Student Number: ....

ROSEVILLE COLLEGE

MATHEMATICS EXTENSION 1

TRIAL EXAMINATION 2004

Time allowed: 2 hours + 5 minutes reading time

Directions

- **Attempt ALL questions.**
- **Show all necessary working, marks may be deducted for careless or untidy work.**
- **Board-approved calculators may be used.**
- **Additional Answer Pages are available.**
- **A table of standard integrals is provided**

QUESTION 1 *(Start a new page)* **MARKS**

(a) Evaluate $\lim_{x \to 0} \frac{\sin 4x}{5x}$. **1**

(b) Evaluate $\int_0^{\sqrt{2}} \frac{dx}{x^2 + 2}$. **3**

(c) If $P(x) = x^4 - 3x^3 + ax^2 - 12$ is divisible by $(x - 3)$, find the value of a. **2**

(d) Use the table of standard integrals to evaluate $\int_3^5 \frac{dx}{\sqrt{x^2 - 4}}$. **2**

(e) Show that $\tan^{-1} x = \cos^{-1} \frac{1}{\sqrt{1 + x^2}}$ **2**

(f) Seven people want to sit around a circular table. Two of them want to sit together. In how many ways can the table be arranged? **2**

QUESTION 2 *(Start a new page)* **MARKS**

(a) How many 7 letter words can be made from the word ADDIDAS? **2**

(b) Find the coefficient of x^{12} in the expansion of $\left(\frac{1}{x^2}+x^3\right)^{14}$. **3**

(c) Find $\frac{d}{dx}\left(\frac{\tan^2 x}{x}\right)$. **2**

(d) (i) Express $\cos\theta - \sqrt{3}\sin\theta$ in the form $A\cos(\theta + \alpha)$ where A and α are constants, $A > 0$ and α is in radians. **2**

(ii) Hence solve the equation $\cos\theta - \sqrt{3}\sin\theta = \sqrt{3}$ for $-\pi \le \theta \le \pi$. **3**

QUESTION 3 *(Start a new page)* **MARKS**

(a) A circle is expanding so that the rate of increase in its radius is 0.75 cms^{-1}.

Find the rate of increase in its area when

(i) the radius is 10cm **2**

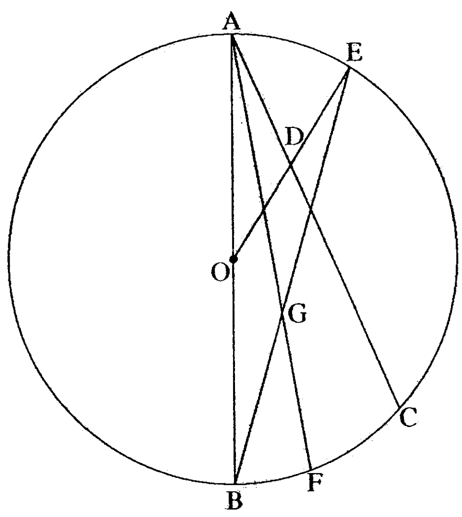
(ii) the circumference is 10cm. **2**

(b) Solve $3^{2x} = 5$. **2**

(c) The equation $2x^3 + 2x^2 + 4x + 1 = 0$ has roots α, β, γ. Find the value of

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$$ **3**

(d) In the figure, *AOB* is the diameter of a circle centre *O*. *D* is a point on chord *AC* such that *DA* = *DO* and *OD* is produced to *E*. *AF* is the bisector of $\angle BAC$ and cuts *BE* in *G*. Prove that *GA* = *GB* **3**



QUESTION 4 *(Start a new page)* **MARKS**

(a) Use mathematical induction to prove that, for all positive integers n, **4**

$$\sum_{r=1}^{n} \frac{r^2}{(2r-1)(2r+1)} = \frac{n(n+1)}{2(2n+1)}$$

(b) (i) Prove the identity **2**

$$\frac{\sin 2\theta}{2\sin\theta} - \cos\theta\cos 2\theta = 2\cos\theta\sin^2\theta$$

(ii) Hence solve the equation **3**

$$\frac{\sin 2\theta}{2\sin\theta} - \cos\theta\cos 2\theta = \cos\theta \quad \text{for } 0 \le \theta \le 2\pi.$$

