The Scots College
2002

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question.
- Answer each question in a SEPARATE writing booklet.
- Extra writing booklets are available.
- Question Papers are to be handed in.

Total marks – 84

- Attempt Questions 1–7.
- All questions are of equal value.

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All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) **Marks**

a. Differentiate $4x\sin^{-1}x$ **2**

b. If $\int_{-a}^{a}\frac{dx}{1+x^2}=\frac{\pi}{2}$ Find the value of a. **2**

c. Find the coordinates of the point P that divides the interval joining (-4, 3) and (2, -7) externally in the ratio 4:3. **2**

d. If $\log_a b = 2.8$ and $\log_a c = 4.1$, find $\log_a bc$. **1**

e. Solve for x: $\frac{x+2}{x}\leq 3,\ x\neq 0$. **2**

f. Evaluate $\int_{0.5}^{1}4x(2x-1)^5\,dx$ by making the substitution $u=2x-1$. **3**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

a. Taking $x = 0.5$ as a first approximation for the root of $\log_e x = -x$, use Newton's method to find a second approximation. (Answer correct to 3 significant figures.) **3**

b. Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ **4**

c. Find the size of α and β in the following diagram (giving reasons). **2**

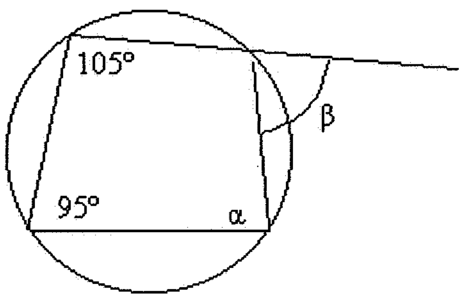


Diagram NOT to scale.

d.

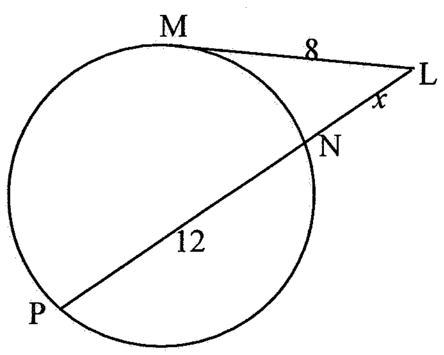


Diagram NOT to scale.

LM is a tangent to the circle, while LNP is a secant intersecting the circle at N and P. Given that LM = 8, NP = 12 and LN = x, find x. **3**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet.

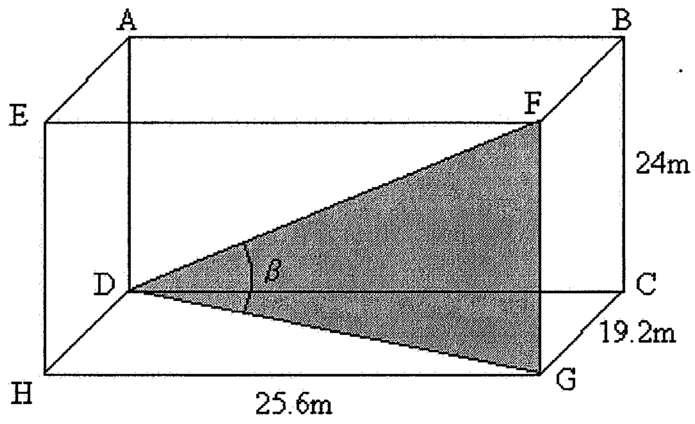
Marks

a. Find, for $0 \le x \le 2\pi$, all solutions for the equation $\cos x = \sin 2x$. **3**

b. If α, β and γ are the roots of the cubic polynomial equation $x^3 + 8x^2 - 4x - 6 = 0$ **3**
Find the value of $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma}$.

c. Find the term independent of x in the expansion of $(3x^4 + \frac{1}{x})^{10}$ **3**

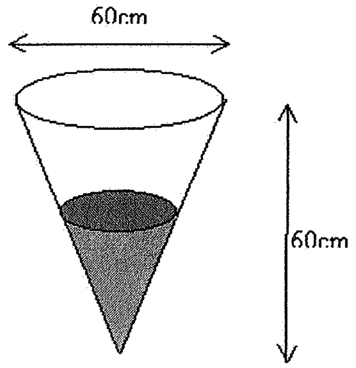
d. Calculate the value of β in the following rectangular prism. **3**
(Answer to the nearest minute.)



