Student Number:

SCEGGS Darlinghurst

Trial Higher School Certificate Examination, 1997

Mathematics

3 Unit

TIME ALLOWED: TWO HOURS

(An additional five minutes reading time is allowed)
No writing or marking of paper is permitted during this time.

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

DIRECTION TO STUDENTS:

- Start each question on a new page.
- All questions may be attempted.
- All questions are of equal value.
- All necessary working should be shown in every question. Marks may not be awarded for careless or badly arranged work.
- A sheet of standard integrals is provided.

Question 1 — Marks

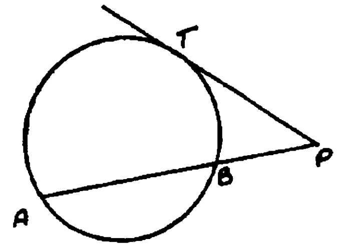
(a) The point $P\ (10, -5)$ divides the interval joining $A\ (6, 1)$ and $B\ (12, -8)$ in the ratio $m : n$. Find values for m and n. — 2

(b) $P(x)$ is a monic cubic polynomial with zeroes 1, $1+\sqrt{2}$ and $1-\sqrt{2}$. Find $P(x)$ in the form: — 2

$$ax^3 + bx^2 + cx + d$$

(c) Find $\int \frac{2x}{1+x^4}\,dx$. You may use the substitution $u = x^2$, if you wish. — 2

(d) — 3



PT is a tangent to the circle at the point of contact T.

$PT = 4\sqrt{3}$ units

$PB = x$ units

$AB = 8$ units

Find the value of x.

(e) (i) Prove that $\frac{dV}{dt} = \frac{d}{dx}\left(\frac{1}{2}V^2\right)$ — 3

(ii) The velocity of a particle is given by $V = 2 - x$, where x is the displacement. Find the acceleration of the particle in terms of x.

Question 2 **Marks**

(a) Sketch the function $y = \tan^{-1}\frac{x}{2}$, showing its domain and range. **2**

(b) Prove that $24^n + 6^n = 2^n 3^n (4^n + 1)$ **2**

(c) Consider the polynominal: **3**

$$P(x) = x^3 - 3x^2 - 2x + 4$$

(i) Show that it has a zero between $x = 3$ and $x = 4$.

(ii) Taking an initial approximation of 3.5, and one application of Newton's Method, find a more accurate approximation, correct to 1 decimal place.

(d) Evaluate exactly: $2\cos^{-1}\left(-\frac{1}{2}\right) - 3\tan^{-1}\left(\sqrt{3}\right)$ **2**

(e) (i) On the same diagram, sketch graphs of $y = x$ and $y = |x + 1|$. **3**

(ii) Using this diagram or otherwise, determine the values of m for which the equation

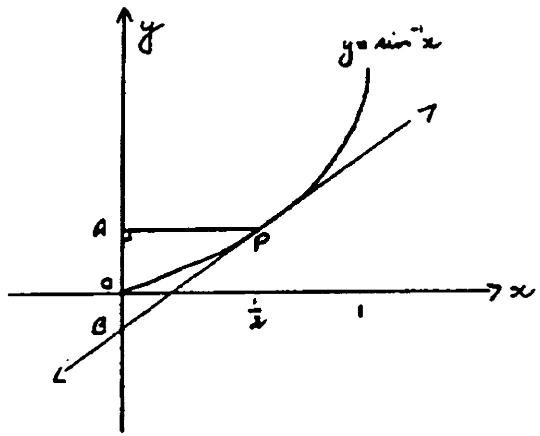
$mx = |x + 1|$ has two distinct solutions.

Question 3 **Marks**

(a) A class consists of 12 boys and 8 girls. A committee of 5 is chosen from the class. Find the probability that it contains at least 4 girls. (Provided working is clear it is not necessary to calculate the answer). **3**

(b) Evaluate $\int \frac{dx}{x(1 + \log_e x)^2}$ using the substitution $u = \log_e x$ **3**

(c) **6**



The curve shown is $y = \sin^{-1} x$ for the domain $0 \le x \le 1$. The straight line is the tangent to the curve at the point P where $x = \frac{1}{2}$. A and B are points on the y axis as shown.

(i) Find the equation of the tangent at P.

(ii) Find the coordinates of A and B.

