Centre Number

Student Number

SCEGGS Darlinghurst

2006

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Total Marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Find $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4-x^2}}$ 2

(b) Find the acute angle between the lines $y = 3x - 1$ and $2x + y - 2 = 0$ 2

(c) Sketch $y = 3\sin^{-1}\frac{x}{2}$ 2

(d) A and B are two points on the Cartesian Plane and the point M divides the interval AB in the ratio $2 : 3$. 1

In what ratio does B divide the interval AM?

(e) Find $\displaystyle\int \sin^2 x\, dx$ 2

(f) Find the term independent of x in the expansion of 3

$$\left(3x^2 + \frac{2}{x}\right)^9$$

Question 2 begins on page 3 ...

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

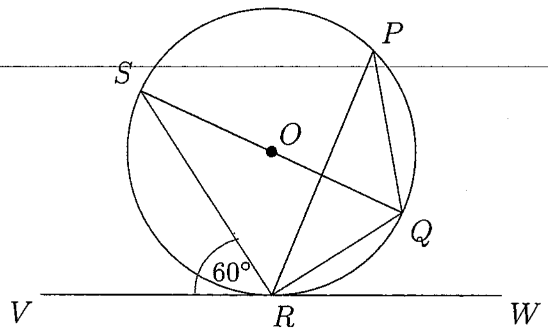
(a) Using the substitution $u = 2x + 1$ find 3

$$\int_{-1}^{0} 2x(2x+1)^3\,dx$$

(b) i. Verify that $(x-4)$ is a factor of the polynomial $P(x) = x^3 - 7x^2 - 6x + 72$. 2

ii. Solve $\dfrac{x+10}{x-4} \le x + 2$ 3

(c) 4



P, Q, R, S are points on the circumference of a circle. SQ passes through the centre of the circle O, VW is a tangent to the circle at R, and $\angle VRS$ is $60°$.

Copy this diagram into your examination booklet and find the value of $\angle QPR$ giving reasons.

Question 3 begins on page 4 . . .

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

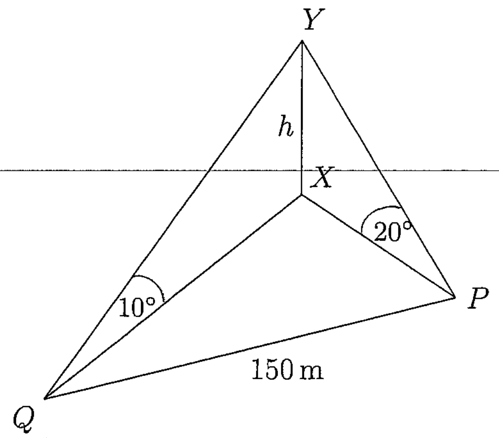
(a) The probability of Julia being late to class is 0.2. She is supposed to attend 20 classes in a week.

The Year Coordinator has said that if Julia is late to three or more classes in a week she will receive a detention.

i. What is the probability that Julia is late exactly twice in a week? 1

ii. What is the probability that Julia receives a detention for being late? 2

(b)



Beatrice is bushwalking with her brother Bertrand along a road PQ which runs directly south-west. From point P, a hill XY has a bearing of 315°T and the angle of elevation to the top of the hill is 20°. On walking 150 metres further along this road to Q, they measure the angle of elevation of the same hill to be 10°.

Let the height of the hill XY be h metres.

i. Find an expressions for QX in terms of h. 1

ii. Show that the height of the hill is given by 2

$$h = \frac{150}{\sqrt{\cot^2 10^\circ - \cot^2 20^\circ}}$$

iii. Hence calculate the height of the hill to the nearest metre. 1

Question 3 continues on page 5 ...

(c) In an episode of *Another One Bites the Dust*, Detective Smith is called to a murder scene at 3:27 am. He measures the victim's body temperature at that time to be 27°C and one hour later it has dropped to 25°C.

The cooling rate of the body is proportional to the difference between the room temperature (21°C) and the temperature, T, of the body. That is, T satisfies the equation

$$\frac{dT}{dt} = -k(T - 21)$$

where k is a constant and t is the number of hours after 3:27 am.

i. Verify that $T = 21 + Ae^{-kt}$ is a solution of this equation, where A is a constant. 1

ii. Find the exact values of A and k. 2

iii. Assuming that the victim's body temperature was 37°C at the time of death, when was the murder committed? Give your answer to the nearest minute. 2

Question 4 begins on page 6 . . .

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) If α, β, and γ are the roots of the equation $2x^3 - x^2 - 5x + 4 = 0$ find the values of

i. $\alpha\beta + \alpha\gamma + \beta\gamma$ 1

ii. $\alpha^2 + \beta^2 + \gamma^2$ 2

(b) i. Sketch the graph of the function $\log_e(x-2)$ 1

ii. The region bounded by the curve $y = \log_e(x-2)$, the y-axis, $y = 0$ and $y = h$ is rotated about the y-axis to create a bowl. Show that the volume of the bowl, V, is given by: 3

$$V = \pi\left(\frac{e^{2h}}{2} + 4e^h + 4h - \frac{9}{2}\right)$$

iii. The bowl is placed with its axis vertical and water is poured in. If water is poured into the bowl at a rate of $50\,\text{cm}^3/\text{sec}$, find the rate at which the water level is rising when the height of the water is 1.5 cm (answer correct to 3 decimal places). 2

(c) i. Gladys and Trent are getting married and need to choose four bridesmaids and four groomsmen from their seven female and seven male siblings/friends. How many ways can this be done if Trent's brother must be included in the bridal party? 1

ii. How many ways can the four bridesmaids and four groomsmen be paired up for the wedding? 1

iii. The reception is at a Chinese restaurant and the bridal party of ten is to be seated at a round table. How many seating arrangements are possible if the only restriction is that the bride and groom are to sit together? 1

Question 5 begins on page 7 ...

Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) i. Write $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$. 2

ii. Hence or otherwise find all solutions of 2

$$\cos x - \sqrt{3}\sin x = 1$$

(b) Consider the function $f(x) = \dfrac{x}{x^2 - 1}$

i. State any vertical asymptotes of the curve $y = f(x)$. 1

ii. Show that $f(x)$ is a decreasing function. 1

iii. Hence sketch the curve $y = f(x)$. Do not use any further calculus. 1

iv. State a possible domain for which $f(x)$ has an inverse function, $f^{-1}(x)$, and on a new set of axes sketch $y = f^{-1}(x)$ for the chosen domain. 2

(c) Using the Principle of Mathematical Induction, prove that for all positive integers n, 3

$$1 + 27 + 189 + \ldots + (2n^2 + 2n - 3)3^{n-1} = (n^2 - 1)3^n + 1$$

Question 6 begins on page 8 ...

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$.

i. Show that the equation of the tangent at P is given by 2

$$px - y - ap^2 = 0$$

ii. The tangent at P cuts the x-axis at X. Find the coordinates of X. 1

iii. Hence show that PX is perpendicular to SX, where S is the focus of the parabola. 2

iv. A circle is drawn through the points S, X and P. 2

Show that the coordinates of the centre of the circle are given by

$$C = \left(ap, \frac{a(1+p^2)}{2}\right)$$

~~Justify your answer.~~

v. Find the locus of C in Cartesian form. 1

(b) i. Write down the binomial expansion of $(1+x)^n$ in increasing powers of x. 1

ii. By integrating both sides of the identity in part (i), show that 2

$$\frac{2^{n+1}-1}{n+1} = \frac{1}{1}\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + \ldots + \frac{1}{n+1}\binom{n}{n}$$

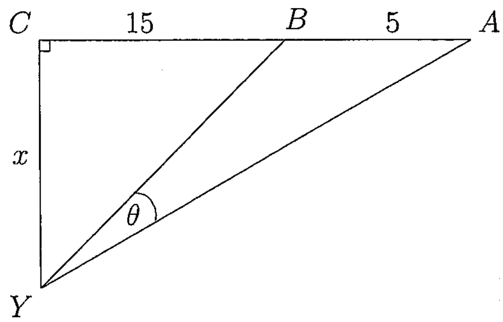
iii. Hence find an expression for the sum 1

$$\frac{2}{1}\binom{n}{0} + \frac{3}{2}\binom{n}{1} + \frac{4}{3}\binom{n}{2} + \frac{5}{4}\binom{n}{3} + \ldots + \frac{n+2}{n+1}\binom{n}{n}$$

Question 7 begins on page 9 ...

Question 7 (12 marks) Use a SEPARATE writing booklet **Marks**

(a)



The diagram shows the try line of a football field with goal posts A and B 5 metres apart. A rugby league player scores a try at C on the try line, 15 metres from the left hand goal posts. After the try, the kicker may score two more points by kicking the ball over the cross-bar of the goal posts from any position along the line CY.

i. If the kicker places the ball at Y, x metres from C, show that the kicking angle θ can be expressed as 1

$$\theta = \tan^{-1}\left(\frac{20}{x}\right) - \tan^{-1}\left(\frac{15}{x}\right)$$

ii. Show that θ is a maximum when $x = 10\sqrt{3}\,\text{m}$ 2

iii. Hence show that the maximum kicking angle is $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{12}\right)$ 2

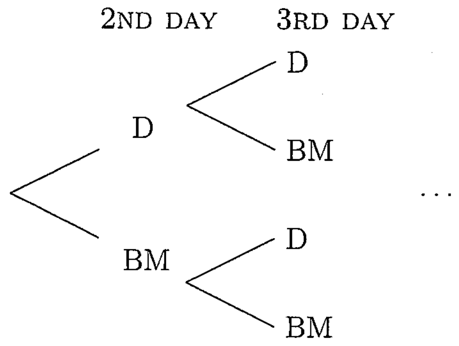
Question 7 continues on page 10 ...

(b) A man inherits two houses on his 21st birthday, one in Darlinghurst (D) and one in the Blue Mountains (BM). The Darlinghurst house has a box containing one red and two white balls. The Blue Mountains house has a similar box containing one red and three white balls. On the first day after his birthday he moves into his Darlinghurst house, and on each day, starting this day, he draws a ball out of the box, notes its colour and replaces it. If he draws out a red ball he spends the next day in his other house. If he draws out a white ball he stays where he is and awaits the outcome of the next drawing.

