The Scots College

2009 Trial HSC Examination

Year 12 Mathematics

Extension 1

General Instructions
- Reading time: 5 mins
- Working time : 2 Hours
- Write using a blue or black pen
- Board approved calculators may be used
- A standard table of integrals is provided
- All necessary working should be shown in every question
- Start each question in a new booklet

Total Marks HSC: 84
Weighting : 40%
Attempt Questions 1—7
All questions are of equal value

Question 1 (12 marks) Start in a SEPARATE booklet **Marks**

(2)

(a) Find $\int \frac{1}{16\,x^2 + 1} \; dx$

(b) Using the substitution $u = x^3 + 3x$, or otherwise, find (3)

$$\int (x^2 + 1)(x^3 + 3x)^{-3}\, dx$$

(c) Solve $\frac{2}{x-3} < 4$. (3)

(d) Evaluate (2)

$$\lim_{x \to 0} \frac{4\, sin2x}{3x}$$

(e) Let A be the point $(-2, -2)$ and let B be the point $(6\,,\ 10)$. Find the coordinates of the point P which divides the interval AB externally in the ratio $1 : 2$. (2)

Question 2 (12 marks) Start in a SEPARATE booklet **Marks**

(a) Find (2)

$$\frac{d}{dx}\ (x\ \sin^{-1} 3x)$$

(b) Let $f(x) = 2x^2 + 3x$. Use the definition (2)

$$f\ '(a) = \lim_{h \to 0} \frac{f\,(a + h) - f\,(a)}{h}$$

to find the derivative of $f(x)$ at the point $x = 5$.

(c) Find (3)

$$\int_0^{\pi/4} \cos^2 5x \ \ dx$$

(d) Solve the equation (2)

$$2 \sin^2 \theta = 9 - 15 \cos \theta \quad , \quad 0^0 \leq \theta \leq 360^0$$

(e) Use the principle of mathematical induction to show that $x^n - y^n$ is divisible by $x - y$ when n is a positive integer. (3)

Question 3 (12 marks) Start in a SEPARATE booklet **Marks**

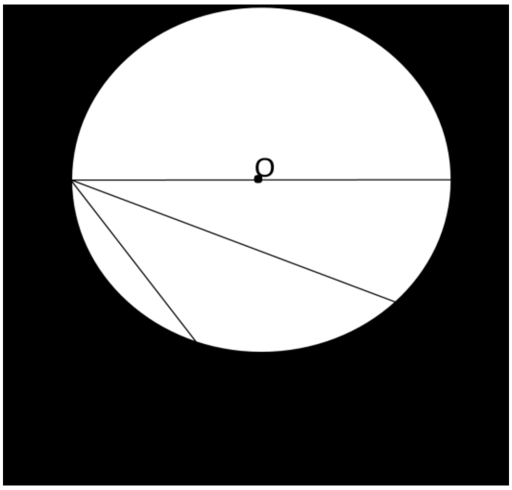
(a) (i) Show that the equation $\log_e x \times \sec x = 1$ has a root between 1 and 2. (1)

(ii) Using the above and one application of Newton's method find a better approximation of the root of the equation in part (i), leaving your answer correct to three decimal places. (3)

(b) (i) Differentiate $y = 5e^{5x}\sin x - e^{5x}\cos x$ with respect to x . (2)

(ii) Hence, or otherwise, find $\int e^{5x}\sin x \, dx$. (1)

(c) The figure below is not to scale. (3)



NP is a tangent to the circle with centre 0 and LM is the diameter. If NM bisects the angle LMP, prove that MNP is a right angled triangle.

(d) Find the exact value of $\sin\left(2\cos^{-1}\left(\frac{3}{5}\right)\right)$ (2)

Question 4 (12 marks) Start in a SEPARATE booklet **Marks**

(a) A curve has parametric equations (2)

$$x = t - \frac{1}{t} \quad , \quad y = t^2 + 1 + \frac{1}{t^2}$$

Find the Cartesian equation for this curve.

(b) (i) Express $3\sin x - 4\cos x$ in the form $R\sin(x - \alpha)$. (3)
Where $R > 0$ *and* $0 < \propto < \frac{\pi}{2}$.

(2)

(ii) Hence or otherwise find the general solution of $4\cos x - 3\sin x = 2.5$

(c)

[redacted]

The conical vessel in the diagram is being filled with water at the rate of $10\ {cm^3}/{min}$. The semi-vertical angle of the vessel is 45^o. The radius of the water surface is $r\ cm$ at any time t minutes, the perpendicular height is $h\ cm$ and the slant edge is $l\ cm$.

(i) Find the volume V of water in terms of the slant edge $\boldsymbol{\ell}$.

(2)

(ii) Find the rate at which the slant edge l is increasing when the radius of the water surface is 7 cm.

