The Scots College

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2000

3/4 UNIT MATHEMATICS

Time Allowed : TWO HOURS
(Including 5 minutes reading time)

Instructions to Candidates:

- All questions are to be attempted
- All questions are of equal value
- All necessary working should be shown for each question
- Non-programmable calculators that are Board approved are permitted
- A table of standard integrals is provided.

Booklet Order:

Booklet 1: Questions 1 and 2
Booklet 2: Questions 3 and 4
Booklet 3: Questions 5, 6 and 7

Question 1 (Begin a new booklet)

(a) Evaluate $\int_1^2 \frac{dx}{\sqrt{4-x^2}}$.

(3 marks)

(b) For $y = -3\sin^{-1}\frac{x}{2}$

i) State the domain and range.

ii) Sketch the curve.

(3 marks)

(c) If $\tan\frac{\theta}{2} = t$, express $1 - \frac{1}{2}\sin\theta\tan\frac{\theta}{2}$ in terms of t.

(3marks)

(d) Find the constants a, b such that $x^2 - 2x - 3$ is a factor of the polynomial $f(x) = x^3 - 3x^2 + ax + b$.

(3 marks)

Question 2

(a) Find $\int \frac{xdx}{1+2x}$ using the substitution $u = 1 + 2x$.

(3 marks)

(b) (i) Show that the equation $\log_e x - \cos x = 0$ has a root between $x = 1$ and $x = 2$.

(ii) By taking $x = 1.2$ as the first approximation, use 1 step of Newton's method to find a better approximation to this root, correct to 2 decimal places.

(3 marks)

(c) Express $3\cos x + 4\sin x$ in the form $A\cos(x-\alpha)$ where A> 0. Hence, or otherwise, solve $3\cos x + 4\sin x = -3$ for $0 \le x \le 360^0$.

(4 marks)

(d) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 2xdx$.

(2 marks)

Question 3 (Begin a new booklet)

(a) Solve the equation $2\ln(3x+1) - \ln(x+1) = \ln(7x+4)$

(3 marks)

(b)

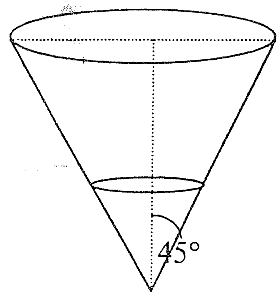


Figure not to scale

Water is being let into the conical vessel shown at a constant rate of 8 cm^3/s.

When the depth is 12 cm, find:

(i) the rate of increase in the depth (in terms of π), and

(ii) the rate of increase in the area of the top surface of the water.

(5 marks)

(c) If $\dfrac{P\sin A}{\tan B} = P\cos A + Q$, show that $P = \dfrac{Q\sin B}{\sin(A-B)}$.

(3 marks)

(d) Differentiate with respect to x: $y = \cos^{-1}(5x-4)$

(1 mark)

Question 4

(a) The acceleration of a particle moving in a straight line is given by :

$$\ddot{x} = 3 - 4x$$

where x is the displacement in metres, from the origin and t is the time in seconds.

If the particle starts from rest at $x = 1$ metres,

(i) Show that the velocity of the particle is given by:

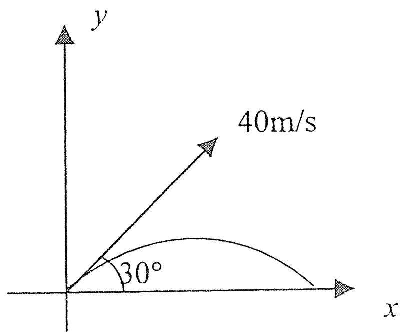
$$v^2 = 2(-2x^2 + 3x - 1)$$

(ii) Identify the second position where the particle will come to rest.

(iii) What will be the acceleration at the second position where the particle comes to rest?

(6 marks)

(b) The diagram shows the path of an object launched at an angle of 30° to the horizontal with an initial speed of 40m/s from O. The acceleration due to gravity is taken as 10m/s^2, and air resistance is ignored.



(i) Given that $\frac{d^2x}{dt^2} = 0$ and $\frac{d^2y}{dt^2} = -10$, derive expressions for the horizontal displacement $x(t)$ and the vertical displacement $y(t)$ of the object from O, t seconds after launching.