Let P_n be the probability that he is in the Darlinghurst house on the nth day after his birthday.

i. Explain why the probability he is in the Blue Mountains house on the nth day is given by $1 - P_n$ 1

ii. Copy the tree diagram below into your writing booklet and complete it by including probability values for each branch given. 1



iii. Hence show that $P_n = \frac{2}{3}P_{n-1} + \frac{1}{4}(1 - P_{n-1})$ 1

iv. Given that $P_1 = 1$ and that the expression in part (iii) simplifies to $P_n = \frac{1}{4} + \frac{5}{12}P_{n-1}$, show that P_4 is given by the expression 2

$$P_4 = \tfrac{1}{4}[1 + (\tfrac{5}{12}) + (\tfrac{5}{12})^2] + (\tfrac{5}{12})^3$$

v. Show that $P_n = \frac{3}{7} + \frac{4}{7}(\frac{5}{12})^{n-1}$ 2

End of paper

Question 1

Calc /4
Comm /2
Reas /1

(a) $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_1^{\sqrt{2}}$ ✓

$= \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$

$= \pi/4 - \pi/6$

$= \pi/12$ ✓

• A standard integral
• Answer must be in radians.

(Calc 2)

(b) $y = 3x - 1 \Rightarrow m_1 = 3$

$2x + y - 2 = 0 \Rightarrow m_2 = -2$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

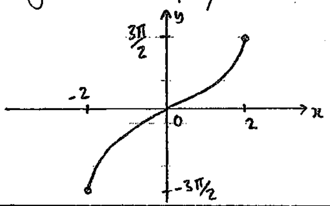
$\tan\theta = \left|\frac{3 - -2}{1 + 3 \times -2}\right|$ ✓

$\tan\theta = 1$

$\theta = 45°$ ✓

• If you're going to use a formula make sure you know it off by heart!!

(c) $y = 3\sin^{-1}\left(\frac{x}{2}\right)$

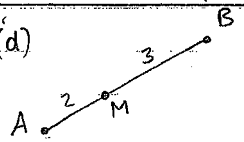


✓ correct D & R

✓ correct shape

(Comm 2)

(d)



B divides AM in ratio $5 : -3$

or $-5 : 3$ ✓

or $5 : 3$ externally

The word externally must be included if you didn't put minus sign.

(Reas 1)

(e) $\int \sin^2 x \, dx = \frac{1}{2}\int 1 - \cos 2x \; dx$ ✓

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$ ✓

• A standard integral question in Ext①. You must know how to do this.
✱ $\int \sin^2 x\,dx =$
$\int \cos^2 x\,dx =$

(Calc 2)

(f) $\left(3x^2 + \frac{2}{x}\right)^9$

$T_{k+1} = \binom{9}{k}(3x^2)^{9-k}\left(\frac{2}{x}\right)^k$

$= \binom{9}{k} 3^{9-k} \; x^{18-2k} \; 2^k \; x^{-k}$

$= \binom{9}{k} 3^{9-k} \; 2^k \; x^{18-3k}$ ✓

For term independent of x : $18 - 3k = 0$

$k = 6$ ✓

∴ Term independent of x : $T_7 = \binom{9}{6} 3^3 \, 2^6$ ✓

$= 145\,152$

OR

$T_{k+1} = \binom{9}{k}(3x^2)^r(2x^{-1})^{9-r}$

$= \binom{9}{k} 3^r x^{2r} \, 2^{9-r} \, x^{-9+r}$

$= \binom{9}{k} 3^r . 2^{9-r} . x^{3r-9}$

For term independent of x: $3r - 9 = 0$

$r = 3$

∴ The term is $\binom{9}{3} 2^6 . 3^3$

$= 145152.$

Another method is to expand the bracket but that takes longer.

Calc /3
Comm /2
Reas /4

Question 2

(a) $\int_{-1}^{0} 2x(2x+1)^3\,dx$

$u = 2x+1$, $du = 2\,dx$

$x=-1 \Rightarrow u=-1$
$x=0 \Rightarrow u=1$

$= \int_{-1}^{1} \frac{u-1}{2} \times u^3 \, du$ ✓

$= \frac{1}{2}\int_{-1}^{1} u^4 - u^3 \, du$

$= \frac{1}{2}\left[\frac{u^5}{5} - \frac{u^4}{4}\right]_{-1}^{1}$ ✓

$= \frac{1}{2}\left[\left(\frac{1}{5}-\frac{1}{4}\right) - \left(\frac{-1}{5}-\frac{1}{4}\right)\right]$

$= \frac{1}{5}$ ✓

Too many mistakes with these fractions!!!

(Calc 3)

(b)(i) $P(x) = x^3 - 7x^2 - 6x + 72$

$P(4) = 4^3 - 7\times4^2 - 6\times4 + 72$
$= 0$ ✓

$\therefore (x-4)$ is a factor of $P(x)$ by the factor theorem ✓

State a clear conclusion for factor theorem.