(iii) Prove that the area of the triangle APB (which is right angled at A) is:

$$\frac{\sqrt{3}}{12} \text{ units}^2.$$

Question 4 | **Marks**

(a) Given $f(x)=\dfrac{x^3+1}{x^3}$ — 3

(i) Find L such that

$$L=\lim_{x\to\infty}\frac{x^3+1}{x^3}$$

(ii) Find values of x such that $f(x)-L<0.001$

(b) Prove that — 4

$$\tan 2A\cot A-1 = \sec 2A$$

(c) A particle moves in a straight line such that its displacement x centimetres at any time t seconds is given by — 5

$$x=2\sin\left(2t-\frac{\pi}{3}\right)$$

(i) Prove that the motion is Simple Harmonic.

(ii) Find the initial position of the particle.

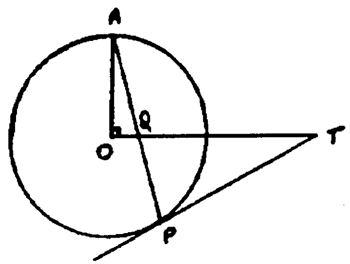
(iii) Find the velocity of the particle when it passes through the centre of motion for the first time.

Question 5 | **Marks**

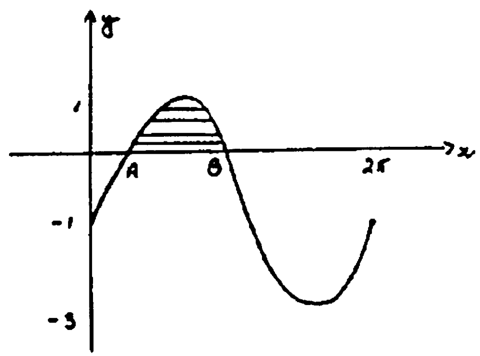
(a) Find the greatest term in the expansion of $(1+4x)^8$ when x has the value of $\frac{1}{3}$. — 4

(b) In the figure below, O is the centre of the circle, $<AOT$ is a right angle and TP is a tangent at the point P. AP meets OT at Q. — 4

Prove that $TP = TQ$.



(c) — 4



The sketch above is of the curve

$$y=2\sin x-1 \text{ for } 0\le x\le 2\pi$$

(i) Find the coordinates of A and B on the x axis.

(ii) Find the volume formed when the shaded area is rotated about the x axis.

Question 6 — **Marks**

(a) At any time, t, the rate of cooling of the temperature T of a body, when the surrounding air temperature is P, is given by: **4**

$$\frac{dT}{dt} = -k(T - P) \text{ for some constant } k.$$

(i) Show that $T = P + Ae^{-kt}$, for some constant A, satisfies this equation.

(ii) A metal ingot has a temperature of 750°C and cools to 500°C in 10 minutes when placed in an air temperature of 21°C. Find how much longer it will take to cool to 250°C. (Answer to the nearest second).

(b) The parametric equations of a curve are: **3**

$$x = 2 + 3\cos\theta$$
$$y = -3 + 3\sin\theta$$

(i) Eliminate θ in order to find the Cartesian equation of the curve.

(ii) Describe the nature of the curve.

5

(c) A projectile is fired horizontally with an initial velocity of V m/s from a position H m above ground level. The acceleration due to gravity is g m/s^2.

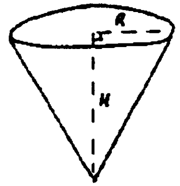
(i) Find the equations of motion for this projectile.

(ii) Find the time taken to reach the ground

(iii) Show that the horizontal range is $V\sqrt{\frac{2H}{g}}$.

Question 7 — **Marks**

(a) A container is in the shape of a vertical cone whose height is double the radius. **7**



(i) Show that the volume of the cone is given by $V = \frac{\pi H^3}{12}$.

(ii) Water is poured in at a rate of 10mm^3/s. Find the rate at which the depth, Hmm, is increasing when the depth of water in the cone is 50mm.

The cone is filled to a depth of 100mm and pouring then stops. A hole is then opened at the vertex of the cone and water flows out at a rate of πH^2 mm^3/s.

(iii) Show that $\frac{dH}{dt} = -4$.

(iv) Hence show that it take 25 seconds to empty the cone.

(b) (i) Sketch the curve $y = \frac{1}{x^2 + 1}$, finding any asymptotes and stationary points. **5**

(ii) Find the area bounded by this curve, the x axis and the values $x = -1$ and $x = 1$.

(iii) Prove that the area between this curve and the x axis is always less than π units^2.

