THE SCOTS COLLEGE

2003
TRIAL HSC EXAMINATION

MATHEMATICS
EXTENSION 1

GENERAL INSTRUCTIONS

- Reading time - 5 minutes
- Working time - 2 hours
- Write using blue or black pen
- Board approved calculators may be used
- A table of integrals is provided
- All necessary working should be shown

- Start each question on a new booklet
- Attempt Questions 1 - 7
- All questions are of equal value

QUESTION 1

(a) Find the acute angle between the lines $2x - y = 0$ and $x + 3y = 0$, giving the answer correct to the nearest minute. [2]

(b) Solve the inequality $\dfrac{x}{x-3} \le 3$ [3]

(c) If u, v and w are the roots of $x^3 - 4x + 1 = 0$, find the value of $\dfrac{1}{u} + \dfrac{1}{v} + \dfrac{1}{w}$. [3]

(d) Solve the equation $\sin 2x = \tan x$ for $0 \le x \le \pi$. [4]

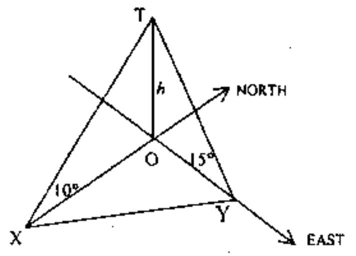
QUESTION 2 [START A NEW BOOKLET]

(a) A is the point $(-2, 1)$ and B is the point (x, y). The point $P(13, -9)$ divides AB externally in the ratio 5 : 3. Find the values of x and y. [3]

(b) **(i)** Show that the equation of the normal to the parabola $x^2 = 4ay$ at the point $T\left(2at, at^2\right)$ is $x + ty = 2at + at^3$. [2]

(ii) Hence show that there is only one normal to the parabola which passes through its focus. [1]

(c) A surveyor at X observes a tower due north. The angle of elevation to the top of the tower is 10°. He then walks 400m to a position Y which is due east of the tower. The angle of elevation from Y to the top of the tower is 15°.



(i) Write an expression for OY in terms of h. [1]

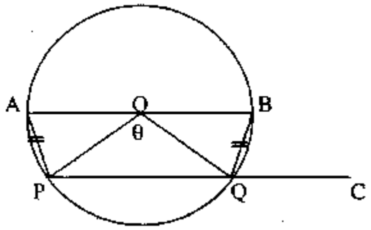
(ii) Calculate h to the nearest metre. [4]

(iii) Find the bearing of Y from X. [1]

QUESTION 3 [START A NEW BOOKLET]

(a) Evaluate $\int_0^{2\pi} \cos^2 2x \, dx$. [3]

(b)



The points A, B, P and Q lie on the circle with centre at O.

AB is a diameter and PC passes through Q.

AP is equal to BQ and ∠POQ = θ

(i) Express ∠AOP in terms of θ. [1]

(ii) Prove that AB is parallel to PC. [2]

(c) By graphing or some other justification, simplify [3]

(i) $\sin^{-1} x + \sin^{-1}(-x)$

(ii) $\tan^{-1} x + \tan^{-1}(-x)$

(iii) $\sin^{-1} x - \cos^{-1}(-x)$

(d) Find $\int_0^2 2x\sqrt{1-\frac{x}{2}}\, dx$ using the substitution $u = 1 - \frac{x}{2}$ [3]

QUESTION 4 [START A NEW BOOKLET]

(a) The surface area of a cube is increasing at a rate of 10cm^2 per second. Find the rate of increase of the volume of the cube when the edge of the cube has length 12cm. [4]

(b) N is the number of animals in a certain population at time t years. The population size N satisfies the equation $\frac{dN}{dt} = -k(N-1000)$ for some constant k.