(Comm 2)

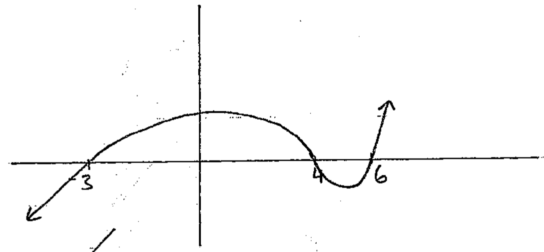
OR:

$$\begin{array}{r} x^2 - 3x - 18 \\ x-4 \,\big)\, x^3 - 7x^2 - 6x + 72 \\ x^3 - 4x^2 \\ \hline -3x^2 - 6x + 72 \\ -3x^2 + 12x \\ \hline -18x + 72 \\ -18x + 72 \\ \hline 0 \end{array}$$ ✓

$\therefore x^3 - 7x^2 - 6x + 72 = (x-4)(x^2-3x-18)$
$\therefore (x-4)$ is a factor of $P(x)$ ✓

(ii) $\frac{x+10}{x-4} \le x+2$ $\qquad x \ne 4$

$(x+10)(x-4) \le (x+2)(x-4)^2$
$0 \le (x+2)(x-4)^2 - (x+10)(x-4)$
$0 \le (x-4)[(x+2)(x-4) - (x+10)]$
$0 \le (x-4)[x^2 - 3x - 18]$
$0 \le (x-4)(x-6)(x+3)$ ✓



$x \ne 4$ ✓, $-3 \le x \le 4$, $x \ge 6$

$\therefore -3 \le x < 4$, $x \ge 6$ ✓

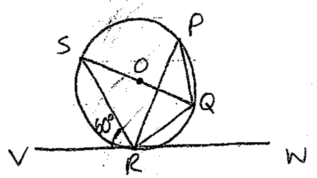
(*) Don't forget to write where the function is undefined first
An easy mark!

• Mark paid for the expansion to give $x^3-7x^2-6x+72$ ...
It must be correct expr...

• If you expand, think about the connection this part has with part (i).

(Reas 4)

(c)



$\angle SQR = \angle SRV = 60°$ ($\angle$ bet. tang. & chord = $\angle$ in alt. se...) ✓
$\angle SRQ = 90°$ ($\angle$ in semicircle $= 90°$) ✓
$\angle QSR = 180° - 90° - 60°$ ($\angle$ sum $\triangle QSR = 180°$) ✓
$= 30°$
$\angle QPR = \angle QSR = 30°$ ($\angle$ in same segment =) ✓

$\therefore \angle QPR = 30°$

(*) Well done but please learn the correct wording for reasons and angles in the circle.

Question 3 Calc 1 Reas 2

Comments: Binomial Probability questions are quite straightforward & are not the type of questions to be skipped.

(a) (i) let p = prob(late) = 0·2

q = prob(not late) = 0·8

$n = 20$

Consider $(p+q)^{20}$

Prob (late exactly twice) = $\binom{20}{2} p^2 q^{18}$

$= \binom{20}{2} 0{\cdot}2^2\, 0{\cdot}8^{18}$ ✓

$\doteqdot 0{\cdot}1369$ (to 4 d.p.)

(ii) Prob (Julia receives a detention)

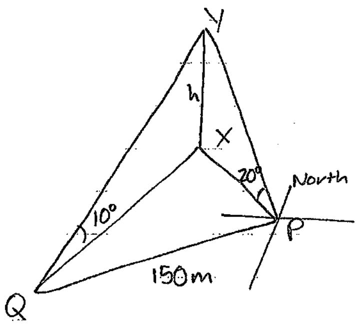
= Prob (Julia late ≥ 3 times)

= 1 − Prob (Julia late 0 time) − Prob (late once) − Prob (late 2x) ✓

$= 1 - \binom{20}{0} 0{\cdot}8^{20} - \binom{20}{1} 0{\cdot}2 \times 0{\cdot}8^{19} - \binom{20}{2} 0{\cdot}2^2 \times 0{\cdot}8^{18}$ ✓

$\doteqdot 0{\cdot}7939$ (to 4 d.p.)

(b)



Comments:
b) i) Well done
ii) It is ∠QPX that is 90°, not ∠QXP. Read the description in the question carefully
iii) There is no excuse for not being able to get this mark — learn to use your calculator!

(i) $QX = \dfrac{h}{\tan 10°}$ ✓

$QX = h\cot 10°$

(ii) $PX = h\cot 20°$ ✓

∠QPX = 90° (since X is on a bearing of 315° from P & PQ runs south west)

∴ by Pythagoras in △PQX

$150^2 + h^2\cot^2 20° = h^2\cot^2 10°$

$150^2 = h^2(\cot^2 10° - \cot^2 20°)$

$h = \dfrac{150}{\sqrt{\cot^2 10° - \cot^2 20°}}$ ✓

(iii) $h = 20{\cdot}0238\ldots \doteqdot 20$ (to the nearest metre). ✓

(c) 3:27 am $t = 0$ $T = 27°C$

$t = 1$ $T = 25°C$

$\dfrac{dT}{dt} = -k(T-21)$

(i) $T = 21 + Ae^{-kt}$

$\dfrac{dT}{dt} = -kAe^{-kt}$

$= -k(T-21)$ (since $T - 21 = Ae^{-kt}$)

∴ it is a solution. ✓

(ii) $t = 0$ $T = 27$

⇒ $27 = 21 + Ae^{-k\times 0}$

$6 = A$ ✓

$t = 1$ $T = 25$

$25 = 21 + 6e^{-k\times 1}$

$2/3 = e^{-k}$

$k = -\ln 2/3$

$k = \ln 3/2$ ✓

Comments:
- Generally very well done
- Many students could not solve an equation that required logarithms
- Conversion of 2.419…h to 2 hr 25 min (on calculator) was done well.

