THE SCOTS COLLEGE

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1999

3/4 UNIT MATHEMATICS

TIME ALLOWED: TWO HOURS
[plus 5 minutes reading time]

INSTRUCTIONS TO CANDIDATES:

- ALL QUESTIONS ARE TO BE ATTEMPTED.
- ALL QUESTIONS ARE OF EQUAL VALUE.
- ALL NECESSARY WORKING SHOULD BE SHOWN FOR EACH QUESTION.
- NON-PROGRAMMABLE CALCULATORS ARE PERMITTED.
- A TABLE OF STANDARD INTEGRALS IS PROVIDED.

BOOKLET ORDER:

BOOKLET 1 : QUESTIONS 1 & 2
BOOKLET 2 : QUESTIONS 3 & 4
BOOKLET 3 : QUESTIONS 5, 6 & 7

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Exam continues over

QUESTION 1

(a) Differentiate $e^x \cos(e^{-x})$. [2 MARKS]

(b) If $\log_2 3 = b$ write $\log_4 27$ in terms of b. [1 MARK]

(c) If u, v and w are the roots of $x^3 - 4x + 1 = 0$, find the value of $\frac{1}{u} + \frac{1}{v} + \frac{1}{w}$. [2 MARKS]

(d) Find the exact value of $\tan^{-1}\sqrt{3} - \tan^{-1}(-1)$. [2 MARKS]

(e) Solve the inequality $\frac{5}{2x-1} \geq 1$. [3 MARKS]

(f) Find the distance between $y = 2x + 1$ and $2x - y + 8 = 0$. [2 MARKS]

QUESTION 2

(a) Find the primitive of $\sin^2 \frac{x}{2}$. [2 MARKS]

(b) Find the value of $\int_{\frac{\pi}{8}}^{\frac{7\pi}{6}} \tan 2x \sec^2 2x$ using the substitution $u = \tan 2x$. [3 MARKS]

(c) P $(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$. The tangent at P cuts the axis of the parabola at M and G is the foot of the perpendicular from P to the axis of the parabola. Prove that M and G are equidistant from the vertex for all positions of P. [4 MARKS]

(d) Solve for $0^0 \leq x \leq 360^0$

$$\sin x + \cos x = \frac{4}{5}.$$

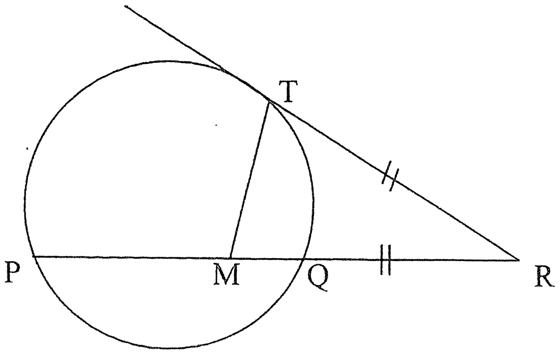
[3 MARKS]

[START A NEW BOOKLET]

QUESTION 3

(a) A particle moves along the x axis with acceleration $35+6t-6t^2$. If the particle is initially at rest at the origin, find its maximum displacement in the positive direction. [5 MARKS]

(b)



In the diagram RT is a tangent to the circle and RQP is a secant cutting the circle in Q and P. M is a point on PQ so that RM = RT.

Copy the diagram and prove that MT bisects the angle PTQ. [4 MARKS]

(c) Prove by induction that 7^n+5 is divisible by 3 where n is any positive integer. [3 MARKS]

QUESTION 4

(a) Prove for any acute angle θ that $\cot\theta = \sqrt{\dfrac{1+\cos 2\theta}{1-\cos 2\theta}}$. Hence show that the exact value of $\cot\dfrac{\pi}{8}$ is $\sqrt{2}+1$. [4 MARKS]

(b) Two sides of a triangle are 10cm and 18cm respectively. If the angle between them is increasing at the rate of $\frac{1}{10}$ radian per day, find how fast the area of the triangle is increasing when the angle is $\dfrac{\pi}{3}$. [3 MARKS]

(c) The rate of growth of the number of goats on a farm is given by $\dfrac{dN}{dt} = k\ (N - 200)$ where N is the number of goats. [5 MARKS]

(i) Show that $N = 200 + Ae^{kt}$ is a solution of the differential equation.

