(Marks)

Question 1

(a) Evaluate $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ (2)

(b) Expand $(x+3)^4$ (2)

(c) Differentiate $x^2 \cos^{-1} 3x$ (2)

(d) Find the remainder when the polynomial $P(x) = x^3 - 4x + 2$ is divided by $x + 4$ (1)

(e) Sketch the curve $y = x^3 - 4x$ and hence solve $x^3 - 4x \geq 0$ (2)

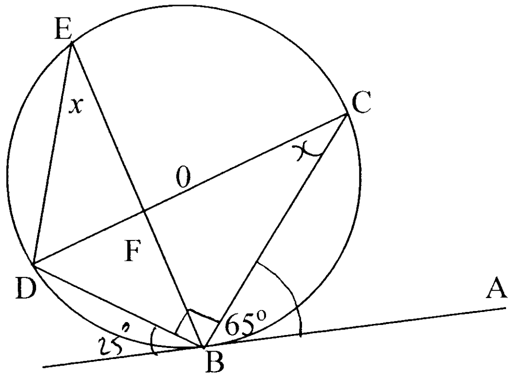
(e) Use the substitution $u = \log_e x$ to find the exact value of $\int_e^{e^2} \frac{1}{x \log_e x} dx$ (3)

(Marks)

Question 2
Start a new page

(a) Sketch $y = 3\sin^{-1} 2x$. Your graph must clearly indicate the domain and range. (2)

(b)



O is the centre of the circle.
AB is a tangent to the circle, touching the circle at B.
$\angle CBA = 65^o$
Find x, giving reasons (2)

(c) The polynomial $P(x) = x^3 - 2x^2 + kx + 24$ has roots α, β, γ.

(i) Find the value of $\alpha + \beta + \gamma$ (1)

(ii) Find the value of $\alpha\beta\gamma$ (1)

(iii) It is known that two of the roots are equal in magnitude but opposite in sign.

Find the third root and hence find the value of k (2)

(d) (i) If $f(x) = e^{x+2}$, find the inverse function $f^{-1}(x)$ (2)

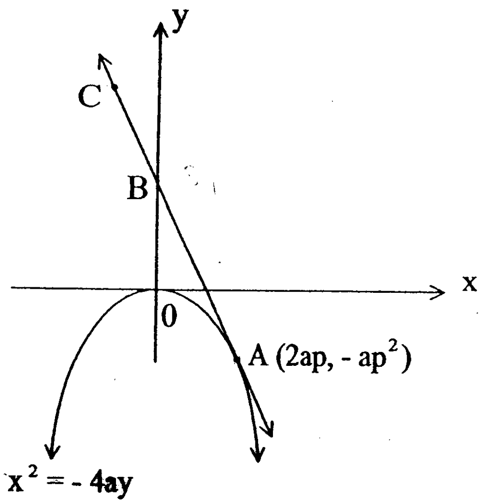
(ii) On the same axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ (2)

(Marks)

Question 3
Start a new page

(a) Find the term independent of x in the binomial expansion of $(2x - \frac{1}{x^2})^6$ (2)

(b) The point A $(2ap, -ap^2)$ is a variable point on the parabola $x^2 = -4ay$. The tangent at A meets the y axis at B. The point C lies on the tangent and divides AB externally in the ratio 3 : 1



(i) Show that the equation of the tangent at A is $px + y = ap^2$ (2)

(ii) Find the coordinates of the points B and C (3)

(iii) Show that the locus of C is a parabola (1)

(c)

(i) Show that the function $f(x) = xe^x - 1$ has a zero between $x = 0$ and $x = 1$ (1)

(ii) Using $x = 0.5$ as a first approximation, use Newton's Method once to obtain a second approximation to the zero correct to 3 significant figures. (3)

(Marks)

Question 4
Start a new page

(a) Solve $\sin 2\vartheta = \sin \vartheta$ for $0 \le \vartheta \le 2\pi$ (2)

(b) Find the exact value of

$$\int_0^{\frac{\pi}{12}} \sin^2 2x \, dx$$ (3)

(c) At any time t, the rate of cooling of the temperature T of a body is given by

$$\frac{dT}{dt} = -k(T - S)$$

where S is the temperature of the surroundings and k is a constant.