(i) Verify where A is constant, that $N = 1000 + Ae^{-kt}$ is a solution of the equation. [2]

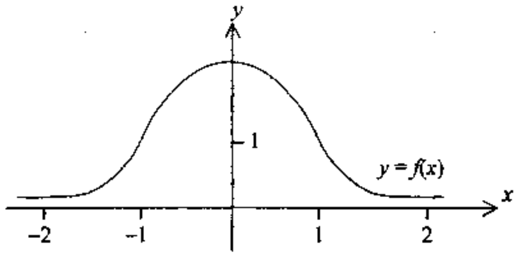
(ii) Initially there are 2500 animals but after 2 years there are only 2200 left. Find the values of A and k, to 2 decimal places. [2]

(iii) Find when the number of animals has fallen to 1300. [2]

(iv) Sketch the graph of the population size against time. [2]

QUESTION 5 [START A NEW BOOKLET]

(a) The graph below shows the derivative of $y = 2\tan^{-1} x$.



(i) Where does $y = 2\tan^{-1} x$ have its greatest slope and what is this slope? [2]

(ii) Calculate the x values correspond with $\frac{dy}{dx} = \frac{1}{3}$? [1]

(iii) Write an integral that represents the area in the first quadrant bounded by this curve, the x axis and $x = k$, where $k > 0$. [1]

(iv) By considering the limit as $k \to \infty$ determine the total area bounded by this curve and the x axis. [1]

(b) (i) Sketch the graph of function $f(x) = e^x - 4$. [1]

(ii) On the same diagram sketch the graph of the inverse function f^{-1}. [2]

(iii) Explain why the x coordinate of any point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^x - x - 4 = 0$. [1]

(iv) Show that the equation $e^x - x - 4 = 0$ has a root between $x = 1$ and $x = 2$. Use one application of Newton's method to approximate the root, to 2 decimal places. [3]

QUESTION 6 [START A NEW BOOKLET]

(a) Prove by Mathematical Induction that $1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$ for all positive integers n. [5]

(b) A particle moves in a straight line so that its displacement x from a fixed point 0 at time t is given by $x = 3\sin 2t + 4\cos 2t$.

(i) If the motion is expressed in the form of $x = R\sin(2t + \alpha)$ where α is in radians, evaluate the constants R and α, to 2 decimal places. [3]

(ii) Show that the motion is Simple Harmonic. [1]

(iii) What is the period of oscillation? [1]

(iv) Determine the maximum displacement from the centre of motion. [2]

QUESTION 7 [START A NEW BOOKLET]

(a) A projectile has an initial velocity V and an angle of projection θ.

(i) Assuming $\frac{d^2y}{dt^2} = -10$, $\frac{d^2x}{dt^2} = 0$ and initially $x = 0$, $y = 10$, find expressions for x and y. [3]

(ii) If $V = 13ms^{-1}$ and $\theta = \tan^{-1}\left(\frac{5}{12}\right)$ find the range of the projectile. [2]

(b) **(i)** Use the Chain Rule to show that

$$\frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$ [1]

(ii) The acceleration due to gravity is inversely proportional to the square of the distance x from the centre of the earth.

This can be written as $\frac{dv}{dt} = \frac{-k}{x^2}$. Find k if $\frac{dv}{dt} = -g$ when $x = R$. [1]

(iii) Hence show that $v^2 = \frac{2R^2g}{x} + u^2 - 2gR$ where the initial velocity of a rocket is u ms^{-1}, g is the acceleration due to gravity and R is the radius of the earth. [2]

(iv) Find the maximum distance that the rocket will travel from the centre of the earth. (Answer in terms of g, R and u). [2]

(v) Taking $g = 9.8$ms^{-2}, $R = 6400$km find the value of u in ms^{-1} for which the rocket will escape the gravity of the earth. [1]

2003 Ext 1 Trial Paper

Question 1

a) $m_1 = 2, \quad m_2 = -\frac{1}{3}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{2 + \frac{1}{3}}{1 - 2(\frac{1}{3})}\right|$

$= |7|$

$\therefore \theta = 81°52'$

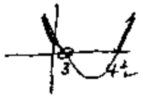
b) $\frac{x}{x-3} \leq 3$

$x(x-3) \leq 3(x-3)^2$

$x^2 - 3x \leq 3x^2 - 18x + 27$

$2x^2 - 15x + 27 \geqslant 0$

$(2x-9)(x-3) \geqslant 0$



$\therefore x < 3$ or $x \geqslant \frac{9}{2}$

c) $x^3 - 4x + 1 = 0$.