(ii) If initially there were 300 goats and 2 years later there were 420, find the number of goats on the farm at the end of 4 years.

[START A NEW BOOKLET]

QUESTION 5

(a) State the domain and range of the function $y = 4\sin^{-1} 2x$. Sketch this function. [4 MARKS]

(b) Show that $\frac{d}{dx}(x\tan^{-1} x) = \frac{x}{1+x^2} + \tan^{-1} x$. Hence evaluate $\int_0^1 \tan^{-1} x \, dx$, giving your answer correct to 3 significant figures. [5 MARKS]

(c) If $y = x\tan^{-1} x - \log_e \sqrt{1+x^2}$ show that $(x^2+1)\frac{d^2y}{dx^2} = 1$. [3 MARKS]

QUESTION 6

(a) A particle is projected with initial velocity 40 m/sec in a direction making 30^0 with the horizontal. Using $g = 9.8\text{m/s}^2$, find: [7 MARKS]

(i) the height of the particle after 3 seconds;

(ii) the cartesian equation of its path.

(b) A particle moves so that its distance x cm from a fixed point 0 after t seconds is given by $x = 4\cos 3t$. [5 MARKS]

(i) Show that the particle is moving in Simple Harmonic Motion.

(ii) What is the period of the motion?

(iii) Find the speed when the particle is first 1cm from 0.

QUESTION 7

(a) Find the area bounded by the curve $y = \frac{1}{\sqrt{16-x^2}}$, the x axis and the ordinates x = -2 and x = 2. [3 MARKS]

(b) Show that $f(x) = 4 - 10x + 3\sin x$ has a zero between $x = 0$ and $x = 1$. Use $x = 0.5$ and one application of Newton's method to find a better approximation. (Answer correct to 3 dec. pl) [5 MARKS]

(c) Find the volume of the solid formed by rotating $\frac{x^2}{9} + \frac{y^2}{4} = 1$ about the x axis. [4 MARKS]

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0.$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0.$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0.$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0.$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0.$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0.$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0.$$

$$\int \frac{1}{\sqrt{(a^2 - x^2)}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a.$$

$$\int \frac{1}{\sqrt{(x^2 - a^2)}} \, dx = \ln \left\{ x + \sqrt{(x^2 - a^2)} \right\}, \; |x| > |a|$$

$$\int \frac{1}{\sqrt{(x^2 + a^2)}} \, dx = \ln \left\{ x + \sqrt{(x^2 + a^2)} \right\}$$