End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

a. Sand is being poured into a conical container at a constant rate of $36\text{cm}^3\text{s}^{-1}$. The height and diameter of the container are both 60cm. After t seconds the depth of the sand in the container is h cm.



i. Show that the depth of sand in the container after 5 seconds is 8.826cm correct to 2d.p. **2**

ii. Find the rate at which the depth of sand is changing after after 5 seconds. (answer to 3d.p.) **2**

iii. Find the rate at which the surface area, S, of the sand in the container is changing when the depth of the sand is 20cm. **2**

b. i. Find the domain and range of the function $y = 4\cos^{-1}\left(\frac{x}{3}\right)$. **1**

ii. Sketch the graph of the function $y = 4\cos^{-1}\left(\frac{x}{3}\right)$ showing clearly the intercepts on the coordinate axes and the coordinates of any endpoints. **2**

iii. Find the area of the region in the first quadrant bounded by the curve $y = 4\cos^{-1}\left(\frac{x}{3}\right)$ and the coordinate axes **3**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

a. The parabola given by $x = 2at$ and $y = at^2$ has points P and Q where $t = p$ and $t = q$ respectively.
Prove:

i. The equation of the chord PQ is given by $y - \frac{(p+q)x}{2} + apq = 0$ **2**

ii. The equation of the tangent at P is given by $y - px + ap^2 = 0$ **2**

iii. The tangents at the ends of any focal chord meet on the directrix and are perpendicular to each other. **3**

b. At time t the temperature T° Celsius of a piece of iron in a room of constant temperature 30° Celsius, is decreasing according to the equation
$\frac{dT}{dt} = -k(T-30)$ for some constant k > 0.

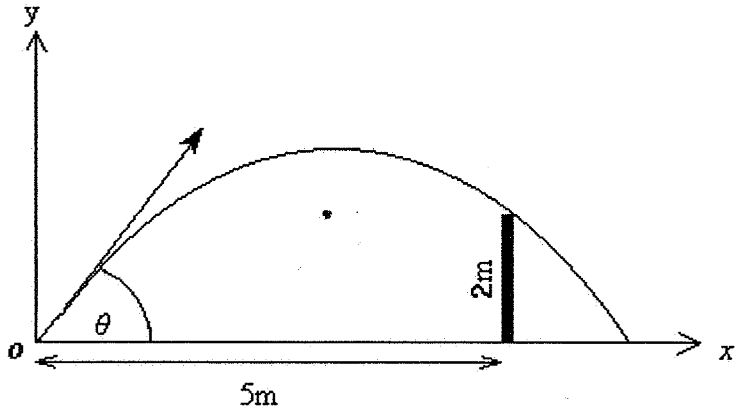
i. Verify that $T = 30 + Ae^{-kt}$, A constant, is a solution of the equation. **2**

ii. The initial temperature, T, of the piece of iron is 100°C and it falls to 60°C after 20 minutes. Find the temperature of the body after a further 10 minutes. (Answer to nearest degree.) **3**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

a. A ball is shot out of a small cannon on the ground at point O with velocity 15ms^{-1}, fired at an angle of θ with the horizontal.



i. Taking acceleration due to gravity to be a constant 10ms^{-2}, show that the equation for the horizontal (x) and vertical (y) components of the particles displacement from the origin O are given by $x = 15t\cos\theta$ and $y = -5t^2 + 15t\sin\theta$ **2**

ii. Show that the Cartesian equation for displacement is given by **2**

$$y = \frac{-x^2}{45}\sec^2\theta + x\tan\theta$$

iii. The ball just clears a 2 metre high fence that is 5 metres from the origin . Find two values of θ (measured in degrees) for this to happen. (Give your answer to the nearest minute) **3**

b. A particle is moving such that its acceleration is given by $\ddot{x} = -16x$. The particle has an initial displacement of 3m and an initial velocity of 12ms^{-1}.

i. Show the velocity is given by $\dot{x} = \left|4\sqrt{18 - x^2}\right|$ **3**

ii. Find the equation for the particle displacement, x, over time t. **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

a. Use the principal of mathematical induction to prove that, for every positive integer n, $13 \times 6^n + 2$ is divisible by 5. **4**

b. A particle moves in a straight line. Its displacement x metres from the origin, after t seconds is given by $x = \sin^2 5t + 2$, $t > 0$.