$u + v + w = 0$

$uv + uw + vw = -4$

$uvw = -1$

$\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = \frac{vw + uw + uv}{uvw}$

$= \frac{-4}{-1}$

$= 4$

d) $\sin 2x = \tan x \quad 0 \leq x \leq \pi$

$2\sin x\cos x = \frac{\sin x}{\cos x}$

$2\sin x\cos^2 x - \sin x = 0$

$\sin x(2\cos^2 x - 1) = 0$

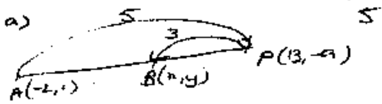
$\therefore \sin x = 0$ or $\cos x = \pm\frac{1}{\sqrt{2}}$

$x = 0, \pi$ $\quad x = \frac{\pi}{4}, \pi - \frac{\pi}{4}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

Question 2

a)



$m : n$
$5 : -3$

$x_1 = \frac{nx_1 + mx_2}{m+n}$ $\quad y = \frac{ny_1 + my_2}{m+n}$

$13 = \frac{-3(-2) + 5x}{5-3}$ $\quad -9 = \frac{-3(1) + 5y}{5-3}$

$26 = 6 + 5x$ $\quad -18 = -3 + 5y$

$5x = 20$ $\quad 5y = -15$

$x = 4$ $\quad y = -3$

b) $x^2 = 4ay$

(i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

At $x = 2at$,

$\frac{dy}{dx} = t$

$\therefore$ grad of normal $= -\frac{1}{t}$

$\therefore$ eqn of normal

$y - at^2 = -\frac{1}{t}(x - 2at)$

$ty - at^3 = -x + 2at$

$x + ty = 2at + at^3$

(ii) $S(0,a)$

$at = 2at + at^3$

$at + at^3 = 0$

$at(1 + t^2) = 0$

$\therefore t = 0$ or $t^2 = -1$ no solution.

c) (i) $\tan 15° = \frac{h}{OY}$

$OY = \frac{h}{\tan 15°}$

(ii) $\tan 10° = \frac{h}{OX}$

$OX = \frac{h}{\tan 10°}$

$\therefore OX^2 + OY^2 = 400^2$

$\frac{h^2}{\tan^2 10°} + \frac{h^2}{\tan^2 15°} = 160000$

$h^2\left(\frac{1}{\tan^2 10°} + \frac{1}{\tan^2 15°}\right) = 160000$

$h^2 = 3471.345..$

$h = 58.918...$

$h = 59\text{m}$

(iii) $\cancel{\tan}\theta = \frac{OY}{400}$

$\sin\theta = \frac{59}{400\tan 15°}$

$\theta = 033°T$.

Question 3.

a) $\cos^2 x = \frac{1}{2}(\cancel{2}\cos 2x - 1)$

$\cos^2 2x = \frac{1}{2}(\cancel{2}\cos 4x + 1)$

$\therefore \int_0^{2\pi} \cos^2 2x \, dx$

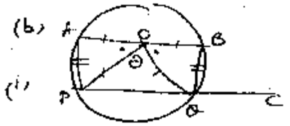
$= \frac{1}{2}\int_0^{2\pi} \cancel{2}\cos 4x + 1 \, dx$

$= \frac{1}{2}\left[\frac{\sin 4x}{\cancel{2}4} + x\right]_0^{2\pi}$

$= \frac{1}{2}\left(\frac{\sin 8\pi}{4} + 2\pi - \left(\frac{\sin 0}{2} - 0\right)\right)$

$= \pi$.