(iii) $T = 37°C$ $t = ?$

$37 = 21 + 6e^{-\ln 3/2\, t}$

$16/6 = e^{-\ln 3/2\, t}$

$\ln(16/6) = -\ln 3/2\, t$

$t = \dfrac{\ln(16/6)}{-\ln(3/2)}$

$= -2{\cdot}419\ldots$ hrs ✓

$\doteqdot -2$ hrs 25 min

∴ Murder was committed 2 hrs 25 min before 3:27 am, ie. 1:02 am ✓

Question 4

Calc /5
Comm /1
Reas /3

(a) $2x^3 - x^2 - 5x + 4 = 0$

(i) $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} = \frac{-5}{2}$ ✓

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ ✓

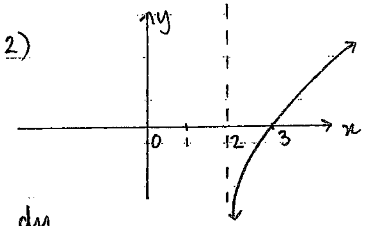
$= \left(\frac{1}{2}\right)^2 - 2\left(-\frac{5}{2}\right)$

$= \frac{1}{4} + 5$

$= 5\frac{1}{4}$ ✓

mark awarded for correct substitution of $\alpha+\beta+\gamma = \frac{1}{2}$ (squared)

(b) (i) $y = \log_e(x-2)$



✓

You must show the vertical asymptote at $x=2$ to be awarded this mark.

(Comm 1)

(ii) $V = \pi \int_0^h x^2 \, dy$

$= \pi \int_0^h (e^y + 2)^2 \, dy$ ✓

$= \pi \int_0^h e^{2y} + 4e^y + 4 \, dy$

$= \pi \left[\frac{e^{2y}}{2} + 4e^y + 4y \right]_0^h$ ✓

$= \pi \left[\left(\frac{e^{2h}}{2} + 4e^h + 4h \right) - \left(\frac{1}{2} + 4 + 0 \right) \right]$

$= \pi \left(\frac{e^{2h}}{2} + 4e^h + 4h - 4\frac{1}{2} \right)$ ✓

(Calc 3)

iii)

$\frac{dv}{dt} = 50 \text{ cm}^3/\text{s}$

$\frac{dh}{dt} = ?$

when $h = 1.5$ cm.

$$\boxed{\frac{dv}{dt} = \frac{dv}{dh} \cdot \frac{dh}{dt}}$$

$\frac{dh}{dt} = \frac{\frac{dv}{dt}}{\frac{dv}{dh}} = \frac{dh}{dv} \times \frac{dv}{dt}$

using part (ii)

$V = \pi \left(\frac{e^{2h}}{2} + 4e^h + 4h - 4\frac{1}{2} \right)$

$\frac{dV}{dh} = \pi \left(e^{2h} + 4e^h + 4 \right)$

$\frac{dh}{dV} = \frac{1}{\pi(e^{2h} + 4e^h + 4)}$ ✓

To be awarded this mark you must show evidence of the chain rule and a link to $\frac{dV}{dh}$ from part (ii).

$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$

$= \frac{1}{\pi(e^{2h} + 4e^h + 4)} \cdot 50$

$= \frac{1}{\pi(e^3 + 4e^{1.5} + 4)} \cdot 50$

$\doteq 0.379$ cm/sec to 3 dp. ✓

(Calc 2)

c) (i)

\# ways to choose bridal party $= \binom{6}{3}\binom{7}{4} = 700$ ✓

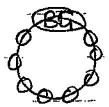
(↑ males, ↑ females)

(ii) \# ways to be paired up $= 4 \times 3 \times 2 \times 1 \ldots$

(4: ways to choose ~~male for~~ 1st maid; 3: ways to choose ~~male for~~ 2nd maid)

$= 4!$ ✓

(iii)



\# ways to be seated $= 1 \times 8! \times 2!$

(1: sit B&G as couple; 8!: sit others around them; 2!: swap B&G)

$= 8!\,2!$

$= 80640$ ✓

(Reas 3)

Question 5

Calc /1
Comm /2
Reas /2

(a) (i) $\cos x - \sqrt{3}\sin x$

$R\cos x\cos\alpha - R\sin x\sin\alpha = R\cos(x+\alpha)$

$R\cos\alpha = 1$
$R\sin\alpha = \sqrt{3}$ $\Rightarrow$ $R = 2$ ✓, $\alpha = \pi/3$ ✓

$\therefore \cos x - \sqrt{3}\sin x = 2\cos(x + \pi/3)$

(ii) $\cos x - \sqrt{3}\sin x = 1$

$2\cos(x+\pi/3) = 1$

$\cos(x+\pi/3) = 1/2$

$(x+\pi/3) = 2\pi n \pm \cos^{-1}(\tfrac{1}{2})$ for all integers n

$x + \pi/3 = 2\pi n \pm \pi/3$

$x = 2\pi n \pm \pi/3 - \pi/3$

$x = 2\pi n,\ 2\pi n - 2\pi/3$ for all integers n

Comments: If you are to use a formula for the general solution it must be done at this step