i. Find the time when the particle is first at $x = \frac{5}{2}$. **2**

ii. In what direction is the particle travelling when it is first at $x = \frac{5}{2}$? **1**

iii. Express the acceleration of the particle in terms of x. **2**

iv. Hence, or otherwise, show that the particle is undergoing simple harmonic motion. **2**

v. State the period of the motion. **1**

End of paper

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iii. Express the acceleration of the particle in terms of x. **2**

iv. Hence, or otherwise, show that the particle is undergoing simple harmonic motion. **2**

v. State the period of the motion. **1**

End of paper

QUESTION ONE SOLUTIONS

a. $\frac{d}{dx}4x.\sin^{-1}x$

$= \sin^{-1}x \times 4 + 4x \times \frac{1}{\sqrt{1-x^2}}$

$= 4\sin^{-1}x + \frac{4x}{\sqrt{1-x^2}}$

b) $\int_{-a}^{a} \frac{dx}{1+x^2}$

$= \left[\tan^{-1}x\right]_{-a}^{a}$

$= \tan^{-1}a - \tan^{-1}(-a)$

so $\tan^{-1}a - \tan^{-1}(-a) = \frac{\pi}{2}$

$2\tan^{-1}a = \frac{\pi}{2}$

$\tan^{-1}a = \frac{\pi}{4}$ $\quad \therefore a = 1$

c) $x_1 = -4, y_1 = 3$

$x_2 = 2, y_2 = -7$

$m = 4\ , n = -3$

$P = \left(\frac{4.2 + (-3).(-4)}{4 + (-3)}, \frac{4.(-7) + (-3).3}{4 + (-3)}\right)$

$P = (20, -37)$

d) $\log_a bc$ $= \log_a b + \log_a c$

$= 2.8 + 4.1$

$= 6.9$

e) $\frac{x+2}{x} \le 3$ mult. b.s. by x^2

$x(x+2) \le 3x^2$

$0 \le 2x^2 - 2x$

$0 \le 2x(x-1)$

$x < 0$ and $x \ge 1$ (note $x \ne 0$)

f) let $u = 2x - 1$ so

$du = 2dx$ and $2x = u + 1$

when $x = 0.5$, $u = 0$ and when $x = 1$, $u = 1$

so $\int_{0.5}^{1} 4x(2x-1)^5\,dx$ becomes

$= \int_{0.5}^{1} 2x(2x-1)^5\,2dx$

$= \int_{0.5}^{1} 2x(2x-1)^5\,2dx$

$= \int_{0}^{1} (u+1)(u)^5\,du$

$= \int_{0}^{1} (u^6 + u^5)\,du$

$= \left[\frac{u^7}{7} + \frac{u^6}{6}\right]_0^1$

$= \frac{13}{42}$

QUESTION TWO SOLUTIONS

a. let $f(x) = \log_e x + x$

then $f'(x) = \frac{1}{x} + 1$

let $x_1 = 0.5$

now $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_2 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$x_2 = 0.5 - \frac{\ln(0.5) + 0.5}{3}$

$x_2 = 0.564$ (to 3 Sig. figs)

b. Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

LHS $= \cos(2\theta + \theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= \cos\theta(2\cos^2\theta - 1) - \sin\theta(2\sin\theta\cos\theta)$

$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta)$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$ (=RHS)

c. $\alpha = 75°$ (opposite angles in a cyclic quad are supp.) and $\beta = 95°$ (Exterior angle equal to opp. int. angle)

d. $LM^2 = LN.LP$

$8^2 = x.(12+x)$

$64 = 12x + x^2$

$x^2 + 12x - 64 = 0$

$(x+16)(x-4) = 0$

so $x = -16$ or $x = 4$

since $x > 0$ the only solution will be $x = 4$

QUESTION 3 SOLUTIONS

a.

$\cos x = \sin 2x$

so $0 = \sin 2x - \cos x$

$0 = 2\sin x \cos x - \cos x$

$0 = \cos x(2\sin x - 1)$

$\therefore 2\sin x - 1 = 0$

and $\cos x = 0$

now $0 \le x \le 2\pi$

so for $2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

and for $\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

So there are 4 solutions

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

b.