(i) Verify that $T = S + Ae^{-kt}$ is a solution of $\frac{dT}{dt} = -k(T - S)$ where A is a constant (1)

(ii) A cup of hot tea cooled from 100°C to 70°C in 10 minutes in a room where the temperature was 20°C. Find the values of A and k (2)

(d) Prove by mathematical induction that

$$11 \times 2! + 19 \times 3! + 29 \times 4! + \ldots + (n^2 + 5n + 5)(n+1)! = (n+4)(n+2)! - 8$$

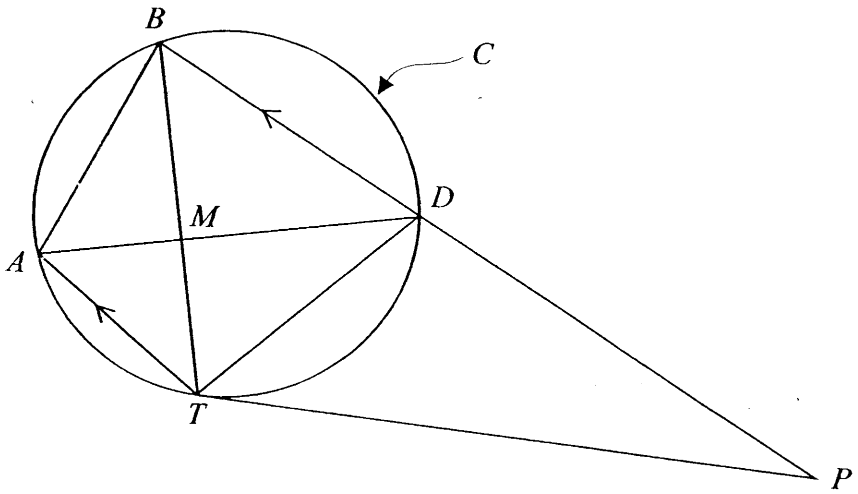
for $n = 1, 2, 3 \ldots$

(4)

(Marks)

Question 5
Start a new page

(a) Find the acute angle between the lines $2x + y = 4$ and $x - y = 2$.
Answer to the nearest degree. (2)



The diagram above shows a circle C. The points A, B, D and T lie on C and the point P is an exterior point.
PT is a tangent to C at T. The line AT is parallel to the line BP and point D lies on BP. The lines AD and BT intersect at M.

Copy or trace the diagram onto your page.

(i) Prove that ΔPTB is similar to ΔBAT (3)

(ii) Show that $AB = DT$ (2)

(b) A particle is moving in a straight line with acceleration given by

$$\frac{d^2x}{dt^2} = 9(x-2)$$

where x is the displacement in metres from an origin O after t seconds. Initially, the particle is 4 metres to the right of O so that x =4 and has velocity $v = -6$

(i) Show that $v^2 = 9(x-2)^2$ (2)

(ii) Find an expression for v and hence find x as a function of t. (3)

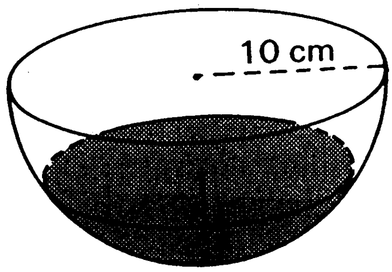
(Marks)

Question 6
Start a new page

(a) The circle $x^2 + y^2 = 100$ is rotated about the y axis from $y = 10$ to $y = 10 - a$.

Use $V = \pi \int_{10-a}^{10} (100 - y^2)\,dy$ to show that the volume, V, is $V = \frac{\pi}{3} a^2 (30 - a)$ (3)

(b)



The volume of water in a hemispherical bowl of radius 10cm is given by $V = \frac{\pi}{3} x^2 (30 - x)$ where $x$cm is the depth of the water at any time t.