(c) Consider the function $f(x) = \cos^{-1}(x-1)$

(i) Find the domain of the function. **1**

(ii) Sketch the graph of the curve $y = f(x)$ showing clearly the coordinates of the endpoints **2**

QUESTION 5 *(Start a new page)* **MARKS**

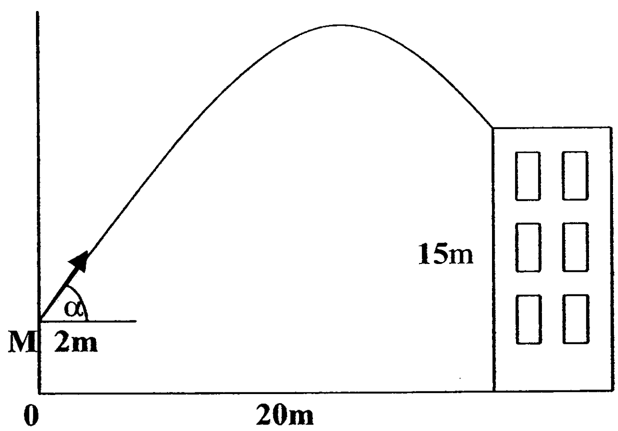
(a) Corn cobs are cooked by immersing them in boiling water. On being removed, a corn cob cools in the air according to the equation
$\frac{dT}{dt} = -k(T - T_O)$ where t is time in minutes, T is temperature in °C and T_O is the temperature of the air, while k is a positive constant.

(i) Verify that $T = T_O + Ae^{-kt}$ is a solution of the above equation where A is a constant. **2**

(ii) If the temperature of the boiling water is 100°C and that of the air is a constant 25°C, find the values of A and k if a corn cob cools to 70°C in 3 minutes. **2**

(iii) How long should a person wait to enjoy the food at a temperature of 50°C? **2**

(b) A girl of height 2 metres throws a ball from M to the roof of a 15 metre high building. She throws the ball at an initial velocity of 25m/s,and she is 20m from the base of the building.



(i) Derive the equations of motion of the ball **3**

(Assume $\ddot{x} = 0$ and $\ddot{y} = -10$)

(ii) Between which two angles of projection must she throw the ball to ensure that it lands on the roof of the building? **3**

QUESTION 6 *(Start a new page)* **MARKS**

(a) Let $f(x)=\tan^{-1}\left(\dfrac{2}{x}\right)-\tan^{-1}(-2x)$ where $x>0$.

Find $f'(x)$. **3**

(b) A particle is moving in a straight line. At time t seconds, it has displacement x metres from a fixed point O on the line, velocity v ms^{-1} and acceleration a ms^{-2}. The particle starts from O and you are given that $v=(2-x)^2$.

(i) Find an expression for a in terms of x **2**

(ii) Find an expression for x in terms of t **3**

(iii) Find the distance from O when the particle has a speed of 1ms^{-1}. **2**

(c) At a football club a team of 13 players is to be chosen from a pool of 32 players consisting of 20 Australian-born players and 12 players born elsewhere. What is the probability that the team will consist of all Australian-born players? **2**

QUESTION 7 *(Start a new page)* **MARKS**

(a) (i) Express cos 2A in terms of $\sin^2 A$ **1**

(ii) Hence find the exact value of $\int_0^{\pi} \sin^2 \frac{x}{4}\, dx$ **2**

(b) A particle moves along a straight line such that its displacement x m from an origin 0 at time t seconds is given by

$$x = 4\sin \frac{\pi}{2} t$$

(i) Show that this is Simple Harmonic Motion **2**
(ii) State the amplitude and period of this motion **1**
(iii) Calculate the maximum speed attained by this particle **1**

(c) Suppose that $(5 + 2x)^{12} = \sum_{k=1}^{12} a_k x^k$

(i) Use the binomial theorem to write an expression for a_k **2**

(iii) Show that $\frac{a_{k+1}}{a_k} = \frac{24-2k}{5k+5}$ **3**

End of paper

Question (1)

