SCEGGS, DARLINGHURST

Mathematics

3 Unit (Additional)

and

3/4 Unit (Common)

2000

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

TIME ALLOWED : 2 Hours

(+ 5 minutes reading time)

INSTRUCTIONS:

- Attempt ALL SEVEN questions and show all necessary working.
- Marks will be deducted for careless or badly arranged work.
- ALL questions are of equal value.
- START EACH QUESTION ON A NEW PAGE.
- Make sure your student number is on each page.
- Approved calculators and templates may be used.
- Standard Integrals are printed on the last page. These may be removed for your convenience.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or format of the Higher School Certificate Examination.

Question 1 : (12 Marks)

(a) Solve : $\frac{2x}{x+1} \le 1$ **3**

(b) Consider each different arrangement of the letters of the word **INFINITE**. **3**

(i) How many different words are possible?

(ii) If one of these words is chosen at random, what is the probability that the 3 I's are together?

(c) Explain how you could find the coordinates of the point C that divides the interval joining $A(1,4)$ to $B(-2,10)$ in the ratio 1 : 5 *without* using a formula. **2**

(d) Give an example of a value of x in radians for which $\sin^{-1}(\sin x) \neq x$ **1**

(e) Prove that : **3**

$$n! + (n-1)! + (n-2)! = n^2(n-2)!$$

Question 2 : Start a new page (12 Marks)

(a) Find: $\int \cos^2 5x\,dx$ **2**

(b) Find the exact volume of the solid of revolution formed when the area between the curve $y = \frac{1}{\sqrt{x^2+9}}$, the x axis and the lines $x = 0$ and $x = 3\sqrt{3}$ is rotated about the x axis. **3**

(c) Evaluate $\int_0^1 \frac{4x}{(4x+1)^2}\,dx$ using the substitution $u = 4x+1$ **4**

(d) Find $\lim_{x\to 0} \frac{\sin 4x}{\tan 2x}$ **3**

Question 3 : **Start a new page** (12 Marks)

(a) Let $f(x) = x^3 + 3x^2 - 10x - 24$ 3

(i) Calculate $f(-2)$

(ii) Hence, express $f(x)$ as the product of three linear factors.

(b) Let α, β and γ be the roots of the equation $x^3 - 3x + 5 = 0$ 4

Find the values of :

(i) $\alpha + \beta + \gamma$

(ii) $\alpha\beta\gamma$

(iii) $(\alpha - 1)(\beta - 1)(\gamma - 1)$

(c) Consider the parabola $x^2 = 4ay$ 5

(i) Show that the equation of the normal to this parabola at the point $P(2ap, ap^2)$ is given by $x + py = ap^3 + 2ap$.

(ii) If this normal meets the parabola again at $Q(2aq, aq^2)$, show that $p^2 + pq + 2 = 0$.

Questions continue over ...

Question 4 : **Start a new page** (12 Marks)

(a) Find the coefficient of x^3 in $\left(3x^2 + \frac{1}{x}\right)^9$ 2

(b) A function is defined as $f(x) = 1 + e^{2x}$ 5

(i) Write down the range of this function.

(ii) Show that the inverse function can be defined as $f^{-1}(x) = \frac{1}{2}\ln(x - 1)$

(iii) On the same set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$

(iv) The equation of the normal to the curve $y = f^{-1}(x)$ at the point where $f^{-1}(x) = 0$ is given by the equation $2x + y - 4 = 0$. Show that the point of intersection of this normal and $y = f(x)$ can be derived from the equation $e^{2x} + 2x = 3$.

(v) By taking $x = 0.4$ as the first approximation of the root to $e^{2x} + 2x = 3$, use one application of Newton's Method to find a better approximation of the root, correct to 3 significant figures.

Questions continue over ...

Question 5 : **Start a new page** (12 Marks)

(a) Solve $2^{2x+1} - 5(2^x) + 2 = 0$ 3

(b) Prove that $2^{10n+3} + 3$ is divisible by 11 for all non-negative integers by Mathematical Induction. 4

(c) A particle moves in a straight line with Simple Harmonic Motion. At time t seconds, its displacement x metres from a fixed point O is given by: 5

$$x = 5\sin\frac{\pi}{2}\left(t + \frac{1}{3}\right)$$

(i) Show that $\ddot{x} = -\frac{\pi^2}{4}x$

(ii) State the period and the amplitude of the motion.

(iii) Find the magnitude of the acceleration when $x = 2\frac{1}{2}$

Questions continue over ...