(b) $f(x) = \dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)}$

(i) Vertical asymptotes: $x = 1$ & $x = -1$ ✓

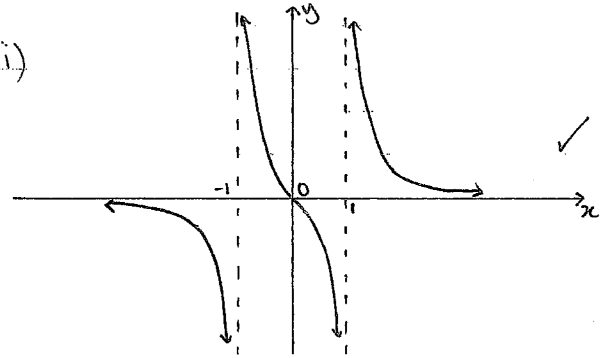
Comments: The equations of the asymptotes are NOT $x \neq 1$ & $x \neq -1$

(ii) $f'(x) = \dfrac{(x^2-1)\cdot 1 - x(2x)}{(x^2-1)^2}$

$= \dfrac{-x^2-1}{(x^2-1)^2}$ ✓

< 0 $\therefore f(x)$ is decreasing

(iii)



✓

Comments: you should check your sketch is consistent with the information in parts i & ii

(iv) Various answers

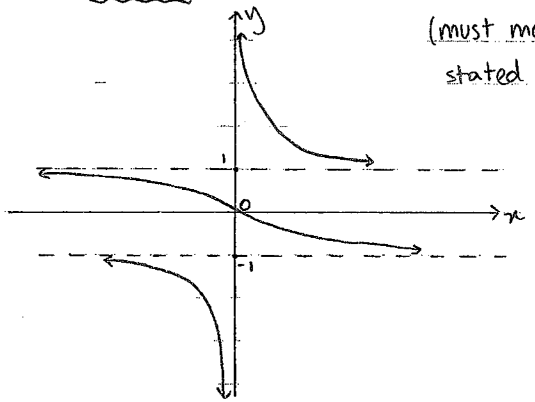
eg. $x \geq 0,\ x \neq 1$

$-1 < x < 1$

$x \leq 0,\ x \neq -1$ ✓

Comments: Answers that included $x=1, x=-1$ in the domain were not accepted eg. $-1 \leq x \leq 1$. Be careful!

Inverse relation looks like: (must match domain stated above)



✓

(c) Prove $1 + 27 + 189 + \cdots + (2n^2+2n-3)3^{n-1} = (n^2-1)3^n + 1$

Test $n=1$ LHS $= (2\times1^2 + 2\times1 - 3)3^{1-1}$
$= 1$

RHS $= (1^2-1)3^1 + 1$
$= 1$

$\therefore$ True for $n = 1$

Assume true for $n = k$

ie. assume $1 + 27 + 189 + \cdots + (2k^2+2k-3)3^{k-1} = (k^2-1)3^k +$

Prove true for $n = k+1$

ie. prove $1+27+189+\cdots+(2k^2+2k-3)3^{k-1} + (2(k+1)^2+2(k+1)-3)3^k = ((k+1)^2-1)3^k + 1$

ie. $1+27+189+\cdots+(2k^2+2k-3)3^{k-1} + (2k^2+6k+1)3^k = (k^2+2k)3^{k+1} + 1$

$LHS = (k^2-1)3^k + 1 + (2k^2+6k+1)3^k$ (using assumption)

$= 3^k(k^2-1+2k^2+6k+1) + 1$

$= 3^k(3k^2+6k) + 1$

$= 3^{k+1}(k^2+2k) + 1$

$= RHS$

$\therefore$ If true for $n=k$, it will be true for $n=k+1$.
Since true for $n=1$, it will be true for $n=2, 3, 4, \ldots$ etc.
$\therefore$ by the PMI it will be true for all positive integers n.

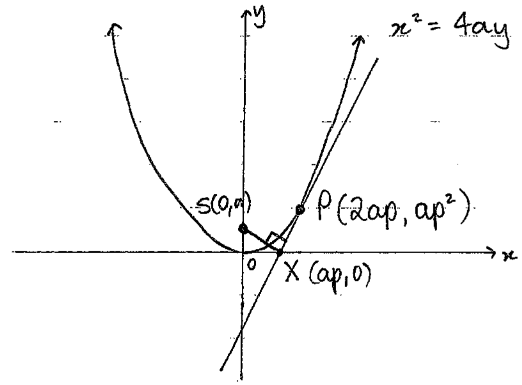
> Comments:
> There was certainly some fudging going on!
> And index rules still got people into trouble.

Question 6

Calc /2
Reas /5

It is always recommended to DRAW A CLEAR DIAGRAM. It is far easier to visualise the question.

(a)



(i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$y' = \frac{x}{2a}$

$m_T @ P = \frac{2ap}{2a} = p$ ✓

$Pt = (2ap, ap^2)$

Equation of tangent at P

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$0 = px - y - ap^2$ ✓

Standard bookwork. A very easy question!

(Calc 2)

(ii) x int. $y = 0$

$0 = px - 0 - ap^2$

$ap^2 = px$

$ap = x$ ✓

$\therefore X = (ap, 0)$

Another very easy mark. The equation is given on the paper and all you have to do is substitute $y=0$.