(a) $\lim_{x \to 0} \frac{\sin 4x}{5x} = \frac{4x}{5x}$

$= \frac{4}{5}$ → ①

(b) $\int_0^{\sqrt{2}} \frac{dx}{x^2+2} = \left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}}\right]_0^{\sqrt{2}}$ → ① → ①

$= \frac{1}{\sqrt{2}}\left[\tan^{-1} 1 - \tan^{-1} 0\right]$

$= \frac{1}{\sqrt{2}} \cdot \frac{\pi}{4}$

$= \frac{\pi}{4\sqrt{2}}$ ①

(c) $P(x) = x^4 - 3x^3 + ax^2 - 12$

$P(3) = 0$

$0 = 81 - 81 + 9a - 12$ → ①

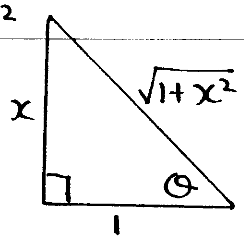
$\therefore a = \frac{4}{3}$ → ①

(d) $\int_3^5 \frac{dx}{\sqrt{x^2-4}} = \left\{\ln(x + \sqrt{x^2-4})\right\}_3^5$ → ①

$= \ln(5+\sqrt{21}) - \ln(3+\sqrt{5})$

$= \ln \frac{5+\sqrt{21}}{3+\sqrt{5}}$ → ①

(e) $\tan^{-1} x = \cos^{-1} \frac{1}{\sqrt{1+x^2}}$



let $\theta = \tan^{-1} x$

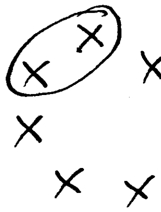
$\therefore \tan\theta = x$

$\therefore \cos\theta = \frac{1}{\sqrt{1+x^2}}$ ①

$\therefore \theta = \cos^{-1} \frac{1}{\sqrt{1+x^2}}$

$\therefore \tan^{-1} x = \cos^{-1} \frac{1}{\sqrt{1+x^2}}$ ①

(f)



As if 6 in a circle

$\therefore$ No ways to sit together

$= 5! \times 2$ → ① → ①

$= 240$

Question (2)

(a) $\frac{7!}{3! \times 2!}$ → ① → ①

$= 420$

(b) $\left(\frac{1}{x^2} + x^3\right)^{14}$

want k such that

$\left(\frac{1}{x^2}\right)^{14-k} \left(x^3\right)^k = x^{12}$ → ① or similar

i.e. $3k - 2(14-k) = 12$

$3k - 28 + 2k = 12$

$5k = 40$

$k = 8$ → ①

$\therefore$ Coefficient is ~~$^{12}C_8 = 495$~~ ?

$^{14}C_8 = 3003$ ①

(c) $\frac{d}{dx} \frac{\tan^2 x}{x} =$ ① derivative $\tan^2 x$. ① Quotient rule.

$= \frac{2 \tan x . \sec^2 x . x - \tan^2 x}{x^2}$

(d) $\cos\theta - \sqrt{3}\sin\theta = A\cos(\theta + \alpha)$

LHS $= 2\cos(\theta + \alpha)$

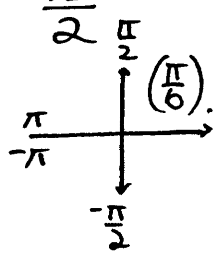
Now $\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$ → ①.

$\therefore$ LHS $= 2\cos\left(\theta + \frac{\pi}{3}\right)$ → ①.

(ii) $2\cos\left(\theta + \frac{\pi}{3}\right) = \sqrt{3}$

$\cos\left(\theta + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$



① ①

$\theta + \frac{\pi}{3} = -\frac{\pi}{6}, \frac{\pi}{6},$

$\theta = -\frac{\pi}{2}, -\frac{\pi}{6}$

①.