$\alpha+\beta+\gamma = \frac{-b}{a} = \frac{-8}{1}$

$\alpha\beta+\alpha\gamma+\gamma\beta = \frac{c}{a} = \frac{-4}{1}$

$\alpha\beta\gamma = \frac{-d}{a} = \frac{-6}{1}$

$\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{4}{3}$

c.i. expansion of $(3x^4 + \frac{1}{2x})^{10}$

$(a+b)^n = \sum_{n=0}^{n} {}^nC_k a^{n-k} b^k$

$T_{k+1} = {}^{10}C_k (3x^4)^{10-k} (\frac{1}{x})^k$

$T_{k+1} = {}^{10}C_k 3^{10-k} \times x^{40-4k} \times x^{-k}$

$T_{k+1} = {}^{10}C_k 3^{10-k} \times x^{40-5k}$

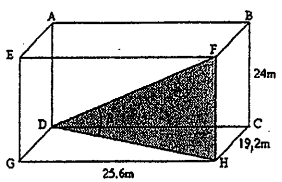
now $40 - 5k = 0$ so $k = 8$

So term independent of x will be

$= {}^{10}C_8 (3x^4)^{10-8} (\frac{1}{x})^8$

$= {}^{10}C_8 3^2$

$= 405$



d. $DH^2 = 25.6^2 + 19.2^2$

$DH^2 = 25.6^2 + 19.2^2$

$DH = 32$

$\therefore \tan\beta = \frac{24}{32}$

$\therefore \beta = 36°52'$

QUESTION FOUR SOLUTIONS

i. Now Volume of Sand in container is

Now $\frac{dV}{dt} = 36$ so after 5 seconds

$V = 180cm^3$ will have poured in.

Note: $h = d$ $(= 2r)$ so $r = h/2$

$V = \frac{1}{3}\pi r^2 h \rightarrow V = \frac{1}{3}\pi(\frac{h}{2})^2 h$

$V = \frac{\pi h^3}{12}$ $\quad \therefore \frac{dV}{dh} = \frac{\pi h^2}{4}$

$\therefore 180 = \frac{\pi h^3}{12}$

$h = \sqrt[3]{\frac{12 \times 180}{\pi}}$

$h = 8.826cm$

ii. We are after $\frac{dh}{dt}$

and we know $\frac{dV}{dt} = 36$

now $\frac{dh}{dt} = \frac{dV}{dt}.\frac{dh}{dV}$

$\frac{dh}{dt} = 36.\frac{4}{\pi h^2}$

when $h = 8.826$cm

$\frac{dh}{dt} = 36.\frac{4}{\pi(8.826)^2}$

$\frac{dh}{dt} = 0.588cms^{-1}$

iii. We are after $\frac{dS}{dt}$.

now S.A. of the sand $S = \pi r^2$

$r = h/2$ so $S = \pi r^2$

$S = \pi(\frac{h}{2})^2$

$S = \frac{\pi h^2}{4}$

so $\frac{dS}{dh} = \frac{\pi h}{2}$

NOW $\frac{dS}{dt} = \frac{dV}{dt}.\frac{dS}{dV}$

$\frac{dS}{dt} = \frac{dV}{dt}.\frac{dS}{dh}.\frac{dh}{dV}$

$\frac{dS}{dt} = 36.\frac{\pi h}{2}.\frac{4}{\pi h^2}$

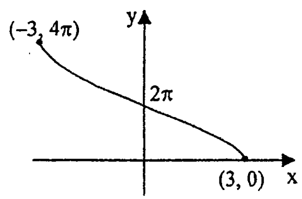
$\frac{dS}{dt} = \frac{72}{h}$

$\frac{dS}{dt} = \frac{72}{20}$

$\frac{dS}{dt} = 3.6cm^2 s^{-1}$

b) i. The domain $-3 \le x \le 3$, The range $0 \le y \le 4\pi$

ii.



iii. $A = \int_0^{2\pi} 3\cos\frac{y}{4}dy$

$A = 3\left[4\sin\frac{y}{4}\right]_0^{2\pi}$

$A = 3\left[4\sin\frac{\pi}{2} - 4\sin 0\right]$

$A = 12$ units^2

QUESTION FIVE SOLUTIONS

a.