The bowl is being filled at a constant rate of $2\pi\, cm^3$ / min

At what rate is the depth increasing when the depth is 2cm? (3)

(c) A particle moves in a straight line and its position at time t seconds is given by

$$x = \sqrt{3} \sin \frac{t}{2} + \cos \frac{t}{2}$$

(i) Write $\sqrt{3} \sin \frac{t}{2} + \cos \frac{t}{2}$ in the form $R \sin \left(\frac{t}{2} + \alpha\right)$ where $R>0$ and $0 \le \alpha \le \frac{\pi}{2}$ (2)

(ii) Hence prove that the particle is moving in simple harmonic motion about $x = 0$. (2)

(iii) For $0 < t < 4\pi$, when is the speed of the particle equal to 0.5ms^{-1}. (2)

(Marks)

Question 7
Start a new page

(a) Evaluate $\lim_{x \to 0} \frac{\sin 6x}{5x}$ (1)

(b) A football is kicked at an angle of α to the horizontal. The position of the ball at time t seconds is given by

$$x = vt \cos \alpha$$

$$y = vt \sin \alpha - \frac{1}{2}gt^2$$

where g m/s^2 is the acceleration due to gravity and v m/s is the initial velocity of projection. (You are NOT required to derive these.)

(i) Show that the equation of the path of the ball is (2)

$$y = x \tan \alpha - \frac{gx^2}{2v^2} \sec^2 \alpha.$$

(ii) Show that the maximum height h reached is given by (3)

$$h = \frac{v^2 \sin^2 \alpha}{2g}$$

(iii) Hence show that $y = x \tan \alpha (1 - \frac{x \tan \alpha}{4h})$. (2)

(iv)If $g = 10$ m/s^2, $\alpha = 30^\circ$ and the ball just clears the head of a player 1.6m tall and 10m away, calculate the maximum height reached by the ball. (2)

(c) The real number x is a solution of the equation $x^2 - x - 1 = 0$. Use the Binomial Theorem to show that the sum S of the series $1 + x + x^2 + ... + x^{2n-1}$ is given by

$$S = \sum_{r=1}^{n} {}^nC_r x^{r+1}$$ (2)

END OF PAPER

Question 1

(a) $\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^3$

$= \sin^{-1}1 - \sin^{-1}0$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$

(b) $(x+3)^4 = x^4 + 4x^3.3 + 6x^2 3^2 + 4x3^3 + 3^4$

$= x^4 + 12x^3 + 54x^2 + 108x + 81$

1
1 2 1
1 3 3 1
1 4 6 4 1

(c) $y = x^2 \cos^{-1}3x$

$y' = uv' + vu'$

$= x^2 . \frac{-1}{\sqrt{1-9x^2}} .3 + \cos^{-1}3x . (2x)$

$= \frac{-3x^2}{\sqrt{1-9x^2}} + 2x\cos^{-1}3x$

(d) $P(-4) = (-4)^3 - 4(-4) + 2$

$= -64 + 16 + 2$

$= -46$

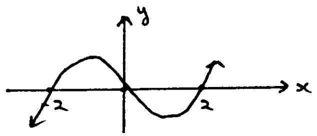
(e) $y = x^3 - 4x$

$= x(x^2-4)$

$= x(x-2)(x+2)$

$x=1,\ y = 1^3 - 4(1)$

$= -3$



$\therefore\ x^3 - 4x \geq 0$

$-2 \leq x \leq 0,\ x \geq 2$

(f) $x = e^2,\ u = \log_e e^2 = 2$

$x = e,\ u = \log_e e = 1$

$u = \log_e x$

$\frac{du}{dx} = \frac{1}{x}$

$\int_e^{e^2} \frac{1}{x\log_e x}dx = \int_1^2 \frac{1}{u}du$

$= [\ln u]_1^2$

$= \ln 2 - \ln 1$

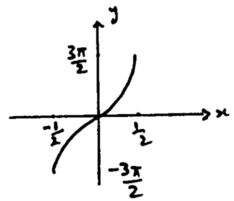
$= \ln 2$

Question 2

(a) $y = 3\sin^{-1}2x$

$-1 \leq 2x \leq 1$

$-\frac{1}{2} \leq x \leq \frac{1}{2}$



(b) $\angle DCB = x$ (angles standing on the same arc are equal)

