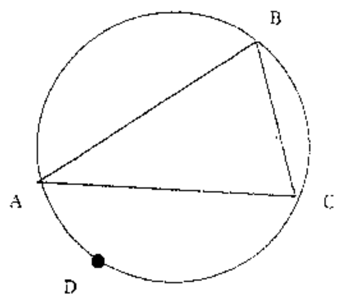
QUESTION 6. Start a new page (12 marks)

(a) A particle moves along a straight line with a velocity given by $\frac{1}{2}v^2 = 18 - 2x^2$, where x is the distance from a fixed point O on the line.

(i) show that the motion is simple harmonic (1)

(ii) find the period and amplitude of the motion of the motion (2)

(b)



ABCD are four points on a circle centre O and radius R units, such that BD is a diameter. A, B, C are joined to form a triangle in which AB=c units, BC=a units and AC=b units. Show, giving reasons, that

(i) $\sin\angle BAC = \dfrac{a}{2R}$ (3)

(ii) Area $\Delta ABC = \dfrac{abc}{4R}$ (3)

(c) (i) Express $\sin x + \sqrt{3}\cos x$ in the form $A\sin(x+\alpha)$ (2)

(ii) Use this to solve $\sin x + \sqrt{3}\cos x = \sqrt{3}$ for $0 \le x \le 2\pi$ (2)

QUESTION 7. Start a new page (12 marks)

(a) Prove that for all positive integers n, $9^{n+2} - 4^n$ is divisible by 5. (4)

(b) Evaluate

$$\int_0^{\frac{1}{2}} \frac{dx}{1+4x^2}$$ (3)

(c) The line $y = 2x + 2$ cuts the line segment AB at some point C. If A is the point (-2,3) and B is the point (4,3) find the ratio of AC:CB. (2)

(d) If $y = \frac{1}{2}(e^x - e^{-x})$, show that $x = \log_e(y + \sqrt{y^2+1})$ (3)

END OF PAPER

Question ①

(a) $\displaystyle\int_0^1 \frac{x}{x^2+2}\,dx$

$u = x^2+2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

when $x=0,\ u=2$

$x=1,\ u=3$

$= \frac{1}{2}\displaystyle\int_2^3 \frac{1}{u}\,du$

$= \frac{1}{2}\left[\ln u\right]_2^3$

$= \frac{1}{2}\left(\ln 3 - \ln 2\right)$

$= \frac{1}{2}\ln\left(\frac{3}{2}\right)$

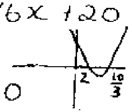
(b) $4(x-2) > 3(x-2)^2$

$4x-8 > 3x^2-12x+12$

$0 > 3x^2-16x+20$

$3x^2-16x+20 < 0$

$(3x-10)(x-2) < 0$



$2 < x < 3\frac{1}{3}$

(c) $\tan\left(2\tan^{-1}\frac{3}{4}\right)$

let $\theta = \tan^{-1}\frac{3}{4}$.

$\therefore \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$= \dfrac{2\times\frac{3}{4}}{1-\frac{9}{16}}$

$= \dfrac{24}{7}$.

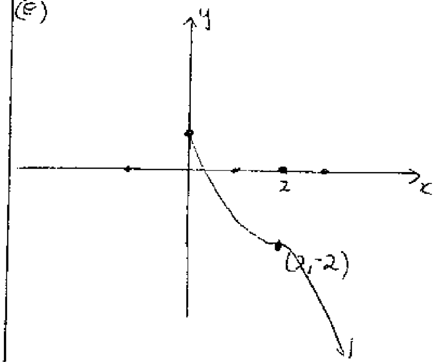
(d) 5R, 4B, 3W

P(R, B, W) or (B R W) or (R, W, B)

$= \left(\frac{5}{12}\times\frac{4}{11}\times\frac{3}{10}\right)\times 6$

$= \frac{3}{11}$

(e)



Question ②

(a) $f(x) = 4\sin^{-1}\left(\frac{x}{3}\right)$

domain $-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$

range $-2\pi \le y \le 2\pi$

(b)(i) $F(x) = x^3+x-3$

$F(1\cdot2) = -0\cdot072 < 0$

$F(1\cdot3) = 0\cdot497 > 0$

$F(1\cdot2)<0$ $F(1\cdot3)>0$ $\therefore$ root betw

(ii) $F(x_1) = 1\cdot2 - \dfrac{F(1\cdot2)}{F'(1\cdot2)}$

$= 1\cdot2 - \dfrac{(-0\cdot072)}{3(1\cdot2)^2+1}$

$\doteqdot 1\cdot213533835$

(c) $P(x) = k(x+2)(x+1)(x-1) = 0$

Now $P(2) = 36$

$\therefore 36 = k(4)(3)(1)$.