(iii) $S(0,a)$
$P(2ap, ap^2)$
$X(ap, 0)$

$$m_{PX} \times m_{SX} = \left(\frac{ap^2-0}{2ap-ap}\right) \times \left(\frac{a-0}{0-ap}\right)$$
$$= \frac{ap^2}{ap} \times \frac{a}{-ap}$$
$$= -1$$
$\therefore PX \perp SX$

Mark awarded for correct method and use of at least 2 out of 3 correct points.

(iv) Since $\angle$ in a semicircle $= 90°$, SP is the diameter of the circle passing through S, X, P

$\therefore$ centre $=$ midpt of SP
$$= \left(\frac{0+2ap}{2}, \frac{a+ap^2}{2}\right)$$
$$= \left(ap, \frac{a(1+p^2)}{2}\right)$$

Use circle geometry facts to justify your answer.
THINK! How does this part connect with part (iii).
(Reas 2)

(v) C: $x = ap$ ①
$y = \frac{a(1+p^2)}{2}$ ②

① $\Rightarrow p = \frac{x}{a}$

sub in ② $\Rightarrow$ $y = \frac{a\left(1 + \frac{x^2}{a^2}\right)}{2}$
$$y = \frac{a}{2} + \frac{x^2}{2a}$$

This is an easy question. It is only worth one mark so if you fill 2 pages something has gone wrong!

HERE'S A COUPLE OF DIFFERENT SOLUTIONS

i) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$

ii) Integrate both sides w.r.t x

$$\int (1+x)^n \, dx = \int \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n \, dx$$
$$\frac{(1+x)^{n+1}}{n+1} = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \ldots + \binom{n}{n}\frac{x^{n+1}}{n+1} + C$$

You must introduce a constant of integration on either side.

Substitute $x=0$ to evaluate C the constant of integration.

$$\frac{1^{n+1}}{n+1} = 0 + C$$
$$\therefore C = \frac{1}{n+1}$$

$$\therefore \frac{(1+x)^{n+1}}{n+1} = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \ldots + \binom{n}{n}\frac{x^{n+1}}{n+1} + \frac{1}{n+1}$$

← LHS.

Substitute $x=1$

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \binom{n}{0} + \binom{n}{1}\frac{1}{2} + \binom{n}{2}\frac{1}{3} + \ldots + \binom{n}{n}\frac{1}{n+1}$$

$$\frac{2^{n+1}-1}{n+1} = \frac{1}{1}\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{1}{n+1}\binom{n}{n}$$

Nice try if you fudge here but it's impossible to get away with!

(b) (i) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$ ✓

A very easy mark.

(ii) $\int_0^1 (1+x)^n\,dx = \int_0^1 \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n\,dx$

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[\binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \dots + \binom{n}{n}\frac{x^{n+1}}{n+1}\right]_0^1$$ ✓

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \binom{n}{0} + \binom{n}{1}\times\frac{1}{2} + \binom{n}{2}\times\frac{1}{3} + \dots + \binom{n}{n}\times\frac{1}{n+1}$$

$$\frac{2^{n+1}-1}{n+1} = \frac{1}{1}\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + \frac{1}{n+1}\binom{n}{n}$$ ✓

This is a neat method. By finding definite integrals you do not have to find constant of integration.

(Reas 2)

(iii) $\frac{2^{n+1}-1}{n+1} = \frac{1}{1}\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + \frac{1}{n+1}\binom{n}{n}$ ① from (ii)

sub $x=1$ into part (i) $\Rightarrow$

$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$ ②

① + ② $\Rightarrow$

$$\boxed{\frac{2^{n+1}-1}{n+1} + 2^n} = \frac{2}{1}\binom{n}{0} + \frac{3}{2}\binom{n}{1} + \frac{4}{3}\binom{n}{2} + \dots + \frac{n+2}{n+1}\binom{n}{n}$$

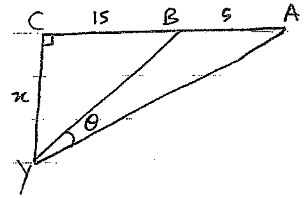
✓

(Reas 1)

Calc /2
Comm /1
Reas /8

Question 7

(a)



(i) $\theta = \angle AYC - \angle BYC$ ✓

$\theta = \tan^{-1}\left(\frac{20}{x}\right) - \tan^{-1}\left(\frac{15}{x}\right)$ since $\tan(\angle AYC) = \frac{20}{x}$

& $\tan(\angle BYC) = \frac{15}{x}$

(ii) For max/min $\frac{d\theta}{dx} = 0$

$$\frac{d\theta}{dx} = \frac{1}{\left(\frac{20}{x}\right)^2+1} \times \left(\frac{-20}{x^2}\right) - \frac{1}{\left(\frac{15}{x}\right)^2+1} \times \left(\frac{-15}{x^2}\right)$$ ✓

$$0 = \frac{-20}{400+x^2} + \frac{15}{225+x^2}$$

$$\frac{20}{400+x^2} = \frac{15}{225+x^2}$$

$4500 + 20x^2 = 6000 + 15x^2$

$5x^2 = 1500$

$x^2 = 300$

$x = \pm 10\sqrt{3}$ m

$x = 10\sqrt{3}$ m (since x is +ve distance) ✓

x	15	$10\sqrt{3}$	20
$\frac{d\theta}{dx}$	0.001…	0	−0.001…
	/	–	\

∴ maximum.

comments:
Many tried to differentiate $\tan^{-1}\left(\frac{20}{x}\right)$ by somehow reversing what is on the std integral page — this only caused grief!
Remember: $\tan^{-1}$(stuff)
↓ differentiates
$\frac{1}{1+(\text{stuff})^2} \times$ inside differentiated.