(ii) Using the equations in (i) above, calculate the time it takes for the object to land at A and the distance OA.

(6 marks)

Question 5 (Begin a new booklet)

(a) Use the method of Mathematical Induction to show that:

$n^3 + 2n$ is divisible by 3 for all positive integers $n \geq 1$.

(5 marks)

(b) (i) Determine the equation of the tangent to the curve C: $y = 2x^2$ at the point $P(t, 2t^2)$.

(ii) The point Q lies on the curve C_1: $y = x^2 + 1$, on the same vertical line (ie with the same x coordinate) as the point P of part (i). Show that the equation of the tangent to C_1 at Q is $y = 2tx + (1 - t^2)$.

(iii) Find the precise locus of the points of intersection of these two tangents, as the common x coordinate t of the points P and Q assume all positive values. Indicate this locus on a sketch.

(7 marks)

Question 6

(a) Let $f(x) = x^2 - 4x$ for all real x.

(i) Explain why $f(x)$ for all $x \geq 2$ has an inverse function, $f^{-1}(x)$.

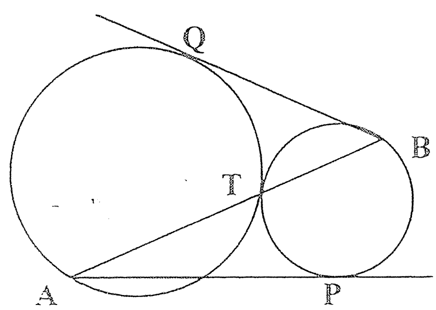
(ii) State the domain and range of $f^{-1}(x)$.

(iii) Find the coordinates of the point where $y = f(x)$ and $y = f^{-1}(x)$ meet.

(iv) If $0 < a < 2$ then find the value, in terms of a, of $f^{-1}(f(a))$.

(6 marks)

(b)



The circles touch at T. ATB is a straight line. AP is a tangent to circle PTB and BQ is a tangent to circle QTA.

Prove that $AP^2 + BQ^2 = AB^2$ **(3 marks)**

(c) Show that $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$

(3 marks)

Question 7

(a) If α, β and γ are the roots of the equation $x^3 - 4x + 1 = 0$ evaluate:

(i) $\alpha + \beta + \gamma$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

(3 marks)

(b) A beaker contains a coloured solution in which the amount of colouring, Q, is known to change at a rate given by $\dfrac{dQ}{dt} = -0.02(Q - 30)$. Initially the beaker contains 70g of colouring and t is in minutes.

(i) Write down an equation for Q in terms of t.

(ii) Find the amount of colouring, to the nearest gram, in the beaker after 45 minutes.

(3 marks)

(c) Assume that the tides rise and fall in Simple Harmonic Motion. A ship needs 10 metres of water to pass down a channel safely. At low tide the channel is 9 metres deep and at high tide the channel is 12 metres deep. Low tide is at 9 am and high tide is at 4 pm.

(i) State the period and amplitude of the motion.

(ii) Between what times can the ship be assured of safe passage?

(6 marks)

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Solns.

uestion 1

$$\int_1^2 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \sin^{-1}(1) - \sin^{-1}\left(\tfrac{1}{2}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{6}$$

$$= \frac{\pi}{3}$$

(3)

$$y = -3\sin^{-1}\frac{x}{2}$$

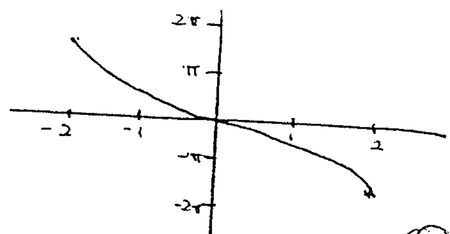
i) Domain: $-1 \leq \frac{x}{2} \leq 1$

$-2 \leq x \leq 2$

Range: $+3\left(-\frac{\pi}{2}\right) \leq y \leq 3\left(\frac{\pi}{2}\right)$

$-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$

ii)



(3)

$$1 - \frac{1}{2}\left(\frac{2t}{1+t^2}\right)(t)$$

$$= 1 - \frac{2t^2}{2(1+t^2)}$$

$$= \frac{2+2t^2-2t^2}{2(1+t^2)}$$

$$= \frac{1}{(1+t^2)}$$

(3)