$\angle DBC = 90°$ (angle in a semicircle)

$\angle BDC = 65°$ (alternate $\angle$ theorem)

$x + 90 + 65 = 180°$ (angle sum of a Δ)

$x = 25°$

(c) $P(x) = x^3 - 2x^2 + kx + 24$ $a=1,\ b=-2,\ c=k,\ d=24$

i) $\alpha+\beta+\gamma = -\frac{b}{a}$

$= 2$

ii) $\alpha\beta\gamma = -\frac{d}{a}$

$= -24$

iii) let $\beta = -\alpha$

$\alpha - \alpha + \gamma = 2$

$\gamma = 2$

$P(2) = 2^3 - 2(2)^2 + k(2) + 24 = 0$

$k = -12$

(d) i) $f(x) = e^{x+2}$

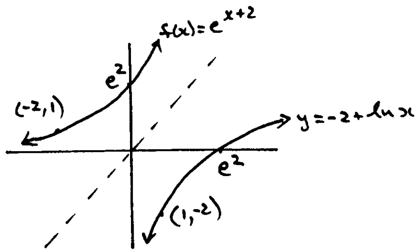
$x = e^{y+2}$

$\ln x = y + 2$

$y = -2 + \ln x$

$f^{-1}(x) = -2 + \ln x$

ii)



Comments: For full marks, intercepts to be shown.

Question 3

(a) $\frac{T_{r+1}}{T_r} = \frac{{}^6C_r(2x)^{6-r}(-x^{-2})^r}{{}^6C_{r-1}(2x)^{6-(r-1)}(-x^{-2})^{r-1}}$

$0 = \frac{6-r-2r}{7-r-2r+2}$

$0 = \frac{6-3r}{11-3r}$

$r = 2$

$\therefore {}^6C_2 = (2x)^4\left(\frac{1}{x^2}\right)^2$

$= 15.16x^4 \cdot \frac{1}{x^4}$

$= 240$

(b) i) $x^2 = -4ay$

$y = \frac{x^2}{-4a}$

$y' = \frac{2x}{-4a}$

$= \frac{x}{-2a}$

at $x = 2ap$, $y' = \frac{2ap}{-2a}$

$= -p$

$y - y_1 = m(x - x_1)$

$y + ap^2 = -p(x - 2ap)$

$y + ap^2 = -px + 2ap^2$

$px + y = ap^2$

ii) at B, $x = 0$, $y = ap^2$ $\therefore B(0, ap^2)$ (1)

$m:n$ $3:-1$ $A(2ap, -ap^2)$ $B(0, ap^2)$

$C = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

$= \left(\frac{3\times0-1\times2ap}{3-1}, \frac{3\times ap^2-1.-ap^2}{3-1}\right)$

$= (-ap, 2ap^2)$

iii) $x = -ap$ $\therefore y = 2a\left(\frac{x}{-a}\right)^2$

$p = \frac{x}{-a}$

$y = 2a\frac{x^2}{a^2}$

$x^2 = \frac{1}{2}ay$ which is a parabola

c) i) $f(0) = 0 - 1 = -1$

$f(1) = e - 1 \doteqdot 1.7$

Since $f(0)$ and $f(1)$ have different signs, $f(x)$ has a zero between $x=0$ & $x=1$

ii) $f'(x) = e^x + xe^x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{0.5e^{0.5}-1}{e^{0.5}+0.5e^{0.5}} = 0.571$