END OF PAPER

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① a) $10 = \frac{12m+6n}{m+n}$

$10m + 10n = 12m + 6n$

$2m = 4n$

$\frac{m}{n} = \frac{4}{2}$

$m = 2, \quad n = 1$

b) Let $\alpha = 1, \ \beta = 1+\sqrt{2}$

$\gamma = 1-\sqrt{2}$

$\alpha+\beta+\gamma = 3$

$\alpha\beta+\alpha\gamma+\beta\gamma = 1+\sqrt{2}+1-\sqrt{2} + 1-2$

$= 1$

$\alpha\beta\gamma = 1-2 = -1$

$\therefore P(x) = x^3 - 3x^2 + x + 1$

c) $\int \frac{2x}{1+x^4}\,dx = \tan^{-1}(x^2)+c$

d) $PT^2 = PA \times PB$

$48 = (x+8)x$

$= x^2 + 8x$

$\therefore x^2+8x-48=0$

$(x+12)(x-4)=0$

$x = -12$ or 4.

only solution is $x = 4$.

e) (i) $\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$

$= v\frac{dv}{dx}$

But $v = \frac{d}{dv}\left(\frac{1}{2}v^2\right)$

$\therefore \frac{dv}{dt} = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\cdot\frac{dv}{dx}$

$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

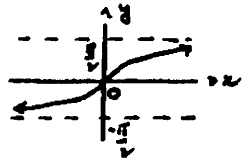
(ii) $v = 2-x$

$v^2 = 4-4x+x^2$

$\frac{1}{2}v^2 = 2-2x+\frac{x^2}{2}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2+x$

② a)



b) $24^n + 6^n = (4\times6)^n + 6^n$

$= 6^n\left[4^n+1\right]$

$= 2^n 3^n\left[4^n+1\right]$

c) (i) $P(3) = 3^3 - 3.3^2 - 2.3 + 4$

$= -2$

$P(4) = 4^3 - 3.4^2 - 2.4 + 4$

$= 12$

since the sign has changed there is a zero between $x=3$ and $x=4$.

(ii) $P'(x) = 3x^2 - 6x - 2$

$P'(3\cdot5) = 3.(3\cdot5)^2 - 6.3\cdot5 - 2$

$= 13\cdot75$

$P(3\cdot5) = (3\cdot5)^3 - 3.(3\cdot5)^2 - 2.3\cdot5 + 4$

$= 3\cdot125.$

$x_1 = 3\cdot5 - \frac{3\cdot125}{13\cdot75}$

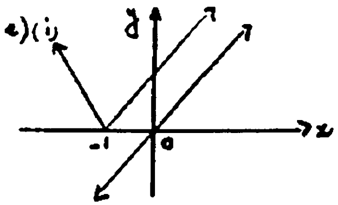
$= 3\cdot3$ (1dp) which is a better approximation.

d) $2\cos^{-1}\left(-\frac{1}{2}\right) - 3\tan^{-1}(\sqrt{3})$

$= 2\times\frac{2\pi}{3} - 3\times\frac{\pi}{3}$

$= \frac{\pi}{3}$

e) (i)



(ii) $-1 < m < 0$

③ a) Total no. of groups:

$\binom{20}{5}$

no. with 4 girls:

$\binom{8}{4}\times\binom{12}{1}$

no. with 5 girls

$\binom{8}{5}$

Probability :-

$$\frac{\binom{8}{4}\times\binom{12}{1}+\binom{8}{5}}{\binom{20}{5}}$$

b) Let $u = \log_e x$

$\frac{du}{dx} = \frac{1}{x}.$

$I = \int \frac{du}{(1+u)^2}$

$= \int (1+u)^{-2}\,du$

$= \frac{(1+u)^{-1}}{-1} + c$

$= \frac{-1}{1+u} + c$

$= \frac{-1}{\log_e x+1} + c$

c) $y = \sin^{-1} x$

(i) $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

if $x = \frac{1}{2}$, $\frac{dy}{dx} = \frac{1}{\sqrt{\frac{3}{4}}} = \frac{2}{\sqrt{3}}$

$y = \frac{\pi}{6}.$

equation: $y - \frac{\pi}{6} = \frac{2}{\sqrt{3}}\left(x - \frac{1}{2}\right)$

$2x - \sqrt{3}y - 1 + \frac{\sqrt{3}\pi}{6} = 0$

(ii) A is $\left(0, \frac{\pi}{6}\right)$

if $x=0$, $\sqrt{3}y = \frac{\sqrt{3}\pi}{6} - 1$

$y = \frac{\pi}{6} - \frac{1}{\sqrt{3}}$

B is $\left(0, \frac{\pi}{6} - \frac{1}{\sqrt{3}}\right)$

$AB = \frac{\pi}{6} - \frac{\pi}{6} + \frac{1}{\sqrt{3}}$

$= \frac{1}{\sqrt{3}}$ units.