$x^2 - 2x - 3 = (x-3)(x+1)$

$\therefore$ $(x-3)$ is a factor ie $f(3)=0$

$(x+1)$ is a factor ie $f(-1)=0$

(3)

$f(x) = x^3 - 3x^2 + ax + b$

$f(3) = 27 - 27 + 3a + b = 0$

$3a + b = 0 \Rightarrow b = -3a$

$f(1) = -1 - 3 - a + b$

$= -4 - a + b = 0$

sub $b = -3a$

$-4 - a - 3a = 0$

Solution $a = -1$

$b = 3$

Question 2

) $\int \frac{x}{1+2x}\,dx$ let $u = 1+2x \Rightarrow x = \frac{u-1}{2}$

$du = 2\,dx$

$dx = \frac{1}{2}\,du$

$= \int \frac{\frac{u-1}{2}}{u}\,\frac{1}{2}\,du$

$= \int \frac{u-1}{4u}\,du$

$= \int \frac{u}{4u}\,du - \int \frac{1}{4u}\,du$

$= \int \frac{1}{4}\,du - \int \frac{1}{4u}\,du$

$= \frac{1}{4}u - \frac{1}{4}\ln u + c$ but $u = (1+2x)$

$= \frac{1+2x}{4} - \frac{\ln(1+2x)}{4} + c$ 3

i) $\log_e x - \cos x = 0$

$x=1$ $\log_e 1 - \cos 1 = -0.54$

$x=2$ $\log_e 2 - \cos 2 = 1.11$

$\therefore$ a root lies between $x=1$ (-ve answer) & $x=2$ (+ve answer)

ii) $z_1 = 1.2$ $f(x) = \log_e x - \cos x$

$f'(x) = \frac{1}{x} + \sin x$

$z_2 = z_1 - \frac{f(z_1)}{f'(z_1)}$

$= 1.2 - \frac{f(1.2)}{f'(1.2)}$

$= 1.2 - \left(\frac{\log_e 1.2 - \cos 1.2}{\frac{1}{1.2} + \sin 1.2}\right)$ 3

$= 1.30$ (to 2 dec pl)

$3\cos x + 4\sin x$

$(\sqrt{9+16})\cos(x-\alpha)$

$5\cos(x-\alpha)$ $\tan\alpha = \frac{4}{3}$

$5\cos(x-53°8')$ $\alpha = 53°8'$

$\cos x + 4\sin x = -3$

$5\cos(x-53°8') = -3$

$\cos(x-53°8') = \frac{-3}{5}$ $233°08'$

$x - 53°8' = 126°52'$ or 4

$$\int_0 \cos^2 2x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 4x)\,dx$$

$$= \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{4}\sin 2\pi\right) - \left(0 + \frac{1}{4}\sin 0\right)\right]$$

$$= \frac{\pi}{4}$$

2

uestion 3

$$2\ln(3x+1) - \ln(x+1) = \ln(7x+4)$$

$$\ln(3x+1)^2 - \ln(x+1) = \ln(7x+4)$$

$$\ln\left(\frac{(3x+1)^2}{(x+1)}\right) = \ln(7x+4)$$

$$\frac{(3x+1)^2}{(x+1)} = (7x+4)$$

$$(3x+1)^2 = (7x+4)(x+1)$$

$$9x^2+6x+1 = 7x^2+4x+7x+4$$

$$2x^2-5x-3=0$$

$$(2x+1)(x-3)=0$$

$2x+1=0$ or $x-3=0$

$2x=-1$ $\quad$ $x=3$

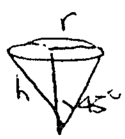
$x=-\frac{1}{2}$

But $\ln(3x+1)$ is not defined for $x=-\frac{1}{2}$

$\therefore$ Solution $x=3$

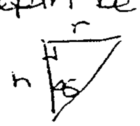
3

)



let the depth be 'h' cm

$r=h$



$$\tan 45° = \frac{r}{h}$$

$$1 = \frac{r}{h}$$

$$r = h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi h^3$$

$$\frac{dV}{dh} = \pi h^2$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$8 = \pi h^2 \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{8}{\pi h^2}$$

$$= \frac{8}{\pi \times 12^2}$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