$\therefore k = 3$

$\therefore P(x) = 3(x+2)(x+1)(x-1)$

$= (3x+6)(x^2-1)$

$= 3x^3-3x+6x^2-6$

$P(x) = 3x^3+6x^2-3x-6$

(d) At $P(2ap, ap^2)$ $m = p$

(i) $\therefore$ $y-ap^2 = p(x-2ap)$

$y-ap^2 = px-2ap^2$

$y = px-ap^2$

K is where $y = -a$.

$-a = px-ap^2$.

$px = ap^2-a$.

$x = \dfrac{ap^2-a}{p}$.

$\therefore$ K is $\left(\dfrac{ap^2-a}{p}, -a\right)$

(ii) $m_{PS} = \dfrac{ap^2-a}{2ap} = \dfrac{p^2-1}{2p}$

$m_{SK} = \dfrac{a+a}{\frac{-ap^2+a}{p}}$ $\quad \dfrac{-2ap}{ap^2-a}$

$= \dfrac{2ap}{a-ap^2}$

$= \dfrac{2p}{1-p^2}$

Since $m_{PS}\times m_{SK} = -1$

$\angle PSK$ is $90°$

Question ③

(a) $\ddot{x} = 4(1+x)$

$\dfrac{d(\frac{1}{2}v^2)}{dx} = 4(1+x)$

(i) $\frac{1}{2}v^2 = 4x+2x^2+c$

when $v=2,\ x=0$

$2 = c$

$\therefore \frac{1}{2}v^2 = 4x+2x^2+2$

$v^2 = 8x+4x^2+4$

$v = \sqrt{4x^2+8x+4}$

$v = 2\sqrt{x^2+2x+1}$

$v = 2\sqrt{(x+1)^2}$

$v = 2(x+1)$.

(ii) $x = e^{2t}-1$

Now $\dfrac{dx}{dt} = 2(x+1)$

$\dfrac{dt}{dx} = \dfrac{1}{2(x+1)}$

$t = \dfrac{1}{2}\displaystyle\int \frac{1}{x+1}\,dx$

$t = \frac{1}{2}\ln(x+1)+k$

when $t=0,\ x=0,\ \therefore k=0$

$2t = \ln(x+1)$

$e^{2t} = x+1$

$\therefore x = e^{2t}-1$

(iii) $\dot{x} = 2e^{2t}$

$\ddot{x} = 4e^{2t}$

when $t=1$

$\ddot{x} = 4e^2\ m/s^2$

b) $\sin 2\theta = \sin\theta$

$2\sin\theta\cos\theta = \sin\theta$

$2\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta(2\cos\theta - 1) = 0$

$\sin\theta = 0, \quad \cos\theta = \frac{1}{2}$

$\therefore \theta = n\pi + (-1)^n \sin^{-1} 0, \quad 2\pi n \pm \cos^{-1}\frac{1}{2}$

$= n\pi + (-1)^n . 0, \quad 2\pi n \pm \frac{\pi}{3}$

$\theta = n\pi, \; 2\pi n \pm \frac{\pi}{3}$

c)(i) $\int \frac{dx}{\sqrt{9-4x^2}}$

$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$

(ii) $\int \sin^2 x\, dx$

$= \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$

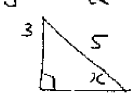
$= \frac{x}{2} - \frac{1}{4}\sin 2x + c$

Question ④

a) $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$

let $x = \cos^{-1}\frac{4}{5}$

$y = \cos^{-1}\frac{3}{5}$



$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$= \frac{4}{5} \times \frac{3}{5} - \frac{3}{5} \times \frac{4}{5}$

$= 0$

$\therefore x + y = \cos^{-1}(0)$

$x + y = \frac{\pi}{2}$

$\therefore \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{3}{5} = \frac{\pi}{2}$