NOTE: $\ln x = \log_e x, \; x > 0$

THE SCOTS COLLEGE 1999 3UNIT TRIAL

1 (a) Let $y = e^x \cos e^{-x}$

$\frac{dy}{dx} = e^x \cos e^{-x} + e^x e^{-x} \sin e^{-x}$

$= e^x \cos e^{-x} + \sin e^{-x}$

(b) $\log_4 27 = \frac{\log 27}{\log 4}$

$= \frac{3 \log 3}{2 \log 2}$

$= \frac{3}{2} \log_2 3$

$= \frac{3b}{2}$

(c) $\frac{1}{u} + \frac{1}{v} + \frac{1}{w}$

$= \frac{vw + uw + uv}{uvw}$

$= \frac{-4}{-1}$

$= 4$

(d) $\tan^{-1}\sqrt{3} - \tan^{-1}(-1)$

$= \frac{\pi}{3} - \left(-\frac{\pi}{4}\right)$

$= \frac{7\pi}{12}$

(e) $\frac{5}{2x-1} \geq 1 \qquad x \neq \frac{1}{2}$

$(2x-1)^2 \frac{5}{2x-1} \geq (2x-1)^2$

$10x - 5 \geq 4x^2 - 4x + 1$

$4x^2 - 14x + 6 \leq 0$

$2x^2 - 7x + 3 \leq 0$

$(2x-1)(x-3) \leq 0$

$x \neq \frac{1}{2}, \therefore \frac{1}{2} < x \leq 3$

(f) $y = 2x + 1 \quad (0, 1)$ is on line

$p = \frac{0 - 1 + 8}{\sqrt{5}} = \frac{7}{\sqrt{5}} = \frac{7\sqrt{5}}{5}$ u

2 (a) $\int \sin^2 \frac{x}{2}\, dx$

$= \int \frac{1 - \cos x}{2}\, dx$

$= \frac{x}{2} - \frac{\sin x}{2} + c$ 2

(b) $u = \tan 2x$

$du = 2 \sec^2 2x\, dx$

when $x = 7\pi/6$, $u = \sqrt{3}$

when $x = \frac{\pi}{8}$, $u = 1$

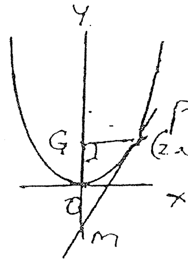
$\int_{\frac{\pi}{8}}^{\frac{7\pi}{6}} \tan 2x \sec^2 2x\, dx$

$= \int_1^{\sqrt{3}} u \cdot \frac{1}{2}\, du$

$= \left[\frac{u^2}{4}\right]_1^{\sqrt{3}}$ 3

$= \frac{3}{4} - \frac{1}{4}$

$= \frac{1}{2}$

(c) 

$x^2 = 4ay$

$P(2ap, ap^2)$ Gradient of tangent $= p$

Tangent at P is

$y - ap^2 = p(x - 2ap)$

$y = px - ap^2$

when $x = 0$ (axis of para

$y = -ap^2$

$\therefore$ M is $(0, -ap^2)$

G is $(0, ap^2)$ 4

$OG = ap^2$

$OM = ap^2$

$\therefore OG = OM$

2 (a) $\sin x + \cos x = \frac{4}{5}$

$$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = \frac{4}{5} \quad (t = \tan\frac{x}{2})$$

$$10t + 5 - 5t^2 = 4 + 4t^2$$

$$9t^2 - 10t - 1 = 0$$

$$t = \frac{10 \pm \sqrt{136}}{18}$$

$t = -0.0923$ or $t = 1.2034$ $(0° \le \frac{x}{2} \le 180°)$

$\frac{x}{2} = 174°44'$ or $\frac{x}{2} = 50°17'$

$x = 100°34', 349°27'$

(c)

(i) $n = 1$

$7 + 5 = 12$ (divisible by 3)

(ii) assume true for $n = k$

Let $7^k + 5 = 3m$ (m an integ

(iii) when $n = k+1$

$$7^{k+1} + 5 = 7.7^k + 5$$
$$= 7(3m - 5) + 5$$
$$= 21m - 30$$

∴ divisible by 3

It is true for $n = 1$ and therefore $n = 2$, $n = 3$ and so on

Hence true for all n.

3.

(a) $a = 35 + 6t - 6t^2$

$$\frac{dv}{dt} = 35 + 6t - 6t^2$$

$$v = 35t + 3t^2 - 2t^3 + c$$

$v = 0$ when $t = 0$ ∴ $c = 0$

$$v = 35t + 3t^2 - 2t^3$$

max. displacement when $v = 0$

$$t(35 + 3t - 2t^2) = 0$$

$$t(7 + 2t)(5 - t) = 0$$

$t = 0$ or $t = 5$ $\quad (t \neq -\frac{7}{2})$

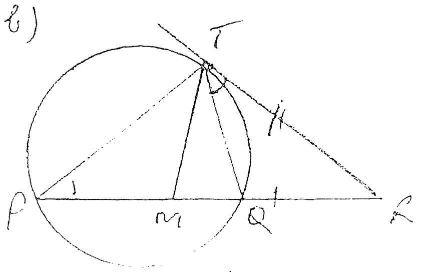
displacement $x = \frac{35t^2}{2} + t^3 - \frac{t^4}{2} + c$