Question 4

a) $\sin 2\theta = \sin\theta$ for $0 \le \theta \le 2\pi$

$2\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta(2\cos\theta - 1) = 0$

$\sin\theta = 0$ $\cos\theta = \frac{1}{2}$

$\theta = 0, \pi, 2\pi,$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

b) $\int_0^{\frac{\pi}{12}} \sin^2 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{12}} 1 - \cos 4x\,dx$ $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, $\sin^2 2x = \frac{1}{2}(1-\cos 4x)$

$\frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/12}$

$= \frac{1}{2}\left[\left(\frac{\pi}{12} - \frac{1}{4}\sin\frac{\pi}{3}\right) - \left(0 - \frac{1}{4}\sin 0\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{12} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2} - 0\right]$

$= \frac{\pi}{24} - \frac{\sqrt{3}}{16}$

c) i) $T = S + Ae^{-kt}$

$\frac{dT}{dt} = -k.Ae^{-kt}$

$= -k(T - S)$

ii) $A = 80$, $S = 20$, $T = 70$ $t = 10$ $100 = 20 + Ae^0$, $A = 80$

$70 = 20 + 80e^{-10k}$

$50 = 80e^{-10k}$

$\frac{5}{8} = e^{-10k}$

$\frac{\ln(\frac{5}{8})}{-10} = k$

$k = 0.0470003$

iii) Let $n=1$, LHS $= 11\times2!$

$= 22$

RHS $= 5\times3! - 8$

$= 30 - 8$

$= 22$

$=$ LHS

$\therefore$ True for $n=1$

Assume true for $n = k$

ie. $11\times2! + 19\times3! + \ldots + (k^2+5k+5)(k+1)! = (k+4)(k+2)! - 8$

To prove true for $n = k+1$

ie $11\times2! + \ldots + ((k+1)^2 + 5(k+1) + 5)((k+1)+1)! = ((k+1)+4)(k+1+2)! - 8$

ie. $11\times2! + \ldots + ((k+1)^2 + 5k + 10)(k+2)! = (k+5)(k+3)! - 8$

LHS $= (k+4)(k+2)! - 8 + (k^2+2k+1+5k+10)(k+2)!$

$= (k+4)(k+2)! + (k^2+7k+11)(k+2)! - 8$

$= (k+2)!\,[(k+4)+(k^2+7k+11)] - 8$

$= (k+2)!\,(k^2+8k+15) - 8$

$= (k+2)!\,(k+5)(k+3) - 8$

$= (k+3)!\,(k+5) - 8$

$=$ RHS.

Since the statement is true for $n=1$, it is true for $n=1+1=2$ & therefore true for all integers $n \geq 1$

QUESTION 5

a) $2x+y=4$

$y=-2x+4 \quad m_1=-2$

$x-y=2$

$y=x-2 \quad \therefore m_2=1$

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$= \left|\frac{-2-1}{1+-2\times 1}\right|$

$= \frac{3}{1}$

$=3$

$\therefore \theta = 72^\circ$

b) $\angle BTA = \angle TBP$ (alternate $\angle$s, $BD \parallel AT$)

$\angle BAT = \angle BTP$ (alternate segment theorem)

$\therefore \Delta BAT \parallel\!| \Delta PTB$ (corresponding angles equal)

ii) In ΔABT, ΔADT

AT is common

$\angle ABT = \angle ADT$ (angles on same arc are equal)

$\angle BAD = \angle BTD$ (angles on arc BD are equal)

$\angle TAD = \angle TBD$ (angles on arc TD are equal)

$\angle DBT = \angle ATB$ (alternate $\angle$'s $BD \parallel AT$)

$\angle BAT = \angle DTA$

$\therefore \Delta ABT \equiv \Delta ADT$ (AAS)