Area of triangle $= \frac{1}{2}\cdot AB\cdot AP$

$= \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\cdot\frac{1}{2}$

$= \frac{1}{4\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\sqrt{3}}{12}$ units^2

④ a) (i) $\lim_{x \to \infty} \frac{x^3+1}{x^3} = \lim_{x \to \infty} 1 + \frac{1}{x^3}$

$= 1$

(ii) $\frac{x^3+1}{x^3} - 1 < 0.001$

$\frac{x^3+1-x^3}{x^3} < 0.001$

$\frac{1}{x^3} < 0.001$

if $x > 0$, $x^3 > 1000$

$x > 10$

if $x < 0$, $x^3 < 1000$

$x < 10$

solution: $x < 0$, $x > 10$

b) LHS $= \tan 2A \cot A - 1$

$= \frac{2\tan A}{1-\tan^2 A} \cdot \frac{1}{\tan A} - 1$

$= \frac{2}{1-\tan^2 A} - 1$

$= \frac{2-1+\tan^2 A}{1-\tan^2 A}$

$= \frac{1+\tan^2 A}{1-\tan^2 A}$

$= \frac{\cos^2 A + \sin^2 A}{\cos^2 A - \sin^2 A}$

$= \frac{1}{\cos 2A}$

$= \sec 2A.$

c) $x = 2\sin\left(2t - \frac{\pi}{3}\right)$

(i) $\dot{x} = 4\cos\left(2t - \frac{\pi}{3}\right)$

$\ddot{x} = -8\sin\left(2t - \frac{\pi}{3}\right)$

$\ddot{x} = -4x$

$\ddot{x} = -2^2 . x$ which is of the form of S.H.M.

(ii) when $t = 0$,

$x = 2\sin\left(-\frac{\pi}{3}\right)$

$= -2 . \frac{\sqrt{3}}{2}$

$x = -\sqrt{3}$ cm.

(iii) $\dot{x} = 4\cos\left(2t - \frac{\pi}{3}\right)$

when $x = 0$, $\sin\left(2t - \frac{\pi}{3}\right) = 0.$

Acute angle is 0

$2t - \frac{\pi}{3} = 0$ or π, 2π ....

$2t = \frac{\pi}{3}$ or $\frac{4\pi}{3}$, ...

the first time through 0 is $t = \frac{\pi}{6}$

$\dot{x} = 4\cos\left(\frac{\pi}{3} - \frac{\pi}{3}\right)$

$= 4\cos 0$

velocity is 4 cm/s.

⑤ a) $\frac{T_{k+1}}{T_k} = \frac{8-k+1}{k} \cdot \frac{4}{3}$

$= \frac{36-4k}{3k}$

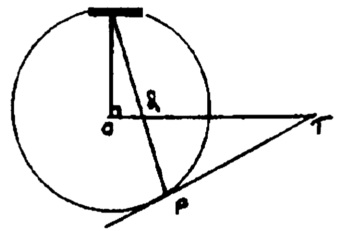
if $T_{k+1} > T_k$, $36 - 4k > 3k$

$7k < 36$

$k < 5\frac{1}{7}$

$\therefore \frac{T_6}{T_5} > 1$, $T_6 > T_5$

$T_6 = 57344 x^5$ or $\binom{8}{5} 4^5 x^5$.



Construction: Join OP

Proof: OA = OP (radii)

$\therefore \angle OAP = \angle OPA$ (opp. equal sides in ΔOAP)

Let $\angle OAP = \angle OPA = x°$.

$\angle OPT = 90°$ (radius perp. to tangent PT)

$\therefore \angle QPT = 90° - x°$.

In ΔAOQ, $\angle AOQ = 90°$ (data)

$\therefore \angle AQO = 180° - 90° - x°$

$= 90° - x°$

$\therefore \angle AQO = \angle PQT$ (vert. opp.)

$\therefore \angle PQT = \angle QPT$

$\therefore$ TQ = TP (opp. equal angles in ΔTQP)

c) (i) if $2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$

$\therefore x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$\therefore$ A is $\left(\frac{\pi}{6}, 0\right)$ B is $\left(\frac{5\pi}{6}, 0\right)$

(ii) $V = \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2\sin x - 1)^2 \, dx$

$= \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 4\sin^2 x - 4\sin x + 1 \, dx$

$= \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 3 - 2\cos 2x - 4\sin x \, dx$

$= \pi \left[3x - \sin 2x + 4\cos x\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$

$= \pi\left[\frac{15\pi}{6} - \sin\frac{5\pi}{3} + 4\cos\frac{5\pi}{6} - \frac{3\pi}{6} + \sin\frac{\pi}{3} - 4 \ldots\right.$

$= \pi\left[2\pi + \frac{\sqrt{3}}{2} - 4\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - 4\frac{\sqrt{3}}{2}\right]$

$= \pi\left[2\pi - 3\sqrt{3}\right]$ units^3.