Question 6 : **Start a new page** (12 Marks)

(a) N is the number of aardvarks in a certain population at time t years. The population size N satisfies the equation $\frac{dN}{dt} = -k(N - 1000)$, for some constant k. 6

(i) Verify by differentiation that $N = 1000 + Ae^{-kt}$ (where A is a constant) is a solution of the equation, $\frac{dN}{dt} = -k(N - 1000)$.

(ii) Initially there are 2500 aardvarks but after 2 years there are only 2200 left. Find the values of A and k.

(iii) Sketch the graph of population size against time.

(b) During the Euro2000 soccer tournament, Brett is standing 25 metres away from the goal line. He kicks a soccer ball off the ground at an angle of 30° to the horizontal with an initial velocity of V m/s. The ball hits the top bar which is 2.4 metres directly above the goal line. Neglecting air resistance and assuming that acceleration due to gravity is 10m/s^2, find: 6

(i) the horizontal and vertical components of the displacement of the ball in terms of the initial velocity, V.

(ii) the Cartesian equation of the motion for the path of the ball.

(iii) the initial velocity of the ball, correct to 1 decimal place.

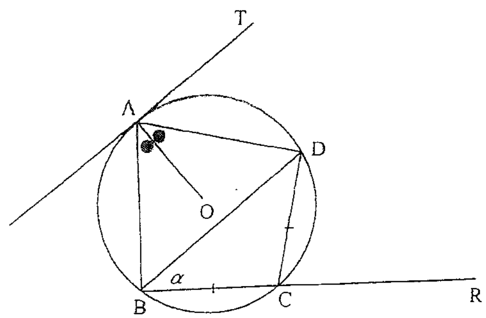
Questions continue over ...

Question 7 : **Start a new page** (12 Marks)

(a) (i) Solve $(2x-1)(2x-\sqrt{3})<0$ 3

(ii) Hence solve $(2\sin\theta-1)(2\sin\theta-\sqrt{3})<0$ for $0\le\theta\le 2\pi$

(b) Points A, B, C and D lie on a circle centre O. The line TA is a tangent to the circle at A, and BC is produced to R. The interval OA bisects $\angle BAD$, and $BC = CD$. The size of $\angle DBC$ is α. 4



NOT TO SCALE

Copy or trace the diagram.

(i) Explain why $\angle DCR = 2\alpha$

(ii) Show that $\angle OAD = \alpha$

(iii) Prove that $\angle ABC$ is a right angle.

(c) (i) Write down the formula for the coefficient of x^r in the expansion of $(1+x)^n$, where r and n are positive integers and $1\le r\le n$ 5

(ii) Let s and t be positive consecutive integers with $t = s + 1$. Show that $s^{2n}+2nt-1$ is divisible by t^2.

(iii) Hence find a perfect square that is a factor of $5^{20}+119$.

- END OF EXAMINATION -

3 UNIT MATHEMATICS

2000 TRIAL SOLUTIONS

[QUES]TION 1: 12 marks

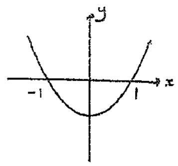
$\frac{2x}{x+1} \le 1 \quad (x \ne -1)$

$2x(x+1) \le (x+1)^2$

$2x^2 + 2x \le x^2 + 2x + 1$

$\therefore x^2 - 1 \le 0$

$\therefore -1 \le x \le 1$



$x \ne -1$

$\therefore -1 < x \le 1$

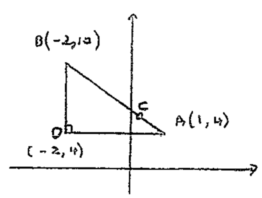
i) INFINITE

Number of different words $= \frac{8!}{3!\,2!}$

$= 3360$

i) Number of possibilities with I's together $= \frac{6!}{2!} = 360$

$\therefore$ Probability $= \frac{360}{3360} = \frac{3}{28}$



...w right angled triangle.

...d $\frac{1}{6}$ of length AD and subtract this distance from the x-value of A - x value

...d $\frac{1}{6}$ of length BD and add this distance to the y-value of A. - y value.