$P(2ap,ap^2)$ and $Q(2aq,aq^2)$

Gradient of Chord PQ

$$\text{Gradient}=\frac{ap^2-aq^2}{2ap-2aq}$$
$$=\frac{a(p^2-q^2)}{2a(p-q)}$$
$$=\frac{a(p-q)(p+q)}{2a(p-q)}$$
$$=\frac{p+q}{2}$$

Eqn of chord

$$y-y_1=m(x-x_1)$$
$$y-ap^2=\frac{p+q}{2}(x-2ap)$$
$$y-ap^2=\frac{(p+q)x}{2}-\frac{2ap^2}{2}-\frac{2apq}{2}$$
$$y-ap^2=\frac{(p+q)x}{2}-ap^2-apq$$
$$y-\frac{(p+q)x}{2}+apq=0$$

ii) Eqn of tangent

$$x^2=4ay \rightarrow y=\frac{x^2}{4a}\rightarrow \frac{dy}{dx}=\frac{x}{2a}$$

when $x=2ap$ gradient $=p$

Eqn of tangent at P

$$y-ap^2=px-2ap^2$$
$$\therefore y-px+ap^2=0$$

iii. If PQ is a focal chord then $pq=-1$

Tng at P $y-px+ap^2=0$---(1)

Tng at Q $y-qx+aq^2=0$---(2)

mult (1) by q and (2) by p

$yq-pqx+ap^2q=0$---(3)

$yp-pqx+aq^2q=0$---(4)

(4)-(3)

$$y(p-q)-apq(p-q)=0$$
$$(p-q)(y-apq)=0$$

$\therefore y=apq$ now $pq=-1$ so $y=-a$

this means all y values are on the directrix

Gradient of tangent through P is p

Gradient of tangent through Q is q

now $pq=-1$ so tangents are perpendicular.

b)i) If $T=30+Ae^{-kt}$ is a solution then

$$\frac{dT}{dt}=-k(T-30)$$

so $\frac{d(30+Ae^{-kt})}{dt}=-k(30+Ae^{-kt}-30)$

$-kAe^{-kt}=-kAe^{-kt}$ so true

ii) When $t=0, T=100$

$\therefore 100=30+Ae^0$

$\therefore A=70$ When $t=20, T=60$

so $60=30+70e^{-20k}$

$$e^{-20k}=\frac{30}{70}$$
$$k=-\ln(\tfrac{3}{7})/20$$

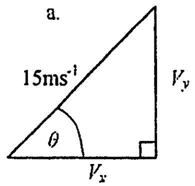
so now after a further 10 minutes $t=30$

$$T=30+70e^{-30\ln(\frac{3}{7})/20}$$
$$T=49.6396$$

$T=50^\circ$ to nearest degree.

Question 6 SOLUTIONS

a.



Horizontal Velocity

$$\frac{V_x}{15}=\cos\theta$$
$$V_x=15\cos\theta$$

now displacement $s=ut+0.5at$

horizontally a = 0

so $x=(15\cos\theta)t+0$

$x=15t\cos\theta$

Vertical Velocity

$$\frac{V_y}{15}=\sin\theta$$
$$V_y=15\sin\theta$$

vertically a = -10

$y=(15\sin\theta)t+0.5\times -10\times t$

$y=-5t^2+15t\sin\theta$

ii. From $x=15t\cos\theta, t=\frac{x}{15\cos\theta}$

subst. into $y=-5t^2+15t\sin\theta$

$$y=-5\left(\frac{x}{15\cos\theta}\right)^2+15\left(\frac{x}{15\cos\theta}\right)\sin\theta$$
$$y=\frac{-5x^2}{225\cos^2\theta}+\frac{15x\sin\theta}{15\cos\theta}$$
$$y=\frac{-x^2}{45}\sec^2\theta+x\tan\theta$$

iii. now when $x=5, y=2$

$$2=\frac{-25}{45}\sec^2\theta+5\tan\theta$$
$$18=-5\sec^2\theta+45\tan\theta$$
$$18=-5(1-\tan^2\theta)+45\tan\theta$$
$$5\tan^2\theta-45\tan\theta+23=0$$

let $x=\tan\theta$ to get $5x^2-45x+23=0$

$$x=\frac{45\pm\sqrt{(-45)^2-4\times5\times23}}{10}$$
$$\tan\theta=\frac{45\pm\sqrt{1565}}{10}$$
$$\theta=\tan^{-1}\left(\frac{45+\sqrt{1565}}{10}\right)$$