$\therefore AB = DT$

Question 5 (continued)

i) $\frac{d^2x}{dt^2} = 9(x-2)$

$\frac{1}{2}v^2 = \int 9(x-2)$

$\frac{1}{2}v^2 = 9\left(\frac{x^2}{2}-2x\right)+c$

$v=-6,\ x=4,\quad \frac{1}{2}(-6)^2 = 9\left(\frac{16}{2}-8\right)+c$

$c=18$

$\frac{1}{2}v^2 = 9\left(\frac{x^2}{2}-2x\right)+18$

$v^2 = 18\left(\frac{x^2}{2}-2x\right)+36$

$= 9x^2-36x+36$

$= 9(x^2-4x+4)$

$= 9(x-2)^2$

ii) $v = \pm 3(x-2)$

but $v<0$ when $x=4$

$\therefore v = -3(x-2)$

$\frac{dx}{dt} = -3(x-2)$

$\frac{dt}{dx} = \frac{1}{-3(x-2)}$

$t = -\frac{1}{3}\ln(x-2)+c$

$t=0,\ x=4,\quad 0 = -\frac{1}{3}\ln(4-2)+c$

$0 = -\frac{1}{3}\ln 2 + c$

$c = \frac{1}{3}\ln 2$

$\therefore t = -\frac{1}{3}\ln(x-2)+\frac{1}{3}\ln 2$

$t-\frac{1}{3}\ln 2 = -\frac{1}{3}\ln(x-2)$

$-3t+\ln 2 = \ln(x-2)$

$e^{\ln 2-3t} = x-2$

$x = 2+e^{\ln 2-3t}$

Question 6

a) $V = \pi \int_{10-a}^{10} (100 - y^2)\,dy$

$= \pi \left[100y - \frac{y^3}{3} \right]_{10-a}^{10}$

$= \pi \left[\left(1000 - \frac{1000}{3}\right) - \left(100(10-a) - \frac{(10-a)^3}{3}\right) \right]$

$= \pi \left[1000 - \frac{1000}{3} - \left(1000 - 100a - \left(\frac{1000 - 300a + 30a^2 - a^3}{3}\right)\right) \right]$

$= \pi \left[-\frac{1000}{3} + 100a + \frac{1000}{3} - 100a + 10a^2 - \frac{a^3}{3} \right]$

$= \pi \left[10a^2 - \frac{a^3}{3} \right]$

$= \frac{\pi}{3} a^2 (30 - a)$

b) Find $\frac{dx}{dt}$ when $x = 2$ $\frac{dV}{dt} = 2\pi$

$V = \frac{\pi}{3} x^2 (30 - x)$

$= \frac{\pi}{3} x^2 . 30 - \frac{\pi}{3} x^3$

$= \pi 10 x^2 - \frac{\pi}{3} x^3$

$\frac{dV}{dx} = 20\pi x - \pi x^2$ At $x = 2$, $\frac{dV}{dx} = 40\pi - 4\pi = 36\pi$

$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$

$= \frac{1}{36\pi} \times 2\pi$

$= \frac{1}{18}$ cm/min

c) i) $R = \sqrt{(\sqrt{3})^2 + (1)^2}$

$= 2$

$\tan \alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$\therefore \sqrt{3} \sin \frac{t}{2} + \cos \frac{t}{2} = 2 \sin\left(\frac{t}{2} + \frac{\pi}{6}\right)$

ii) $x = 2 \sin\left(\frac{t}{2} + \frac{\pi}{6}\right)$

$v = \frac{1}{2} . 2 \cos\left(\frac{t}{2} + \frac{\pi}{6}\right)$

$= \cos\left(\frac{t}{2} + \frac{\pi}{6}\right)$

$a = -\frac{1}{2} \sin\left(\frac{t}{2} + \frac{\pi}{6}\right)$

$= -\frac{1}{4} \times 2 \sin\left(\frac{t}{2} + \frac{\pi}{6}\right)$

$= -\frac{1}{4} x$ which is in SHM

iii) $0.5 = \cos\left(\frac{t}{2} + \frac{\pi}{6}\right)$

$\frac{t}{2} + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \ldots$

$\frac{t}{2} = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}$

$t = \frac{\pi}{3}, 3\pi$

Question 7

a) $\lim_{x \to 0} \frac{\sin 6x}{5x} = \lim_{x \to 0} \frac{\sin 6x}{6x} \times \frac{6}{5}$