Question (3)

(a) $\frac{dr}{dt} = 0.75$

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

(i) $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 2\pi r \times 0.75$ → ①

for $r=10$ $= 2 \times \pi \times 10 \times 0.75$

$= 15\pi \, cm^2/s$ } ①

(ii) $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$r^2 = \frac{A}{\pi}$

$r = \frac{A}{\sqrt{\pi}}^{1/2}$

$\frac{dr}{dA} = \frac{1}{2\sqrt{A\pi}}$

① → $= 2\sqrt{A\pi} \times 0.75$

when $A = 10$

$\frac{dA}{dt} = 2\sqrt{10} \times 0.75$

$= 1.5\sqrt{10}\sqrt{A} \, cm^2/s$ } ①

(b) $3^{2x} = 5$

$\ln 3^{2x} = \ln 5$

$2x \ln 3 = \ln 5$ } ①

$2x = \frac{\ln 5}{\ln 3}$

$x = \frac{1}{2} \frac{\ln 5}{\ln 3}$

$x = 0.73248676$ } ①

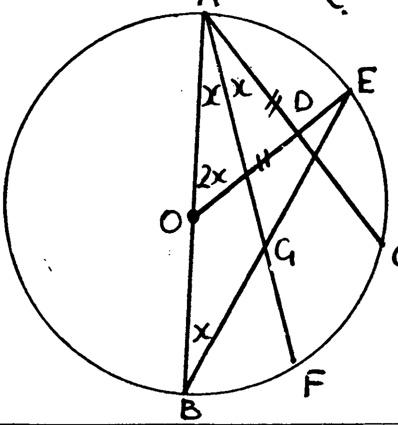
(c) $2x^3 + 2x^2 + 4x + 1 = 0$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

② → $= (2) \div -\frac{1}{2}$ ①

$= -4$

(d)



let $\angle OAF = x = \angle GAD$ (given)

$\therefore \triangle ADO$ is isosceles ($AD = DO$)

$\therefore \angle ADO = \angle OAD = 2x$ (equal $\angle$'s of isos $\triangle$)

$\therefore \angle ABE = x$ (angle at centre is double $\angle$ at circ)

In $\triangle AGB$

$\therefore \angle GAB = \angle ABG$ (both x)

$\therefore \triangle AGB$ is isosceles

$\therefore GA = GB$ (equal sides of isos. $\triangle$)

Question (4)

$\sum_{r=1}^{n} \frac{r^2}{(2r-1)(2r+1)} = \frac{n(n+1)}{2(2n+1)}$

$\frac{1}{3} + \frac{4}{15} + \dots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$

Step ① Prove for $n = 1$

when $n = 1$

$\frac{1}{3} = \frac{1(2)}{2(3)}$

$\therefore LHS = RHS$

True for $n = 1$

Step (2) Assume true for $n = k$

$\therefore \frac{1}{3} + \dots + \frac{k^2}{(2k-1)(2k+1)} = \frac{k(k+1)}{2(2k+1)}$ ①

Step ③ Prove for $n = k+1$

To Prove

$\frac{1}{3} + \dots \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$

$= \frac{(k+1)(k+2)}{2(2k+3)}$ $\frac{1}{2}$

$LHS = \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$ $\frac{1}{2}$

$= \frac{k(k+1)(2k+3) + (k+1)^2 \cdot 2}{2(2k+1)(2k+3)}$

$= \frac{(k+1)(k(2k+3) + 2(k+1))}{2(2k+1)(2k+3)}$

$= \frac{(k+1)(2k^2+3k+2k+2)}{2(2k+1)(2k+3)}$ ($\frac{1}{2}$)

$= \frac{(k+1)(2k+5k+2)}{2(2k+1)(2k+3)}$

$= \frac{(k+1)(2k+1)(k+2)}{2(2k+1)(2k+3)}$

$= \frac{(k+1)(k+2)}{2(2k+3)}$ (1)

$=$ RHS

True for $n=k+1$ when true for $n=k$

Step (4)

Since true for $n=1$ from step (3) its true for $n=2$ and hence $n=3$ and so on for all n. ($\frac{1}{2}$)

(b) (i) $\frac{\sin 2\theta}{2\sin\theta} - \cos\theta\cos 2\theta = 2\cos\theta\sin^2\theta$

(1)

LHS $= \frac{2\sin\theta\cos\theta}{2\sin\theta} - \cos\theta(\cos^2\theta - \sin^2\theta)$

$= \cos\theta - \cos\theta(1-2\sin^2\theta)$

$= \cos\theta(1-1+2\sin^2\theta)$ (1)

$= 2\cos\theta\sin^2\theta$

$=$ RHS.