(b)



(i) $\triangle AOP \equiv \triangle BOQ$ (SSS)

$\therefore \angle AOP = \angle BOQ$

$\therefore \angle AOP = \frac{180 - \theta}{2}$

$= 90° - \frac{\theta}{2}$

(ii) $\angle OPQ = \frac{180 - \theta}{2}$ (base $\angle$s of isosceles $\triangle$)

$= 90 - \frac{\theta}{2}$

$\therefore \angle AOP = \angle OPQ$ & they are alternate

$\therefore AB \parallel PQ$.

c) (i) $\sin^{-1}(x) + \sin^{-1}(-x)$

$= \sin^{-1}(x) - \sin^{-1}(x)$

$= 0$

(ii) $\tan^{-1}(x) + \tan^{-1}(-x)$

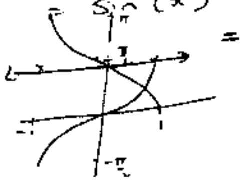
$= \tan^{-1}(x) - \tan^{-1}(x)$

$= 0$

(iii) $\sin^{-1}(x) - \cos^{-1}(-x)$

$= \sin^{-1}(x) - \cos^{-1}(x)$

$= -\frac{\pi}{2}$



d) $u = 1 - \frac{x}{2}$ $x = 0$ $u = 1$

$x = 2$, $u = 0$

$x = 2(1-u)$

$\frac{du}{dx} = -\frac{1}{2}$

$$\int_0^2 2x\sqrt{1-\frac{x}{2}}\,dx$$

$$= 2.-2\int_1^0 2(1-u)\sqrt{u}\,.du$$

$$= -8\int_0^1 u^{\frac{1}{2}}(1-u)\,du$$

$$= 8\int_1^0 u^{\frac{1}{2}} - u^{3/2}\,du$$

$$= 8\left[\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right]_0^1$$

$$= 8\left(\frac{2}{3} - \frac{2}{5} - 0\right)$$

$$= \frac{32}{15}$$

Question 4

a) $\frac{dA}{dt} = 10$ $x = 12$

$\frac{dV}{dt} = ?$

$A = 6x^2$ $V = x^3$

$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$

$10 = 12x \cdot \frac{dx}{dt}$

$x = 12$, $10 = 144 \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{10}{144}$

$\frac{dx}{dt} = \frac{5}{72}$

$\therefore \frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$

$= 3x^2 \cdot \frac{dx}{dt}$

$= 3(12)^2 \cdot \frac{5}{72}$

$= 30\text{ cm}^3/\text{s}$

(b)

(i) $N = 1000 + Ae^{-kt}$

$\frac{dN}{dt} = Ae^{-kt}.-k$

$= -k(N - 1000)$

(ii) $t = 0$, $N = 2500$

$t = 2$, $N = 2200$

$2500 = 1000 + A$

$A = 1500$

$2200 = 1000 + 1500e^{-2k}$

$1500e^{-2k} = 1100$

$e^{-2k} = \frac{11}{15}$

$-2k = \ln\left(\frac{11}{15}\right)$

$k = \frac{\ln\left(\frac{11}{15}\right)}{-2}$

$k = 0.16$ (2dp)

(iii) $1300 = 1000 + 1500e^{-0.16t}$

$1500e^{-0.16t} = 300$

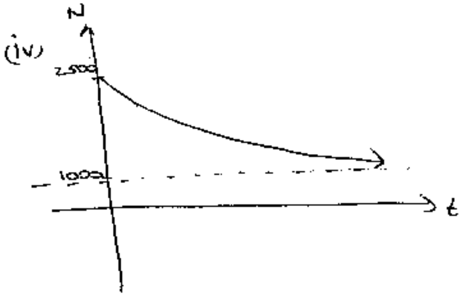
$e^{-0.16t} = \frac{1}{5}$

$-0.16t = \ln\left(\frac{1}{5}\right)$

$t = \frac{\ln\left(\frac{1}{5}\right)}{-0.16}$

$t = 10.06$ years

(iv)