$x = 0$ when $t = 0$ ∴ $c = 0$

$$x = \frac{35t^2}{2} + t^3 - \frac{t^4}{2}$$

$t = 5$, $x = \frac{35(25)}{2} + 125 - \frac{625}{0}$

(b)



TR = MR ∴ ∠RTM = ∠RMT (base ∠s, isos Δ)

∠RTQ = ∠TPQ (Tangent/angle segment)

By subtraction,

∠MTQ = ∠RMT − ∠TPQ

= ∠PTM (ext ∠ sum of int opp)

∴ MT bisects ∠PTQ

4 (a)

$$\frac{1+\cos 2\theta}{1-\cos 2\theta} = \frac{1 + 2\cos^2\theta - 1}{1 - (1 - 2\sin^2\theta)}$$
$$= \frac{2\cos^2\theta}{2\sin^2\theta}$$
$$= \cot^2\theta$$

$$\therefore \sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}} = \cot\theta$$

$$\cot\frac{\pi}{8} = \sqrt{\frac{1+\cos\pi/4}{1-\cos\pi/4}}$$
$$= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}}$$
$$= \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$$
$$= \sqrt{\frac{(2+\sqrt{2})(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}}$$
$$= \frac{2+\sqrt{2}}{\sqrt{2}}$$
$$= 1 + \sqrt{2}$$

(b) $A = \frac{1}{2}ab\sin\theta$

$= 90\sin\theta$

$\frac{dA}{d\theta} = 90\cos\theta = 45$ when $\theta = \frac{\pi}{3}$

$$\frac{dA}{dt} = \frac{dA}{d\theta} \cdot \frac{d\theta}{dt}$$
$$= 45 \cdot \frac{1}{10}$$
$$= 4.5 \text{ cm}^2/\text{day}$$

4 (c)

$$\frac{dN}{dt} = k(N-200)$$

(i) $N = 200 + Ae^{kt}$

$$\frac{dN}{dt} = kAe^{kt}$$
$$= k(N-200)$$

$\therefore N = 200 + Ae^{kt}$ is a solution.

(ii) $N = 200 + Ae^{kt}$

$t=0,\ N = 300$

$300 = 200 + Ae^{0}$

$A = 100$

$\therefore N = 200 + 100e^{kt}$

$t = 2,\ N = 420$

$420 = 200 + 100e^{2k}$

$220 = 100e^{2k}$

$e^{2k} = 2.2$

$k = \frac{\ln 2.2}{2}$

$k = 0.39422$

$\therefore N = 200 + 100e^{0.394t}$

when $t = 4$

$N = 200 + 100e^{4(0.39422)}$

$= 684$ goats

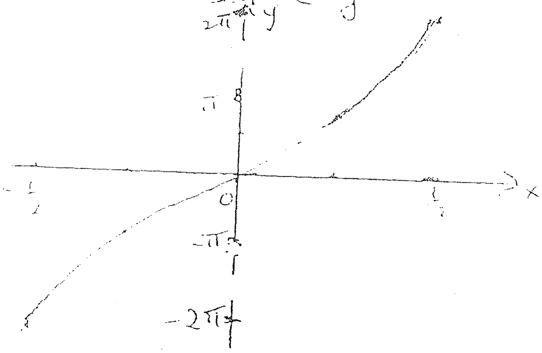
5 (a) $y = 4\sin^{-1}2x$

domain $-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$

range $4(-\frac{\pi}{2}) \le y \le 4(\frac{\pi}{2})$

$-2\pi \le y \le 2\pi$



(b) $\frac{d}{dx}(x\tan^{-1}x) = \tan^{-1}x + x\cdot\frac{1}{1+x^2}$

$= \tan^{-1}x + \frac{x}{1+x^2}$

$\therefore \int\left(\tan^{-1}x + \frac{x}{1+x^2}\right)dx = x\tan^{-1}x$

$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int\frac{x}{1+x^2}dx$

$\int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx$

$= \frac{\pi}{4} - \frac{1}{2}\left[\ln(1+x^2)\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 0)$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