(ii) $\therefore 2\cos\theta\sin^2\theta = \cos\theta$

$2\cos\theta\sin^2\theta - \cos\theta = 0$

$\cos\theta(2\sin^2\theta - 1) = 0$ — (1)

$\therefore \cos\theta = 0 \qquad \sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

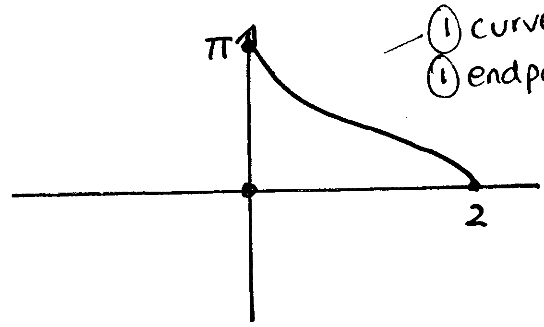
(2)

(c) $f(x) = \cos^{-1}(x-1)$

(i) domain $-1 \le x-1 \le 1$

$0 \le x \le 2$ (1)

(ii)



(1) curve

(1) endpoints.

Question (5)

(a) $\frac{dT}{dt} = -k(T-T_0)$

(i) $T = T_0 + Ae^{-kt}$

$\frac{dT}{dt} = -k.\, Ae^{-kt}$ — (1)

$= -k[T-T_0]$ — (1)

$\therefore$ satisfies equation.

(ii) when $t=0$ $T=100$, $T_0=25$

$\therefore 100 = 25 + A$ (1)

$A = 75$

$\therefore T = 25 + 75e^{-kt}$

when $t=3$, $T=70$

$70 = 25 + 75e^{-3k}$

$45 = 75e^{-3k}$

$-3k = \ln\left(\frac{45}{75}\right)$ (1)

$k = 0.170275207$

(iii) when $T = 50$

$50 = 25 + 75e^{-kt}$

$25 = 75e^{-kt}$ (1)

$-kt = \ln\left(\frac{25}{75}\right)$

$t = 6.45$ mins

7 mins (1).

(b)(i) $\ddot{x} = 0$

$\dot{x} = 25\cos\alpha$

$x = 25t\cos\alpha$ } (1)

$\ddot{y} = -10 \quad (t=0 \ \dot{y} = v\sin\alpha)$

$\dot{y} = -10t + 25\sin\alpha$ — (1)

$y = -5t^2 + 25t\sin\alpha + c \quad \{t=0, y=2\}$

$y = -5t^2 + 25t\sin\alpha + 2$ — (1)

(ii) $t = \dfrac{x}{25\cos\alpha}$ sub into y

$y = \dfrac{-5 \cdot x^2}{25^2\cos^2\alpha} + \dfrac{x\sin\alpha \cdot 25}{25\cos\alpha} + 2$

$y = \dfrac{-x^2}{125}(\sec^2\alpha) + x\tan\alpha + 2$

$y = \dfrac{x^2}{125}(1+\tan^2\alpha) + x\tan\alpha + 2$ (1)

sub in point (20, 15)

$15 = \dfrac{-400}{125}(1+\tan^2\alpha) + 20\tan\alpha + 2$ (1)

$1875 = -400(1+\tan^2\alpha) + 2500\tan\alpha + 250$

$1625 = -400 - 400\tan^2\alpha + 2500\tan\alpha$

$0 = -400\tan^2\alpha + 2500\tan\alpha - 2025$

$0 = 16\tan^2\alpha - 100\tan\alpha + 81$

$\tan\alpha = \dfrac{+100 \pm \sqrt{100^2 - 4\times81\times16}}{32}$

$= \dfrac{+100 \pm \sqrt{4816}}{32}$ (1)

$\alpha \doteqdot 79°$ and $44°$

$44° \leq \alpha \leq 79°$

Question (6)