Question 5

a)(i) $y = 2\tan^{-1}x$

$\frac{dy}{dx} = \frac{2}{1+x^2}$

greatest slope occurs at $x = 0$, $\frac{dy}{dx} = 2$

(ii) $\frac{2}{1+x^2} = \frac{1}{3}$

$6 = x^2 + 1$

$x^2 = 5$

$x = \pm\sqrt{5}$

(iii) $A = \int_0^k f(x)\,dx$

(iv) As $k \to \infty$, ~~Area under~~ ~~Area~~ Area under curve

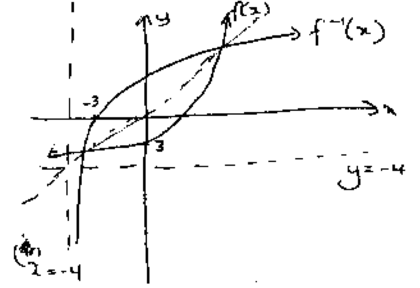
$A = 2\int_0^{\infty} f(x)\,dx$

$= 2.\left[2\tan^{-1}x\right]_0^{\infty}$

$= 2.\ 2.\frac{\pi}{2}$

$= 2\pi$ sq units.

(b)(i) $f(x) = e^x - 4$



(ii) $f(x)$ & $f^{-1}(x)$ are reflections along the line $y = x$.

Points of intersection are $y = e^x - 4$ & $y = x$ hold true

ie $e^x - 4 = x$

$e^x - x - 4 = 0.$

(iv) let $f(x) = e^x - x - 4$

$f(1) = e - 1 - 4 < 0$

$f(2) = e^2 - 2 - 4 > 0$

$\therefore$ root lies between $x = 1$ & $x = 2$

$f'(x) = e^x - 1$ $x_1 = 1.5$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.5 - \frac{f(1.5)}{f'(1.5)}$

$= 1.79$ (2 dp)

Question 6

a) Step 1: Need to prove $n = 1$ is true

LHS = 1, RHS = $\frac{1(3(1)-1)}{2}$

$= 1$

$=$ LHS

$\therefore n = 1$ is true

Step 2: Assume that $n = k$ is true

ie $1 + 4 + 7 + \ldots + (3k-2) = \frac{k(3k-1)}{2}$

Need to prove that $n = k+1$ is true

ie $1 + 4 + 7 \ldots + (3k-2) + (3k+1)$

$= \frac{(k+1)(3k+2)}{2}$

LHS $= 1 + 4 + 7 + \ldots + (3k-2) + (3k+1)$

$= \frac{k(3k-1)}{2} + (3k-1)$

$= \frac{1}{2}(3k^2 - k + 6k - 2)$

$= \frac{1}{2}(3k^2+6k-2)$

$= \frac{1}{2}(3k+1)(k-2)$

$=$ RHS

∴ $n=k+1$ is also true

Step 3 Since $n=1$, $n=k$ & $n=k+1$ are all true

then $n=2, n=3 \ldots$ are true

∴ $1+4+7+\ldots+(3n-2) = \frac{n(3n-1)}{2}$

(b) $x = 3\sin 2t + 4\cos 2t$

(i) $3\sin 2t + 4\cos 2t$

$= R\sin(2t+\alpha)$

$= R\sin 2t\cos\alpha + R\cos 2t\sin\alpha$

∴ $R\cos\alpha = 3$ $\qquad R^2 = 3^2+4^2$

$R\sin\alpha = 4$ $\qquad R = \sqrt{25}$

$\tan\alpha = \frac{4}{3}$ $\qquad R = 5$

$\alpha = 0{\cdot}93$

(ii) $x = 5\sin(2t+0{\cdot}93)$

$\dot{x} = 10\cos(2t+0{\cdot}93)$

$\ddot{x} = -20\sin(2t+0{\cdot}93)$

$= -4x$

∴ motion is S.H.