but as $r=h$ $\quad \frac{dr}{dt} = \frac{dh}{dt}$

$$= \frac{dA}{dh} \times \frac{dh}{dt}$$

$$= 2\pi r \times \frac{1}{18\pi}$$

$$= 24\pi \times \frac{1}{18\pi}$$

$$= 1\frac{1}{3} \text{ cm}^2/\text{s}$$

) $\frac{P\sin A}{\tan B} = P\cos A + Q$

$P\sin A \times \frac{\cos B}{\sin B} = P\cos A + Q$

$P\sin A\cos B = P\cos A\sin B + Q\sin B$

$P\sin A\cos B - P\cos A\sin B = Q\sin B$

$P(\sin A\cos B - \cos A\sin B) = Q\sin B$

$P(\sin(A-B)) = Q\sin B$

$P = \frac{Q\sin B}{\sin(A-B)}$ 3

$y = \cos^{-1}(5x-4)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(5x-4)^2}} \times 5$

$= \frac{5}{\sqrt{1-(25x^2-40x+16)}}$

$= \frac{5}{\sqrt{40x-25x^2-15}}$ 1

uestion 4

$\ddot{x} = 3-4x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3-4x$

$\frac{1}{2}v^2 = \int(3-4x)\,dx$

$= 3x-2x^2+C$

hen $t=0$, $x=1$, $v=0$

$0 = 3-2+C$

$C=-1$

$\frac{1}{2}v^2 = 3x - 2x^2 - 1$

$v^2 = 2(3x-2x^2-1)$

hen the particle comes to rest $v=0$

$0 = -2(2x^2-3x+1)$

$2x^2-3x+1=0$

$(2x-1)(x-1)=0$

$x = \frac{1}{2}$ or $x=1$

he particle will come to rest again

hen $x = \frac{1}{2}$ m. 6

$\ddot{x} = 3-4x$ when $x=\frac{1}{2}$

$\ddot{x} = 3-2$

$= 1$

$$\frac{d^2x}{dt^2} = 0$$

$\ddot{x} = 0$

$\dot{x} = c$

when $t=0$ $\dot{x} = V\cos\theta$

$\therefore c = V\cos\theta$

$\dot{x} = V\cos\theta$

$x = Vt\cos\theta + c$

when $t=0$, $x=0$

$\therefore c=0$

$x = Vt\cos\theta$

$x = (40\cos 30^\circ)t$

$= 20\sqrt{3}t$

$\ddot{y} = -10$

$\dot{y} = -10t + c$

when $t=0$ $\dot{y} = V\sin\theta$

$\therefore c = V\sin\theta$

$\dot{y} = V\sin\theta - 10t$

$y = -5t^2 + Vt\sin\theta + c$

when $t=0$, $y=0$

$\therefore c=0$

$y = -5t^2 + 40\sin 30^\circ t$

$= -5t^2 + 20t$

) $y=0$ $\quad -5t^2+20t=0$

$-5t(t-4) = 0$

$t=0$ or $t=4$ sec $\quad\therefore$ object lands after 4 sec

when $t=4$ $\quad x = 20\sqrt{3} \times 4$

$= 80\sqrt{3}$ m $\quad \therefore OA = 80\sqrt{3}$ m

6

uestion 5

Let $n^3+2n = 3P$

$n=1$ $\quad 1+2=3$ $\quad\therefore$ true for $n=1$

assume it true for $n=k$

ie $k^3+2k = 3P$

rove true for $n=(k+1)$

so $(k+1)^3+2(k+1)$

$= k^3+3k^2+3k+1 \; +2k+2$

$= k^3+2k+3k^2+3k+3$

$= 3P + 3(k^2+k+1)$

$= 3[P+(k^2+k+1)]$

$= 3m$ $\quad$ for $m = [P+(k^2+k+1)]$

$\therefore$ divisible by 3

o since true for $n=1$, $n=k$ & proven for $n=k+1$

hen true for all $n>0$

5

i) $y = 2x^2$

$\frac{dy}{dx} = 4x$ at $x = t$

$m = 4t$

$m = \frac{y - y_1}{x - x_1}$

$\frac{4t}{1} = \frac{y - 2t^2}{x - t}$

$y - 2t^2 = 4t(x - t)$

$y - 2t^2 = 4tx - 4t^2$

$y = 4tx - 2t^2$

i $y = x^2 + 1$

$\frac{dy}{dx} = 2x$ at $x = t$

$m = 2t$

if $x = t$, $y = (t^2 + 1)$

$m = \frac{y - y_1}{x - x_1}$

$\frac{2t}{1} = \frac{y - (t^2+1)}{x - t}$

$y - (t^2+1) = 2t(x - t)$

$y - (t^2+1) = 2tx - 2t^2$

$y = 2tx - t^2 + 1$

$y = 2tx + (1 - t^2)$

i) $\left.\begin{array}{l} y = 4tx - 2t^2 \\ y = 2tx + (1 - t^2) \end{array}\right]$ Solve simultaneously