(3)

Question 5 (12 marks) Start in a SEPARATE booklet **Marks**

(a) For the function $y = 2\cos^{-1}(4x - 1)$

(i) State the largest possible domain and range. (2)

(ii) Sketch a neat graph of the function. (2)

(b) Aaron and David roll a die alternately in a game. The first person to roll a 6 wins the game. Aaron rolls first.

(i) What is the probability of Aaron winning in his second roll? (1)

(ii) What is the probability of Aaron winning the game? (2)

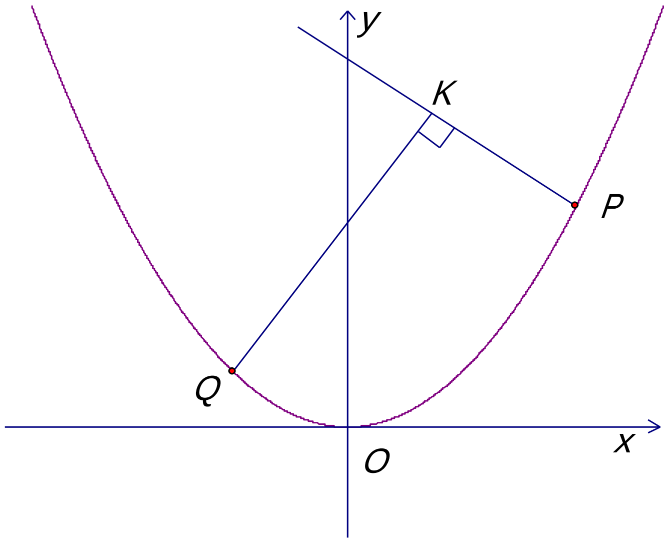
(c) Solve the equation $2x^3 - 7x^2 - x - 1 = 0$, given that the roots are in geometric progression. (3)

(d) If $\cos(2\sin x) = \sin(2\cos x)$, then show that (2)

$$\sin x \pm \cos x = n\pi \pm \frac{\pi}{4}$$

Question 6 (12 marks) Start in a SEPARATE booklet **Marks**

(a)



$P\ (\ 2at\ ,\ \ at^2\ )$ is any point on the parabola $x^2\ =\ 4ay$.

(i) show that the equation of the Normal at P is (2)

$$y\ =\ at^2\ +\ 2a\ -\ \frac{x}{t}$$

(ii) Q is another point on the parabola such that the Normals at P and Q are at right angles to each other and intersect at K . Find the coordinates of the point Q in terms of t. (2)

(iii) Show that the coordinates of K are (3)

$$x\ = at\ -\ \frac{a}{t}\quad \text{and}\quad y\ = at^2\ +\ a\ +\ \frac{1}{t^2}.$$

Question 6 continued……….

(b) The time is 10:20 a.m. and Arun has just finished making two cups of coffee which have a temperature of $95\ ^{o}C$. The coffees are loosing heat according to Newton's law of cooling $\frac{dT}{dt} = -k\,(T-20)$ where T is the temperature of the coffee in degrees Celsius after t minutes, k is a positive constant and the temperature of the room is $20\ ^{o}\ C$.

(i) Show that $T = 20 + 75\,e^{-kt}$ is a solution of the equation . (1)

(ii) At 10:30 a.m. the coffees have cooled to $75\ ^{o}\ C$. Find the value of k to two decimal places. (2)

(iii) The second cup of coffee is for Lawrie who will drink it as long as its temperature is not less than $60^{o}\ C$. What is the latest time that he could arrive to drink his coffee? (2)

Question 7 (12 marks) Start in a SEPARATE booklet **Marks**

(a) In a scientific cloud seeding experiment a projectile is fired from O with a velocity V at angle of θ to level ground. The cloud is at a height h above the ground. The projectile returns to the ground at a horizontal distance of Z metres.

The equations of motion of the projectile are

$$x = Vtcos\theta, \quad y = -\frac{1}{2}gt^2 + Vtsin\theta$$

(You are NOT Required to prove the above equations)

(i) Show that when the projectile returns to the ground level (2)

$$Z = \frac{V^2 \sin 2\theta}{g} \quad metres.$$

(ii) If the projectile MUST pass through the cloud twice then show that (2)

$$h < \frac{V^2 \ sin^2\theta}{2g}$$

(iii) Consider $g = 10\frac{m}{s^2}$, $\theta = 60^o$, $V = 320\frac{m}{s}$ and $h = 3000\ m.$ (2)
Find the times at which the projectile passes through the clouds. Leave your answer correct to one decimal.

(iv) Find the horizontal distance covered between the two times found in part (iii). (2)

Question 7 continued………

(b) A particle is moving so that $\ddot{x} = 32x^3 - 48x^2 + 16x$

Initially the velocity is $24\ m/s$ and $x = 3$ metres.