(iii) period $= \frac{2\pi}{2}$

$= \pi$

(iv) max disp when $\dot{x} = 0$

$10\cos(2t+0{\cdot}93) = 0$

$\cos(2t+0{\cdot}93) = 0$

$2t + 0{\cdot}93 = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$2t = \frac{\pi}{2} - 0{\cdot}93, \frac{3\pi}{2} - 0{\cdot}93$

$t = 0{\cdot}32, 1{\cdot}89 \ldots$

At $t = 0{\cdot}32$,

$x = 5\sin(2(0{\cdot}32)+0{\cdot}93)$

$x = 5$

Question 7

a) $\frac{d^2x}{dt^2} = 0$ $\qquad$ Initially, $\dot{x} = V\cos\theta$, $\dot{y} = V\sin\theta$

$\frac{dx}{dt} = C_1$

$t=0, \frac{dx}{dt} = V\cos\theta$, ∴ $\frac{dx}{dt} = V\cos\theta$

$x = \int V\cos\theta \, dt$

$x = Vt\cos\theta + C_2$

$t=0, x=0, C_2 = 0$. ∴ $x = Vt\cos\theta$

$\frac{d^2y}{dt^2} = -10$

$\frac{dy}{dt} = \int -10 \, dt$

$= -10t + C_3$

$t=0, \frac{dy}{dt} = V\sin\theta$, ∴ $C_3 = V\sin\theta$

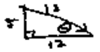
∴ $\frac{dy}{dt} = -10t + V\sin\theta$

$y = \int -10t + V\sin\theta \, dt$

$y = -5t^2 + Vt\sin\theta + C_4$

$t=0, y=10$, ∴ $C_4 = 10$

∴ $y = -5t^2 + Vt\sin\theta + 10$

ii) $V = 13$ 

sub $y=0$, $-5t^2 + t \cdot 13 \cdot \frac{5}{13} + 10 = 0$

$-5t^2 + 5t + 10 = 0$

$t^2 + t - 2 = 0$

$(t-2)(t+1) = 0$

∴ $t = 2$ or $t = -1$ but $t \geq 0$

∴ $t = 2$

When $t = 2$, $x = 13 \cdot 2 \cdot \frac{12}{13} = 24$m

b) (i) $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$

$= v \cdot \frac{dv}{dx}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2} \cdot 2v \cdot \frac{dv}{dx}$

$= v \cdot \frac{dv}{dx}$

∴ $\frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(ii) $\frac{dv}{dt} = \frac{-k}{x^2}$

sub $\frac{dv}{dt} = -g$, $x = R$,

$-g = -\frac{k}{R^2}$

$R^2g = k$

(iii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{R^2g}{x^2}$

∴ $\frac{1}{2}v^2 = \int -\frac{R^2g}{x^2} \, dx$

$\frac{1}{2}v^2 = \frac{R^2g}{x} + c$

When $x = R$, $v = u$

$\frac{1}{2}u^2 = \frac{R^2g}{R} + c$

∴ $c = \frac{1}{2}u^2 - Rg$

∴ $\frac{1}{2}v^2 = \frac{R^2g}{x} + \frac{1}{2}u^2 - Rg$

∴ $v^2 = \frac{2R^2g}{x} + u^2 - 2gR$

iv) max distance, $v = 0$

$\frac{2R^2g}{x} + u^2 - 2gR = 0$

$\frac{2R^2g}{x} = 2gR - u^2$

∴ $x = \frac{2R^2g}{2gR - u^2}$

(v) $g = 9{\cdot}8$, $R = 6400$

as $x \to \infty$, $u^2 = 2gR$

$u^2 = 2(9{\cdot}8)(6400)$

$u = \pm 11200$

but $u > 0$ ∴ $u = 11200\text{ms}^{-1}$