⑥ a) (i) $T = P + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

but $Ae^{-kt} = T - P$

$\therefore \frac{dT}{dt} = -k(T-P)$ as required

(ii) when $t = 0$, $750 = P + A$

$= 21 + A$

$A = 729$

$T = 21 + 729e^{-kt}$

when $t = 10$

$500 = 21 + 729e^{-10k}$

$479 = 729e^{-10k}$

$e^{-10k} = \frac{479}{729}$

$-10k = \ln\frac{479}{729}$

$k = 0.041997813\ldots$

$250 = 21 + 729e^{-kt}$

$229 = 729e^{-kt}$

$e^{-kt} = \frac{229}{729}$

$-kt = \ln\frac{229}{729}$

$t = 27.6$

it takes a further 17 min 34 s.

b) $x = 2 + 3\cos\theta$

$y = -3 + 3\sin\theta$

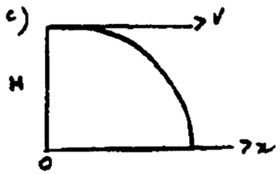
(i) $\cos\theta = \frac{x-2}{3}$

$\sin\theta = \frac{y+3}{3}$

$\therefore (x-2)^2 + (y+3)^2 = 9$

(ii) circle centre $(2,-3)$ radius 3 units.

c)



(i) $\ddot{x} = 0 \qquad \ddot{y} = -g$

$\dot{x} = V \qquad \dot{y} = -gt$

$x = Vt \qquad y = -\frac{gt^2}{2} + H$

(note: finding of constants omitted)

(ii) when $y = 0$,

$-\frac{gt^2}{2} + H = 0.$

$\frac{gt^2}{2} = H$

$t = \sqrt{\frac{2H}{g}}$ s.

(iii) range is x when $y = 0$

i.e. range is $V\sqrt{\frac{2H}{g}}$ m.

7 a) (i) $V = \frac{1}{3}\pi R^2 H$

$R = \frac{H}{2}$

$\therefore V = \frac{1}{3}\pi \cdot \frac{H^2}{4} \cdot H$

$= \frac{\pi H^3}{12}$

(ii) $\frac{dV}{dt} = \frac{dV}{dH} \cdot \frac{dH}{dt}$

$10 = \frac{\pi H^2}{4} \cdot \frac{dH}{dt}$

$= 625\pi \cdot \frac{dH}{dt}$

$\frac{dH}{dt} = \frac{10}{625\pi} = \frac{2}{125\pi}$ mm/s.

(iii) $\frac{dV}{dt} = \frac{dV}{dH} \cdot \frac{dH}{dt}$

$-\pi H^2 = \frac{\pi H^2}{4} \cdot \frac{dH}{dt}$

$\therefore \frac{dH}{dt} = -4.$

(iv) $H = \int -4\,dt$

$H = -4t + c$

when $t = 0$ $H = 100$

$\therefore 100 = c$

$H = -4t + 100$

when $H = 0$, $t = 25$

i.e. it takes 25 s to empty.

b) (i) $y = \frac{1}{x^2+1}$

$y' = -2x(x^2+1)^{-2}$

$= \frac{-2x}{(x^2+1)^2}$

when $y' = 0$, $x = 0$, $y = 1$

if $x < 0$, $y' > 0$

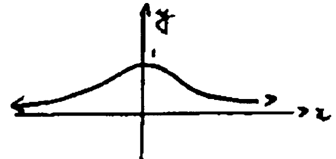
$x > 0$, $y' < 0$

$\therefore (0,1)$ is a max. stationary point.

$\lim_{x\to\infty} \frac{1}{x^2+1} = 0$

$\lim_{x\to-\infty} \frac{1}{x^2+1} = 0.$

$\frac{1}{x^2+1} > 0$ for all real x



(ii) Area $= \int_{-1}^{1} \frac{dx}{1+x^2}$

$= \tan^{-1}(1) - \tan^{-1}(-1)$

$= \frac{\pi}{4} - -\frac{\pi}{4}$

$= \frac{\pi}{2}$ u².

(iii) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

$= \tan^{-1}(\infty) - \tan^{-1}(-\infty)$

$= \frac{\pi}{2} - -\frac{\pi}{2}$

$= \pi.$

$\therefore$ the area between the curve and the x axis approaches but never reaches π u².