(a) $f(x) = \tan^{-1}\left(\dfrac{2}{x}\right) - \tan^{-1}(-2x)$

$f(x) = \tan^{-1}\left(\dfrac{2}{x}\right) + \tan^{-1}(2x)$

$f'(x) = \dfrac{-2x^{-2}}{1 + \dfrac{4}{x^2}} + \dfrac{2}{1+(2x)^2}$ (1) (1) (1) — (1 for only 1 correct)

$= \dfrac{-2}{x^2\left(\dfrac{x^2+4}{x^2}\right)} + \dfrac{2}{1+4x^2}$

$= \dfrac{-2}{x^2+4} + \dfrac{2}{1+4x^2}$

(b)

(i) $v = (2-x)^2$

$v^2 = (2-x)^4$

$\frac{1}{2}v^2 = \frac{1}{2}(2-x)^4$ — (1)

$\dfrac{d(\frac{1}{2}v^2)}{dx} = 2(2-x)^3 \cdot -1$

$a = -2(2-x)^3$ — (1)

(ii) $\dfrac{dx}{dt} = (2-x)^2$

$\dfrac{dt}{dx} = \dfrac{1}{(2-x)^2} = (2-x)^{-2}$

$t = \dfrac{(2-x)^{-1}}{-1 \times -1} + c$ — (1)

when $t=0$, $x=0$

$0 = 2^{-1} + c$

$c = -\frac{1}{2}$ — (1)

$t = \dfrac{1}{2-x} - \dfrac{1}{2}$

$x = \dfrac{4t+1}{2t+1}$ — (1) make x subject

(iii) when $|v| = 1$

$1 = (2-x)^2$

$2 - x = \pm 1$ — (1)

$x = 1$ m or 3m — (1).

c) 32 Players

20 Australian | 12 Others

$P(\text{all Aust}) = \dfrac{{}^{20}C_{13}}{{}^{32}C_{13}}$ — (1) — (1).

$= 0.00022316$

Question (7).

(a)(i) $\cos 2A = 1 - 2\sin^2 A$ (1)

(ii) $\int_0^{\pi} \sin^2 \frac{x}{4}\, dx$

$= \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos\frac{x}{2}\, dx$

$= \left[\frac{x}{2} - 1.\sin\frac{x}{2}\right]_0^{\pi}$ (1)

$= \frac{\pi}{2} - 1 - 0$

$= \frac{\pi}{2} - 1$ (1)

(b) $x = 4\sin\frac{\pi}{2}t$

(i) $x = 4\sin\frac{\pi}{2}t$

$\dot{x} = 2\pi\cos\frac{\pi}{2}t$

$\ddot{x} = -\pi^2\sin\frac{\pi}{2}t$ (1)

$\ddot{x} = -\frac{\pi^2}{4}.x$

in form $\ddot{x} = -n^2 x$

which is SHM. $n = \frac{\pi}{2}$ (1)

(ii) Amplitude $= 4$

Period $= \dfrac{2\pi}{\left(\frac{\pi}{2}\right)}$ ($\frac{1}{2}$ each)

$= 4$ (1)

(iii) max speed. when $\cos\frac{\pi}{2}t = 1$

$\dot{x} = 2\pi\cos\frac{\pi}{2}t$

$\dot{x}_{MAX} = 2\pi$ m/s. (1)

(c) $(5+2x)^{12} = \sum_{k=1}^{12} a_k x^k$

$a_k = {}^{12}C_k\, 5^{12-k}.\, 2^k$ (1) (1)

(ii) $a_{k+1} = {}^{12}C_{k+1}\, 5^{12-k-1}\, 2^{k+1}$

$\therefore \dfrac{a_{k+1}}{a_k} = \dfrac{{}^{12}C_{k+1}\, 5^{11-k}.\, 2^{k+1}}{{}^{12}C_k\, 5^{12-k}\, 2^k}$ (1)

$= \dfrac{\left[\frac{12!}{(k+1)!(12-k-1)!}\right] 2}{\left[\frac{12!}{k!(12-k)!}\right] 5}$

$= \dfrac{12!\, 2}{(k+1)!(12-k-1)!} \times \dfrac{k!(12-k)!}{12!\, 5}$

$= \dfrac{2(12-k)}{5(k+1)}$. (2)

$= \dfrac{24-2k}{5k+5}$

as required.