(i) Show that $v^2 = 16x^2\,(x-1)^2$ (2)

(ii) Hence or otherwise show that (2)

$\int \frac{1}{x(x-1)}\,dx = 4t$, where t is time elapsed.

Standard Integrals

$\int x^n \, dx$	$= \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1; x \neq 0, \text{if } n < 0$
$\int \frac{1}{x} \, dx$	$= \ln x + C, \quad x > 0$
$\int e^{ax} \, dx$	$= \frac{1}{a} e^{ax} + C, \quad a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a} \sin ax + C, \quad a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a} \cos ax + C, \quad a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a} \tan ax + C, \quad a \neq 0$
$\int \sec ax \tan ax \, dx$	$= \frac{1}{a} \sec ax + C, \quad a \neq 0$
$\int \frac{1}{a^2 + x^2} \, dx$	$= \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \quad a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$	$= \sin^{-1} \frac{x}{a} + C, \quad a > 0, \quad -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx$	$= \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx$	$= \ln\left(x + \sqrt{x^2 + a^2}\right) + C$

NOTE : $\ln x \equiv \log_e x, \quad x > 0$

Yr 12 HSC Ext 1 TRIAL

Solutions

Q1 (a)

$\int \frac{1}{16x^2+1}\,dx$

$= \frac{1}{16}\int \frac{1}{x^2+(\frac{1}{4})^2}\,dx$

$= \frac{1}{16} \times \frac{1}{1/4} \tan^{-1}(4x) + c$

$= \frac{1}{4}\tan^{-1}(4x) + c$

(b) $\int (x^2+1)(x^3+3x)^{-3}\,dx$

$u = x^3+3x$

$\frac{du}{dx} = 3x^2+3$

$dx = \frac{du}{3(x^2+1)}$

$\therefore \int (x^2+1)\, u^{-3} \frac{du}{3(x^2+1)}$

$\Rightarrow \frac{1}{3}\frac{u^{-2}}{(-2)} + c$

$\Rightarrow -\frac{1}{6(x^3+3x)} + c$

(c)

$\frac{2}{x-3} < 4$

$2(x-3) < 4(x-3)^2$

$4(x-3)^2 - 2(x-3) > 0$

$2(x-3)[2(x-3)-1] > 0$

$2(x-3)(2x-7) > 0$

$x < 3$ or $x > \frac{7}{2}$

(d) $\lim_{x\to 0} 4\frac{\sin 2x}{3x}$

$= \frac{4}{3} \times 2 \lim_{x\to 0} \frac{\sin 2x}{2x}$

$= \frac{8}{3}$

(e) m : n

1 : −2

(−2, −2) (6, 10)

$\frac{mx_2+nx_1}{m+n}$ $\frac{my_2+ny_1}{m+n}$

$\frac{1(6)-2(-2)}{1-2}$ $\frac{1(10)-2(-2)}{1-2}$

$= -10$ $= -14$

(−10, −14)

Q.2

(a) $\frac{d}{dx}(x\sin^{-1}3x)$

$= \sin^{-1}3x\,(1) + x\frac{1}{\sqrt{1-9x^2}}(3)$

$= \sin^{-1}3x + \frac{3x}{\sqrt{1-9x^2}}$

(b) $f(a) = 2a^2+3a$

$f(a+h) = 2(a+h)^2+3(a+h)$

$= 2a^2+4ah+2h^2+3a+3h$

$f'(a) = \frac{2a^2+4ah+2h^2+3a+3h-2a^2-3a}{h}$

$= 4a+3.$

$f'(5) = 4\times 5+3 = 23$

(c) $\int_0^{\pi/4} \cos^2 5x\,dx$

$\int_0^{\pi/4} \frac{1}{2} + \frac{1}{2}\cos 10x\,dx.$

$\left[\frac{1}{2}x + \frac{1}{2}\sin 10x \times 10\right]_0^{\pi/4}$

$\left[\frac{x}{2} + 5\sin 10x\right]_0^{\pi/4}$

$\frac{\pi}{8} + 5\sin\frac{5\pi}{2} - 0 - 5\times 0$

$\frac{\pi}{8} + 5\sin\frac{5\pi}{2}$

(d) $2\sin^2\theta = 9 - 15\cos\theta$

$2(1-\cos^2\theta) = 9-15\cos\theta$

$2 - 2\cos^2\theta = 9-15\cos\theta$

$2\cos^2\theta - 15\cos\theta + 7 = 0$

$(2\cos\theta-1)(\cos\theta-7) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = 7$

$60°, 300°$

Q.2 (e)

$x^n - y^n$ is divisible by $x-y$

(a) for $n=1$ $x-y$ is divisible by $x-y$

$\therefore$ True for $n=1$

(b) Assume that it is true for $n=k$

i.e. $x^k - y^k$ is divisible by $x-y$ (hypothesis) — (1)

consider $x^{k+1} - y^{k+1} = x^k \cdot x - y^k \cdot y$

$= x^k \cdot x - x^k \cdot y + x^k \cdot y - y^k \cdot y$ *

$= x^k(x-y) + y(x^k - y^k)$.