$= 0.439$ (3 sig. fig)

(c) $y = x\tan^{-1}x - \ln\sqrt{1+x^2}$

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)$

$\frac{dy}{dx} = \tan^{-1}x + x\frac{1}{1+x^2} - \frac{2x}{2(1+x^2)}$

$= \tan^{-1}x + \frac{x}{1+x^2} - \frac{x}{1+x^2}$

$= \tan^{-1}x.$

$\frac{d^2y}{dx^2} = \frac{1}{1+x^2}$

$(x^2+1)\frac{d^2y}{dx^2} = x^2+1 \cdot \frac{1}{1+x^2} = 1$

6

(a) Horizontally $\dot{x} = V\cos\alpha$
$= 40\cos 30^\circ$
$= 20\sqrt{3}$

Vertically $\dot{y} = V\sin\alpha - gt$
$= 40\sin 30^\circ - 9.8t$
$= 20 - 9.8t$

distance $x = V\cos\alpha\, t$. —— ①
$= 20\sqrt{3}.\ 3$
$= 60\sqrt{3}$

$y = V\sin\alpha t - \frac{1}{2}gt^2 + c$

$t=0, y=0$

$\therefore y = V\sin\alpha t - \frac{1}{2}gt^2$

$y = 20t - 4.9t^2$ —— ②

$t=3$, $y = 15.9\,m$.

From ① $t = \frac{x}{20\sqrt{3}}$

In ②

$y = \frac{20x}{20\sqrt{3}} - 4.9\left(\frac{x^2}{1200}\right)$

$y = \frac{x}{\sqrt{3}} - \frac{4.9x^2}{1200}$.

(b)(i) $x = 4\cos 3t$

$\dot{x} = -12\sin 3t$

$\ddot{x} = -36\cos 3t$
$= -9x$

In form $\ddot{x} = -n^2x$ where $n=3$

SHM.

(ii) $T = \frac{2\pi}{n} = \frac{2\pi}{3}$

period

(iii) when $x = 1$

$1 = 4\cos 3t$

$\cos 3t = \frac{1}{4}$

$3t = 1.32$

$t = 0.44$

$\dot{x} = -12\sin 3t$
$= -12\sin 1.32$
$= -11.6\ m/s$

7

(a) $A = \int_{-2}^{2} y\,dx$

$= \int_{-2}^{2} \frac{1}{\sqrt{16-x^2}}\,dx$

$= \left[\sin^{-1}\frac{x}{4}\right]_{-2}^{2}$

$= \sin^{-1}\frac{1}{2} - \sin^{-1}(-\frac{1}{2})$

$= \frac{\pi}{6} - -\frac{\pi}{6} = \frac{\pi}{3}u^2$

(b) $x=0$, $f(0) = 4 - 0 + 0 = 4$

$x = 1$, $f(1) = 4 - 10 + 3\sin 1$
$= -3.476$

change of sign

$\therefore 0 < x < 1$

$f(0.5) = 0.4383 = f(a)$

$f'(x) = -10 + 3\cos x = f'(a)$

$f'(0.5) = -7.3673$

$x_1 = a - \frac{f(a)}{f'(a)}$.

$= 0.5 - \frac{0.4383}{-7.3673}$

$= 0.55949$

$= 0.559$ (3 dec. pl)

(c) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ $\quad \frac{y^2}{4} = 1 - \frac{x^2}{9}$

$V = \pi\int y^2\,dx$ $\quad$ when $y=0$, $x = \pm 3$

$= \pi\int_{-3}^{3}\left(4 - \frac{4x^2}{9}\right)dx$

$= \pi\left[4x - \frac{4x^3}{27}\right]_{-3}^{3}$

$= \pi\left[(12-4) - (-12+4)\right]$

$= 16\pi\ u^3$