(b)(i) $P(1+r)^n = P(1\cdot 12)^1$

(ii) $B_2 = P(1\cdot12)^2 + P(1\cdot12)$

$B_3 = P(1\cdot12)^3 + P(1\cdot12)^2 + P(1\cdot12)$

$= P(1\cdot12)(1\cdot12^2 + 1\cdot12 + 1)$

(iii)

$B_{20} = P(1\cdot12)(1\cdot12^{19} + 1\cdot12^{18} + \dots + 1)$

$1000\,000 = P(1\cdot12)(1\cdot12^{19} + \dots +$

$a=1, n=20, r=1$

$1000000 = P(1\cdot12)\left(\frac{1\cdot12^{20}-1}{0\cdot12}\right)$

$120000 = P(1\cdot12)(1\cdot12^{20}-1)$

$P = \frac{120\,000}{(1\cdot12)(1\cdot12^{20}-1)}$

$= \$12391\cdot77$

(c)(i) $P = N + Ae^{0\cdot1t}$

at $t=0$, $P = 2\cdot7\times10^6$

$2\cdot7\times10^6 = 1\cdot2\times10^6 + A$

$A = 1\cdot5\times10^6$

$\therefore P = 1\cdot2\times10^6 + 1\cdot5\times10^6 e^{0\cdot1t}$

when $t = 3\cdot5$

$P = 1\cdot2\times10^6 + 1\cdot5\times10^6 e^{0\cdot35}$

$= 3328601\cdot323$

$\doteqdot 3\cdot3\times10^6$

when $P = 8\cdot1\times10^6$

$8\cdot1\times10^6 = 1\cdot2\times10^6 + 1\cdot5\times10^6 e^{0\cdot1t}$

$6\cdot9\times10^6 = 1\cdot5\times10^6 e^{0\cdot1t}$

$4\cdot6 = e^{0\cdot1t}$

$\ln(4\cdot6) = 0\cdot1t$

$t = 15\cdot26$

$\therefore$ On 16th day

Question ⑤

(a) $\frac{dA}{dt} = 15$

$\frac{dV}{dt} = \frac{dV}{dr} \times \boxed{\frac{dr}{dt}}$ $\quad V = \frac{4}{3}\pi r^3$

$= 4\pi r^2 \times \boxed{\frac{dr}{dt}}$ $\quad \frac{dV}{dr} = 4\pi r^2$

Now $\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$

$= \boxed{\frac{1}{8\pi r} \times 15}$ $\quad A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$\therefore \frac{dv}{dt} = 4\pi r^2 \times \frac{15}{{}_2\cancel{8}\cancel{\pi r}}$

$= \frac{15r}{2}$

When $r = 5$

$\frac{dv}{dt} = 37\frac{1}{2}\ mm^3/s$

(b) $y = e^{mx}$

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$

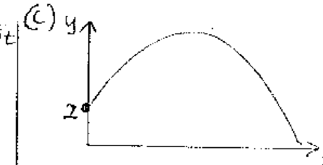
$m^2e^{mx} - me^{mx} - 6e^{mx} = 0$

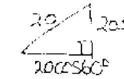
$e^{mx}(m^2 - m - 6) = 0$

$e^{mx}(m-3)(m+2) = 0$

$m = -2, 3$

(c)





$\ddot{x} = 0 \qquad \ddot{y} = -10$

$\dot{x} = 10 \qquad \dot{y} = -10t + 10\sqrt{3}$

$x = 10t \qquad y = -5t^2 + 10\sqrt{3}t + 2$

(i) height reached $\rightarrow \dot{y} = 0$

$-10t + 10\sqrt{3} = 0$

$t = \sqrt{3}\ s$

$\therefore y = -5(3) + 10(3) + 2$

$= -15 + 30 + 2$

$y = 17\ m$

(ii) time of flight

$y = 0$

$-5t^2 + 10\sqrt{3}t + 2 = 0$

$5t^2 - 10\sqrt{3}t - 2 = 0$

$t = \frac{10\sqrt{3}\ \sqrt{300+40}}{10}$

$t = \sqrt{3}, \sqrt{340}$

$t = 31\cdot937s$

(iii) when $t = \sqrt{1020}$

$x = 10\sqrt{1020}\ m \quad (319\cdot37$

Question ⑥

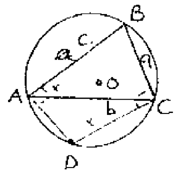
(a) $\frac{1}{2}v^2 = 18 - 2x^2$

(i) $\frac{d\frac{1}{2}v^2}{dx} = -4x \quad$ S.H.M.