$= 1 \times \frac{6}{5}$

$= \frac{6}{5}$

b) i) $x = vt \cos\alpha$

$\therefore t = \frac{x}{v \cos\alpha}$

Sub into $y = vt \sin\alpha - \frac{1}{2} g t^2$

$y = v\left(\frac{x}{v\cos\alpha}\right) \sin\alpha - \frac{1}{2} g \left(\frac{x}{v\cos\alpha}\right)^2$

$= x \tan\alpha - \frac{g x^2}{2v^2 \cos^2\alpha}$

$= x \tan\alpha - \frac{g x^2}{2v^2} \sec^2\alpha$

ii) Ball reaches maximum height when $\dot{y} = 0$

$0 = V \sin\alpha - gt$

$t = \frac{V \sin\alpha}{g}$

Substitute into $y = vt \sin\alpha - \frac{1}{2} g t^2$

$y = v\left(\frac{v \sin\alpha}{g}\right) \sin\alpha - \frac{1}{2} g \left(\frac{v^2 \sin^2\alpha}{g^2}\right)$

$= \frac{v^2 \sin^2\alpha}{g} - \frac{1}{2} \frac{v^2 \sin^2\alpha}{g}$

$= \frac{v^2 \sin^2\alpha}{2g}$

iii) $g = \frac{v^2 \sin^2\alpha}{2h}$

$\therefore y = x \tan\alpha - \frac{\left(\frac{v^2 \sin^2\alpha}{2h}\right) x^2 \sec^2\alpha}{2v^2}$

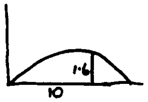
$= x \tan\alpha - \frac{v^2 \sin^2\alpha \, x^2 \sec^2\alpha}{2h . 2v^2}$

$= x \tan\alpha - \frac{x^2 \sin^2\alpha}{\cos^2\alpha . 4h}$

$= x \tan\alpha - \frac{x^2 \tan^2\alpha}{4h}$

$= x \tan\alpha \left(1 - \frac{x \tan\alpha}{4h}\right)$

iv)



$x = 10$
$y = 1.6$

$$y = x\tan\alpha\left(1 - \frac{x\tan\alpha}{4h}\right)$$

$$1.6 = 10\tan 30^\circ\left(1 - \frac{10\tan 30}{4h}\right)$$

$$1.6 = 10\cdot\frac{1}{\sqrt{3}}\left(1 - \frac{\frac{10}{\sqrt{3}}}{4h}\right)$$

$$1.6 = \frac{10}{\sqrt{3}}\left(1 - \frac{10}{4h\sqrt{3}}\right)$$

$$1.6\times\frac{\sqrt{3}}{10} - 1 = -\frac{10}{4h\sqrt{3}}$$

$$0.7228718 = \frac{10}{4h\sqrt{3}}$$

$$h \doteqdot 1.9967$$

$$\doteqdot 2\text{m}$$

c) $1 + x + x^2 + \dots + x^{2n-1}$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{2n} = \frac{1(x^{2n} - 1)}{x - 1}$$

$$= \frac{x^{2n} - 1}{x - 1}$$

$$x^2 - x - 1 = 0$$

$$x^2 = x + 1$$

$$x^{2n} = (x+1)^n$$

$$x^{2n} = \sum_{r=0}^{n} {}^nC_r x^r$$

$$x^{2n} - 1 = \sum_{r=1}^{n} {}^nC_r x^r$$

Also $x^2 - x - 1 = 0$

$$x^2 - x = 1$$

$$x(x-1) = 1$$

$$x = \frac{1}{x-1}$$

$$S_{2n} = \frac{x^{2n} - 1}{x - 1}$$

$$= x\cdot\sum_{r=1}^{n} {}^nC_r x^r$$

$$= \sum_{r=1}^{n} {}^nC_r x^{r+1}$$