... $-y \quad 3$

(e) LHS $= n! + (n-1)! + (n-2)!$

$= (n-2)!\left[n(n-1) + (n-1) + 1\right]$

$= (n-2)!\,(n^2 - n + n - 1 + 1)$

$= n^2 (n-2)!$

$=$ RHS

QUESTION 2: 12 marks

(a) $\int \cos^2 5x \, dx$

$= \frac{1}{2}\int \cos 10x + 1 \, dx$

$= \frac{1}{2}\left[\frac{\sin 10x}{10} + x\right] + C$

$= \frac{1}{20}\sin 10x + \frac{1}{2}x + C$

(b) $V = \pi \int_0^{3\sqrt{3}} \frac{1}{x^2+9}\, dx$

$= \left[\frac{\pi}{3}\tan^{-1}\frac{x}{3}\right]_0^{3\sqrt{3}}$

$= \frac{\pi}{3}\tan^{-1}\sqrt{3} - \frac{\pi}{3}\tan^{-1} 0$

$= \frac{\pi^2}{9}$

(c) $\int_0^1 \frac{4x}{(4x+1)^2}\, dx$ $\quad u = 4x+1$, $\frac{du}{dx} = 4$

when $x=0,\ u=1$; $x=1,\ u=5$

$= \int_1^5 \frac{u-1}{u^2}\cdot\frac{1}{4}\, du$

$= \frac{1}{4}\int_1^5 \frac{1}{u} - \frac{1}{u^2}\, du$

$= \frac{1}{4}\left[\ln u + \frac{1}{u}\right]_1^5$

$= \frac{1}{4}\left[(\ln 5 + \frac{1}{5}) - (\ln 1 + 1)\right]$

$= \frac{1}{4}(\ln 5 - \frac{4}{5})$

$= \frac{1}{4}\ln 5 - \frac{1}{5}$

(d) $\lim_{x\to 0}\frac{\sin 4x}{\tan 2x} = \lim_{x\to 0}\frac{2\sin 2x\cos 2x}{\frac{\sin 2x}{\cos 2x}}$

$= \lim_{x\to 0} 2\cos^2 2x$

$= 2$

QUESTION 3: 12 marks

(a) $f(x) = x^3 + 3x^2 - 10x - 24$

(i) $f(-2) = -8 + 12 + 20 - 24 = 0$

(ii) $\therefore f(x) = (x+2)(x^2 + x - 12)$

$= (x+2)(x+4)(x-3)$

(b) $x^3 - 3x + 5 = 0$

(i) $\alpha + \beta + \gamma = 0$

(ii) $\alpha\beta\gamma = -5$

(iii) $(\alpha-1)(\beta-1)(\gamma-1)$

$= (\alpha\beta - \alpha - \beta + 1)(\gamma - 1)$

$= \alpha\beta\gamma - \alpha\gamma - \beta\gamma + \gamma - \alpha\beta + \alpha + \beta - 1$

$= \alpha\beta\gamma - (\alpha\beta + \beta\gamma + \alpha\gamma) + (\alpha + \beta + \gamma) - 1$

$= -5 + 3 + 0 - 1$

$= -3$

(c) (i) $x^2 = 4ay$

$\therefore y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

At P, $\frac{dy}{dx} = \frac{2ap}{2a} = p$

$\therefore$ Grad norm $= -\frac{1}{p}$

$\therefore$ Eq^n normal:

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$\therefore x + py = ap^3 + 2ap$

(ii) Q lies on normal:

$2aq + paq^2 = ap^3 + 2ap$

$\therefore 2q + pq^2 = p^3 + 2p$

$p^3 - pq^2 + 2p - 2q = 0$

$p(p^2 - q^2) + 2(p - q) = 0$

$(p-q)\left(p(p+q) + 2\right) = 0$

but $p \ne q$ since P and Q are distinct points

$\therefore (p - q) \ne 0$

$\therefore p^2 + pq + 2 = 0$

QUESTION 4: 12 marks

(a) $\left(3x^2 + \frac{1}{x}\right)^9$

$T_{k+1} = {}^9C_k\,(3x^2)^{9-k}\cdot\left(\frac{1}{x}\right)^k$

$= {}^9C_k \cdot 3^{9-k}\cdot x^{18-2k}\cdot x^{-k}$

$= {}^9C_k \cdot 3^{9-k}\cdot x^{18-3k}$

$\therefore$ Coefficient of x^3:

$\Rightarrow 18 - 3k = 3$

$3k = 15$

$k = 5$

$\therefore$ Coefficient $= {}^9C_5 \times 3^{9-5}$

$= 126 \times 3^4$

$= 10206$

(b) $f(x) = 1 + e^{2x}$

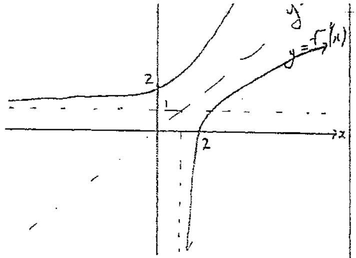
(i) $y > 1$

(ii) $x = 1 + e^{2y}$

$\therefore x - 1 = e^{2y}$

$\ln(x-1) = 2y$

$\therefore y = \frac{1}{2}\ln(x-1)$



(iv) Normal at $(2, 0)$ is

$2x + y - 4 = 0$

ie. $y = 4 - 2x$

$\therefore 4 - 2x = 1 + e^{2x}$ ✓

$\therefore e^{2x} + 2x = 3.$

(v) $f(x) = e^{2x} + 2x - 3$

$f'(x) = 2e^{2x} + 2$ ✓

$f(0.4) = 0.025\ldots$ ✓

$f'(0.4) = 6.451\ldots$

$\therefore a_1 = 0.4 - \dfrac{0.025}{6.451}$

$\doteqdot 0.396$ (correct to 3 significant figures). ✓

(a) $2^{2x+1} - 5(2^x) + 2 = 0$

$2(2^{2x}) - 5(2^x) + 2 = 0$

let $u = 2^x$

$2u^2 - 5u + 2 = 0$ ✓

$(2u - 1)(u - 2) = 0$ ✓

$\therefore u = \frac{1}{2}$ or 2

$\therefore x = -1$ or 1 ✓

(b) $n = 0$:

LHS: $2^3 + 3 = 11$ which is divisible by 11. ✓

Assume true for $n = k$:

ie $2^{10k+3} + 3 = 11M$ for some $M \in \mathbb{Z}^+$

When $n = k+1$:

$2^{10(k+1)+3} + 3$

$= 2^{10k+10+3} + 3$ ✓

$= 2^{10} . 2^{10k+3} + 3$

$= 2^{10}(11M - 3) + 3$ by induction hypothesis ✓

$= 11M.2^{10} - 3.2^{10} + 3$

$= 11M.2^{10} - 3069$

$= 11(2^{10}.M - 279)$ ✓

which is divisible by 11 since $2^{10}.M > 279$ for all $M \in \mathbb{Z}^+$.

$\therefore$ If hypothesis is true for $n = k$, it is also true for $n = k+1$.
Since it is true for $n = 0$, it is also true for $n = 1, 2, \ldots$ and hence all non-negative integers by mathematical induction.

(i) $\dot{x} = \dfrac{5\pi}{2} \cos \dfrac{\pi}{2}(t + \tfrac{1}{3})$ ✓

$\ddot{x} = -\dfrac{5\pi^2}{4} \sin \dfrac{\pi}{2}(t + \tfrac{1}{3})$ ✓

$= -\dfrac{\pi^2}{4} x$

(ii) Amp $= 5$ ✓

Period $= \dfrac{2\pi}{\frac{\pi}{2}} = 4$ ✓

(iii) $\ddot{x} = -\dfrac{\pi^2}{4} \times \dfrac{5}{2}$

$= -\dfrac{5\pi^2}{8}$ ✓

QUESTION 6 : 12 marks

(a) (i) $N = 1000 + Ae^{-kt}$

$\dfrac{dN}{dt} = -k . Ae^{-kt}$ ✓

$= -k(N - 1000)$ ✓

(ii) when $t = 0$, $N = 2500$

$2500 = 1000 + Ae^0$

$\therefore A = 1500$ ✓

when $t = 2$, $N = 2200$

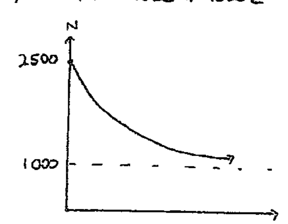
$2200 = 1000 + 1500e^{-2k}$

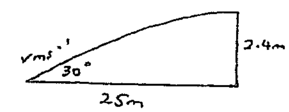
$0.8 = e^{-2k}$

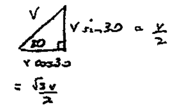
$\therefore k = -\frac{1}{2}\ln 0.8$ ✓

$\doteqdot 0.11157$

(iii) $N = 1000 + 1500e^{-0.1t}$ ✓✓







✓

(i) Horizontal:

$\ddot{x} = 0$

$\therefore \dot{x} = c$

when $t = 0$, $\dot{x} = \dfrac{\sqrt{3}v}{2}$

$\therefore \dot{x} = \dfrac{\sqrt{3}v}{2}$

$x = \dfrac{\sqrt{3}}{2}vt + c$

when $t = 0$, $x = 0$

$\therefore x = \dfrac{\sqrt{3}}{2}vt$

Vertical:

$\ddot{y} = -10$

$\dot{y} = -10t + c$

when $t = 0$, $\dot{y} = \dfrac{v}{2}$

$\therefore \dot{y} = -10t + \dfrac{v}{2}$

$\therefore y = -5t^2 + \dfrac{vt}{2} + c$

when $t = 0$, $y = 0$

$\therefore y = -5t^2 + \dfrac{vt}{2}$ ✓

(ii) $t = \dfrac{2x}{\sqrt{3}v}$

$\therefore y = -5\left(\dfrac{2x}{\sqrt{3}v}\right)^2 + \dfrac{v}{2}\left(\dfrac{2x}{\sqrt{3}v}\right)$

$y = -\dfrac{20x^2}{3v^2} + \dfrac{x\sqrt{3}}{3}$ ✓

(iii) when $x = 25$, $y = 2.4$

$2.4 = -\dfrac{20 \times 25^2}{3v^2} + \dfrac{25\sqrt{3}}{3}$ ✓

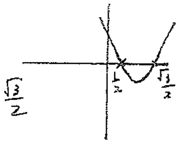
$7.2v^2 = -12500 + 25\sqrt{3}v^2$

$\therefore v^2 = \dfrac{12500}{25\sqrt{3} - 7.2}$

$= 346.248\ldots$

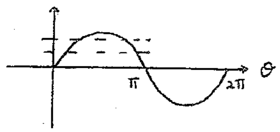
$\therefore v = 18.6\,m/s$ ✓

(i) $(2x-1)(2x-\sqrt{3}) < 0$



$\therefore \quad \frac{1}{2} < x < \frac{\sqrt{3}}{2}$ ✓

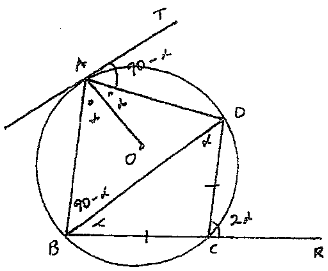
ii) $\therefore \frac{1}{2} < \sin\theta < \frac{\sqrt{3}}{2}$



$\sin\theta = \frac{1}{2}$ $\qquad$ $\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ $\qquad$ $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

$\therefore \quad \frac{\pi}{6} < \theta < \frac{\pi}{3}$ or $\frac{2\pi}{3} < \theta < \frac{5\pi}{6}$ ✓✓



$\angle BDC = \alpha$ ($\angle$ opp = sides in isos. Δ are =)

$\therefore \angle DCR = 2\alpha$ (exterior $\angle$ in Δ = 2 interior opp $\angle$) ✓

$\therefore \angle BAD = 2\alpha$ (exterior $\angle$ in cyclic quad = opp $\angle$)

$\therefore \angle OAD = \alpha$ (OA bisects BAD – given). ✓

$\therefore \angle OAT = 90°$ ($\angle$ between tangent + radius = 90°) ✓

$\therefore \angle DAT = 90 - \alpha$

$\therefore \angle ABD = 90 - \alpha$ ($\angle$ between chord + tangent = $\angle$ in the alt. segment)

$\therefore \angle ABC = 90 - \alpha + \alpha$

$= 90°$. ✓

$\therefore \angle ABC$ is a right angle.

(c) (i) Coefficient $= {}^nC_r$

$= \frac{n!}{(n-r)!\,r!}$ ✓

(ii) $s^{2n} + 2nt - 1$

$= (t-1)^{2n} + 2nt - 1$ ✓

$= \Big(t^{2n} + {}^{2n}C_1 t^{2n-1}(-1)^1 + {}^{2n}C_2 t^{2n-2}(-1)^2 + \dots + {}^{2n}C_{2n-1} t(-1)^{2n-1} + (-1)^{2n}\Big) + 2nt - 1$ ✓

$= \big(t^{2n} - 2nt^{2n-1} + {}^{2n}C_2 t^{2n-2} - \dots + {}^{2n}C_2 t^2 - 2nt + 1\big) + 2nt - 1$

$= t^{2n} - 2nt^{2n-1} + {}^{2n}C_2 t^{2n-2} + \dots + {}^{2n}C_2 t^2$

$= t^2\left(t^{2n-2} - 2nt^{2n-3} + \dots + {}^{2n}C_2\right)$ ✓

$\therefore s^{2n} + 2nt - 1$ is divisible by t^2

(iii) $5^{20} + 119$

$s = 5 \quad \therefore t = 6$

$n = 10$

$\therefore$ By (ii) $5^{20} + 120 - 1$ is divisible by $t^2 = 36$. ✓

END OF SOLUTIONS.