(ii) $T = \frac{2\pi}{2} = \pi$

When $v = 0 \qquad 18 - 2x^2 = 0$

$\therefore a = 3 \qquad x = \pm 3$



$\angle BCD = 90°$ ($\angle$ in semi-circle)
$\angle BDC = \angle BAC$ ($\angle$'s standing on same arc)
$\therefore \sin\angle BDC = \frac{a}{BD}$
$\sin\angle BDC = \frac{a}{2R}$
$\therefore \sin\angle BAC = \frac{a}{2R}$

(ii) Area $\triangle ABC = \frac{1}{2} \cdot c \cdot b \cdot \sin\angle BAC$
$= \frac{1}{2} \cdot c \cdot b \cdot \frac{a}{2R}$
$= \frac{abc}{4R}$

(c) (i) $\sin x + \sqrt{3}\cos x \equiv A\sin(x+\alpha)$
$A = \sqrt{1+3} = 2$.
$\alpha = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$.
$\therefore \sin x + \sqrt{3}\cos x = 2\sin(x + \frac{\pi}{3})$

(ii) $\sin x + \sqrt{3}\cos x = \cancel{5}\sqrt{3}$
$\therefore 2\sin(x+\frac{\pi}{3}) = \cancel{5}\sqrt{3}$

$\sin(x + \frac{\pi}{3}) = \sqrt{3}$

$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}$
$x = 0, \pi/3, 2\pi$.

Question 7

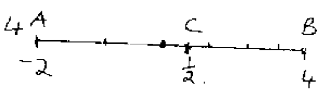
(a) Induction.
Step ① Assume true $n=k$
$9^{k+2} - 4^{k} = 5M$ M +ve int
Step ② Prove true for $n=k+1$
R.T.P.
$9^{k+3} - 4^{k+1}$ is divisible by 5
Step ③ Proof
$9^{k+3} - 4^{k+1} = 9.9^{k+2} - 4^{k+1}$
$= 9(5M + 4^{k}) - 4^{k+1}$
$= 45M + 9.4^{k} - 4.4^{k}$
$= 45M + 5.4^{k}$
$= 5(9M + 4^{k})$
which is divisible by 5
Step ④
Hence statement is true for $n=k+1$ when it is true for $n=k$.
Step ⑤
For $n=1$
$9^2 - 4 = 80$
$\therefore$ True for $n=1$
Step ⑥
Since true for $n=1$ by step ④ it will be true for $n=2$ and then $n=3$ and so on for all integers.

(b) $\int_0^{\frac{1}{2}} \frac{dx}{1+4x^2} = \int_0^{\frac{1}{2}} \frac{dx}{4(\frac{1}{4}+x^2)}$
$= \frac{1}{2}\left[\tan^{-1}2x\right]_0^{\frac{1}{2}}$
$= \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$
$= \frac{1}{2}\left[\frac{\pi}{4}\right]$
$= \frac{\pi}{8}$

(c) $A(-2,3)$. $B(4,3)$
$m_{AB} = 0$.

Equation $y=3$.
sub into $y = 2x+2$
$3 = 2x+2$
$2x = 1$
$x = \frac{1}{2}$.
$\therefore$ C is $(\frac{1}{2}, 3) \rightarrow (m,n)$

$\therefore$ 4



$\therefore AC : CB$
$2\frac{1}{2} : 3\frac{1}{2}$
$5 : 7$

(d) $y = \frac{1}{2}(e^{x} - e^{-x})$
$2y = e^{x} - e^{-x}$
$0 = e^{2x} - 1 - 2y.e^{x}$
$0 = e^{2x} - 2ye^{x} - 1$
$e^{x} = \frac{2y \pm \sqrt{4y^2+4}}{2}$
$e^{x} = \frac{\cancel{2}y \pm \cancel{2}\sqrt{y^2+1}}{\cancel{2}}$
$e^{x} = y \pm \sqrt{y^2+1}$
$e^{x} > 0 \therefore \sqrt{y^2+1} > 0$
$e^{x} = y + \sqrt{y^2+1}$
$x = \ln(y + \sqrt{y^2+1})$