(iii) For max. $x = 10\sqrt{3}$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{20}{10\sqrt{3}}\right) - \tan^{-1}\left(\frac{15}{10\sqrt{3}}\right)$$

$$\theta = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{3}{2\sqrt{3}}\right)$$

$\therefore$ To show:

$$\tan^{-1}\left(\frac{\sqrt{3}}{12}\right) = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{3}{2\sqrt{3}}\right)$$

$$\tan(\text{LHS}) = \tan\left(\tan^{-1}\left(\frac{\sqrt{3}}{12}\right)\right)$$

$$= \frac{\sqrt{3}}{12} \quad ✓$$

$$\tan(\text{RHS}) = \tan(\alpha - \beta)$$

let $\alpha = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$\beta = \tan^{-1}\left(\frac{3}{2\sqrt{3}}\right)$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$= \frac{\frac{2}{\sqrt{3}} - \frac{3}{2\sqrt{3}}}{1 + \frac{2}{\sqrt{3}} \times \frac{3}{2\sqrt{3}}} \quad ✓$$

$$= \frac{\frac{1}{2\sqrt{3}}}{1 + 1}$$

$$= \frac{1}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{12}$$

$\therefore$ LHS = RHS

$\therefore \quad \theta = \tan^{-1}\left(\frac{\sqrt{3}}{12}\right)$ is the maximum kicking angle.

(b) Darlinghurst 1R 2W   R → moves

Blue Mtns 1R 3W   W → stays

(i) Since on any given day he can only be in the D or the BM house ✓

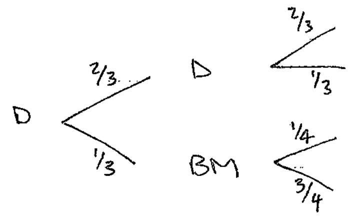
$P(D) + P(BM) = 1$

$\therefore P(\text{BM on nth day}) = 1 - P(\text{city on nth day})$

$= 1 - P_n$

Comments: simply restating what was given in the question was not enough — had to state, somehow that being in D or BM were complementary events & $\therefore$ added to 1

(ii)



✓

Comments: This question just required very careful reading of the question.

(iii) $P_n = \text{Prob (D on nth day)}$

$= \text{Prob (D on (n-1)th day \& draws a W)}$ ✓

$+ \text{Prob (BM on (n-1)th day \& draws a R)}$

$$= \frac{2}{3}P_{n-1} + \frac{1}{4}(1 - P_{n-1})$$

(iv) $$P_n = \frac{1}{4} + \frac{5}{12}P_{n-1}$$

Comments: done well by those who attempted it. It is easier to build the expression from P_1, rather than started with P_4 & accumulate several layers of brackets.

$P_1 = 1$

$$P_2 = \frac{1}{4} + \frac{5}{12} \times 1$$

$$P_3 = \frac{1}{4} + \frac{5}{12}\left(\frac{1}{4} + \frac{5}{12}\right)$$

✓ method

✓ perfect execution

$$= \frac{1}{4} + \frac{5}{12} \times \frac{1}{4} + \left(\frac{5}{12}\right)^2$$

$$P_4 = \frac{1}{4} + \frac{5}{12}\left(\frac{1}{4} + \frac{5}{12} \times \frac{1}{4} + \left(\frac{5}{12}\right)^2\right)$$

$$= \frac{1}{4} + \left(\frac{5}{12}\right) \times \frac{1}{4} + \left(\frac{5}{12}\right)^2 \times \frac{1}{4} + \left(\frac{5}{12}\right)^3$$

$$= \frac{1}{4}\left[1 + \left(\frac{5}{12}\right) + \left(\frac{5}{12}\right)^2\right] + \left(\frac{5}{12}\right)^3$$

(v) $P_n = \frac{1}{4}\left[1+\left(\frac{5}{12}\right)+\left(\frac{5}{12}\right)^2+\cdots+\left(\frac{5}{12}\right)^{n-2}\right]+\left(\frac{5}{12}\right)^{n-1}$

$$= \frac{1}{4} \times \frac{1\left(1-\left(\frac{5}{12}\right)^{n-1}\right)}{1-5/12} + \left(\frac{5}{12}\right)^{n-1}$$ ✓

$$= \frac{1}{4} \times \frac{12}{7}\left(1-\left(\frac{5}{12}\right)^{n-1}\right)+\left(\frac{5}{12}\right)^{n-1}$$

$$= \frac{3}{7} - \frac{3}{7}\times\left(\frac{5}{12}\right)^{n-1}+\left(\frac{5}{12}\right)^{n-1}$$

$$= \frac{3}{7} + \frac{4}{7}\times\left(\frac{5}{12}\right)^{n-1}$$ ✓

<u>Comments</u>:

- there are only $(n-1)$ terms in $1+\left(\frac{5}{12}\right)+\cdots+\left(5/12\right)^{n-2}$