$\theta=83°15'$ (to nearest minute)

$$\theta=\tan^{-1}\left(\frac{45-\sqrt{1565}}{10}\right)$$

$\theta=28°33'$ (to nearest minute)

b.

$$\ddot{x}=-16x$$
$$\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

so $\frac{1}{2}v^2=\int\ddot{x}\,dx$

so $\frac{1}{2}v^2=\int -16x\,dx$

$$\frac{1}{2}v^2=-8x^2+c$$

when $x=3$, $v=12$

so $\frac{1}{2}\times144=-72+c$

$$c=144$$
$$\frac{1}{2}v^2=-8x^2+144$$

$$v=\pm\sqrt{-16x^2+288}$$
$$v=\pm\sqrt{-16x^2+288}$$
$$v=\pm4\sqrt{18-x^2}$$

so $\frac{dx}{dt}=\pm4\sqrt{18-x^2}$

so $\frac{dt}{dx}=\frac{1}{\pm4\sqrt{18-x^2}}$

and $\pm t=-\frac{1}{4}\cos^{-1}\frac{x}{3\sqrt{2}}+c$

when $t=0, x=3$

$$\therefore 0=-\frac{1}{4}\cos^{-1}\frac{3}{3\sqrt{2}}+c$$
$$\therefore c=\frac{\pi}{16}$$

$$\pm t=-\frac{1}{4}\cos^{-1}\frac{x}{3\sqrt{2}}+\frac{\pi}{16}$$
$$\pm4t=-\cos^{-1}\frac{x}{3\sqrt{2}}+\frac{\pi}{4}$$
$$x=3\sqrt{2}\left[\cos\left(\frac{\pi}{4}\pm4t\right)\right]$$

QUESTION SEVEN SOLUTIONS.

a. Prove true for $n = 1$

$13 \times 6^1 + 2 = 100$

which is divisible by 5

Assume true for $n = k$

$13 \times 6^k + 2 = 5M$

where m is a +ve integer

Prove true for $n = k + 1$

$$\begin{aligned} 13 \times 6^{k+1} + 2 &= 13 \times 6^k \times 6 + 2 \\ &= (13 \times 6^k + 2 - 2) \times 6 + 2 \\ &= (5M - 2) \times 6 + 2 \\ &= 30M - 10 \\ &= 5(6M - 2) \end{aligned}$$

which is divisible by 5. If the formula is true for $n = k$, it is true for $n = k + 1$.

Now we know it is true for $n = 1$ so it follows that it must be true for $n = 1+1 = 2$ and because it is true for $n = 2$ it must be true for $n = 3$ and so on for all positive integer values of n.

b.i. $x = \sin^2 5t + 2$

when $x = \frac{5}{2}$.

$\frac{5}{2} = \sin^2 5t + 2$

$\sin^2 5t = \frac{1}{2}$

$\sin 5t = \pm\frac{1}{\sqrt{2}}$

$5t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$........

$t = \frac{\pi}{20}, \frac{3\pi}{20}, \frac{5\pi}{20}$........

Particle first at $x = \frac{5}{2}$

after $\frac{\pi}{20}$

ii. $x = \sin^2 5t + 2$

$v = 10 \sin 5t \cos 5t$

$v = 5 \sin 10t$

when $t = \frac{\pi}{20}$, $v = 5 \sin \pi$,

so v =+5

positive value means particle is travelling in the positive direction at this time.

iii. $x = \sin^2 5t + 2$

$v = 5 \sin 10t$

$a = 50 \cos 10t$

$a = 50(1 - 2\sin^2 5t)$

$a = 50 - 100 \sin^2 5t$

now $x = \sin^2 5t + 2$

$x - 2 = \sin^2 5t$

$a = 50 - 100(x - 2)$

$a = 250 - 100x$

iv. $a = 50 - 100(x - 2)$

is of the form

$a = \text{constant} - n^2(x - a^2)$

Where $n = 10$ and $a^2 = 2$

so $a = \sqrt{2}$

v. $n = 10$

$$\begin{aligned} \text{period } &= \frac{2\pi}{n} \\ &= \frac{2\pi}{10} \\ &= \frac{\pi}{5} \text{ seconds.} \end{aligned}$$