$4tx - 2t^2 = 2tx + (1 - t^2)$

$2tx = 2t^2 - t^2 + 1$

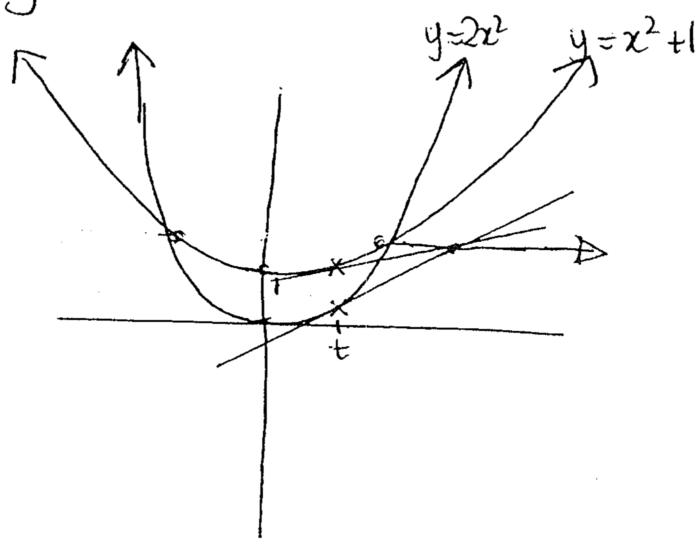
$2tx = t^2 + 1$

$x = \frac{t^2+1}{2t}$

$y = 4t\left(\frac{t^2+1}{2t}\right) - 2t^2$

$y = 2(t^2+1) - 2t^2$

$y = 2$



required locus is $y = 2$

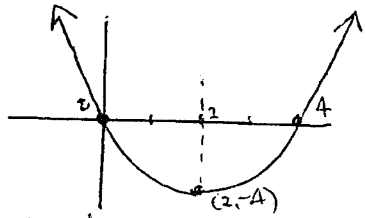
where $x > 1$ or $x < -1$

as x must be positive only $x > 1$

6

Question 6

) $f(x) = x^2 - 4x$
$= x(x-4)$



i) for every y there is one & only one x.

ii) Domain $x \geq -4$

Range $y \geq 2$

iii) $y = x^2 - 4x$ $\quad x = y^2 - 4y$

$y = (y^2-4y)^2 - 4(y^2-4y)$
$y = y^4 - 8y^3 + 16y^2 - 4y^2 + 16y$
$0 = y^4 - 8y^3 + 12y^2 + 15y$
$y(y^3 - 8y^2 + 12y + 15) = 0$
$P(x) = y^3 - 8y^2 + 12y + 15$
$P(3) = 27 - 72 + 36 + 15$
$= 6$
$P(5) = 5^3 - 8\times5^2 + 12\times5 + 15$
$= 0$ $\quad \therefore (y-5)$ is a factor

$$\begin{array}{r} y^2 - 3y - 3 \\ y-5 \overline{) y^3 - 8y^2 + 12y + 15} \\ y^3 - 5y \\ \hline -3y^2 + 12 \\ -3y^2 + 15y \\ \hline -3y + 15 \\ -3y + 15 \\ \hline .. \end{array}$$

$y(y-5)(y^2-3y-3) = 0$
$y = 0$ or $y = 5$ or $y^2 - 3y - 3 = 0$
$\therefore x = 25 - 20$
$x = 5$
$(5, 5)$

6

) $f^{-1}(f(a))$
$= 2 + \sqrt{f(a)+4}$
$= 2 + \sqrt{a^2 - 4a + 4}$
$= 2 + \sqrt{(a-2)^2}$
$= 2 + (\sqrt{a-2})^2$
$= 2 + |a-2|$

$AP^2 = AB.AT$ (square of tangent equals product of secant segments
$BQ^2 = AB.BT$ (" " " ")
$P^2 + BQ^2 = AB.AT + AB.BT$
$= AB(AT + BT)$
$= AB.AB$
$= AB^2$

3

Let $\tan^{-1}\frac{1}{4}=\alpha$

$\tan\alpha=\frac{1}{4}$ $\quad -\frac{\pi}{2}<\alpha<\frac{\pi}{2}$

Let $\tan^{-1}\frac{3}{5}=\beta$

$\tan\beta=\frac{3}{5}$ $\quad -\frac{\pi}{2}<\alpha<\frac{\pi}{2}$

$$\tan\left[\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{3}{5}\right]$$

$$=\tan(\alpha+\beta)$$

$$=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$=\frac{\frac{1}{4}+\frac{3}{5}}{1-\frac{1}{4}\times\frac{3}{5}}$$

$$=\frac{\frac{17}{20}}{\frac{17}{20}}$$

$$=1$$

$$\tan\left[\tan^{-1}\left(\tfrac{1}{4}\right)+\tan^{-1}\left(\tfrac{3}{5}\right)\right]=1$$

$$\tan^{-1}(1)=\tan^{-1}\left(\tfrac{1}{4}\right)+\tan^{-1}\left(\tfrac{3}{5}\right)$$

$$\tan^{-1}\left(\tfrac{1}{4}\right)+\tan^{-1}\left(\tfrac{3}{4}\right)=\frac{\pi}{4}$$

3

Question 7

$x^3-4x+1=0$ $\quad a=1,\ b=0,\ c=-4,\ d=1$

$$\alpha+\beta+\gamma=\frac{-b}{a}=0$$

$$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}=\frac{-4}{1}=-4$$

$$\alpha\beta\gamma=\frac{-d}{a}=\frac{-1}{1}=-1$$

ii) $$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}=\frac{-4}{-1}=4$$

3

$$\frac{dQ}{dt}=-0.02(Q-30)$$

$$Q=30+Ae^{-0.02t}$$

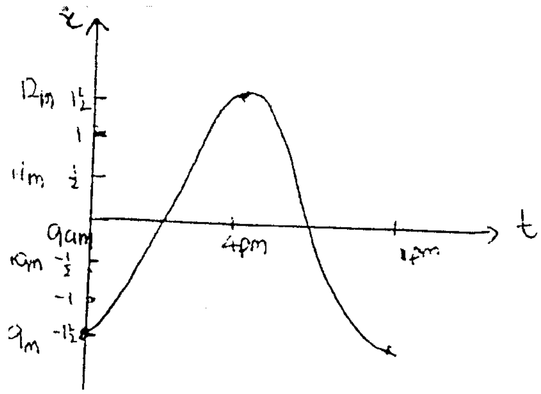
$t=0,\ Q=70$

$70=30+A$ $\quad Q=30+40e^{-0.02t}$

ii) $t=45$ $\quad Q=30+40e^{-0.02\times45}$

$\doteqdot 46g$

3



Let 12m ⇒ $1\frac{1}{2}$ m or x axis
9m ⇒ $-1\frac{1}{2}$ m or x axis

i) Period $= \frac{2\pi}{n}$

$$14 = \frac{2\pi}{n}$$

$$n = \frac{\pi}{7}$$

Amplitude $= 1\frac{1}{2}$ m

∴ Period = 14hrs

$\therefore x = a\cos(nt+\alpha)$

$$x = -1\frac{1}{2}\cos\left(\frac{\pi}{7}t+\alpha\right)$$

when $t=0$, $x = 1\frac{1}{2}$

$$-1\frac{1}{2} = -1\frac{1}{2}\cos\alpha$$

$$\cos\alpha = 1$$

$$\alpha = 0$$

when $x = -\frac{1}{2}$

$$-\frac{1}{2} = -1\frac{1}{2}\cos\left(\frac{\pi}{7}t\right)$$

$$\cos\left(\frac{\pi}{7}t\right) = \frac{1}{3}$$

$$\frac{\pi}{7}t = 1.2309....$$

$$t = 2.74$$

= 2hrs 45min

∴ 9am + 2hrs 45min
= 11.45 am
11 − 2hr 45min
= 8.15 pm

6