The first term is divisible by $(x-y)$

The 2nd term is divisible by $(x-y)$ due to statement / assumption (1).

i.e. $x^{k+1} - y^{k+1}$ is divisible by $x-y$

(c) Hence by the principle of mathematical induction it is true for all positive integral values of n.

Q.3.

(a) If $f(x) = \cos x - \log_e x$

Assume $x_0 = 1.5$.

$f'(x) = -\sin x - \frac{1}{x}$

$x_1 = x_0 - \frac{f(1.5)}{f'(1.5)}$

$= 1.5 - \frac{(-0.3347)}{-1.6642}$

$= 1.29888$

$= 1.299$

If. $f(x) = \log_e x \sec x - 1 =$

$f'(x) = \log_e x \sec x \tan x + \sec x \cdot \frac{1}{x}$

$x_0 = 1.5$.

$x_1 = 1.5 - \frac{f(x_0)}{f'(x_0)}$

$= 1.5 - \frac{4.73}{90.25}$

$= 1.4476$

$= 1.448$

(b) (i) $y = 5e^{5x}\sin x - e^{5x}\cos x$

$= e^{5x}(5\sin x - \cos x)$

$\frac{dy}{dx} = (5\sin x - \cos x)\,5 \times e^{5x} + e^{5x}(5\cos x + \sin x)$

$= e^{5x}[25\sin x - 5\cos x + 5\cos x + \sin x]$

$= e^{5x}[26 \sin x]$ ✓

(ii) From part (i)

$\int e^{5x}\, 26 \sin x \, dx = 5e^{5x}\sin x - e^{5x}\cos x + c$

$\int e^{5x} \sin x \, dx = \frac{e^{5x}}{26}(5\sin x - \cos x) + c$

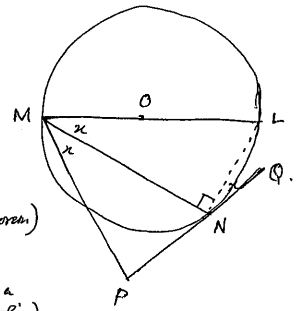
Q.3.

(c)

Construct LN

Let $\angle LMN = \angle NMP = x$ [given]

$\angle LNM = 90^\circ$ (Angle in a semicircle)



$\angle LNQ = \angle NML = x$ (Alt. Seg. & Tangent Theorem)

$\angle PNM = 180 - (90 + x)$ ($\because$ PNQ is a straight line)

$\Rightarrow \angle PNM = 90 - x$

In triangle MNP

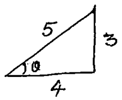
$\angle MPN =$ ~~90°~~

$= 90^\circ$ $\quad\because$ sum of angles in a triangle is 180.

($\cancel{x} + 90 - \cancel{x} + \angle MPN = 180$.
$\angle MPN = 90^\circ$)

Hence proved.

(d) $\sin\left(2\cos^{-1}\left(\frac{3}{5}\right)\right)$



$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2 \times \frac{4}{5} \times \frac{3}{5}$

$= \frac{24}{25}$

Q.4.

(a) $x = t - \frac{1}{t}$, $y = t^2 + 1 + \frac{1}{t^2}$

$y = t^2 - 2t \times \frac{1}{t} + \frac{1}{t^2} + 1 + 2$

$y = \left(t - \frac{1}{t}\right)^2 + 3$

$y = x^2 + 3$

(b)(i) $3\sin x - 4\cos x = R\sin(x - \alpha)$

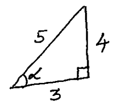
$= R\sin x\cos\alpha - R\cos x\sin\alpha$

$\Rightarrow R\cos\alpha = 3$ — ①

$-R\sin\alpha = -4$ — ②

$R^2 = 3^2 + 4^2$

$R = \pm 5$



② ÷ ① $\Rightarrow$ $\tan\alpha = \frac{4}{3}$

$\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 0.93^c$ $[53^\circ 8']$

$\therefore 3\sin x - 4\cos x = 5\sin\left(x - \tan^{-1}(4/3)\right)$

(ii) $-5\sin\left(x - \tan^{-1}(4/3)\right) = 2.5$

$\sin\left(x - \tan^{-1}(4/3)\right) = -1/2$.

$\sin\left(x - \tan^{-1}(4/3)\right) = -\sin(1/2)$

$x - \tan^{-1}(4/3) = n\pi + (-1)^n\left(-\sin^{-1}(1/2)\right)$

$x = n\pi + \tan^{-1}(4/3) + (-1)^n\left(-\sin^{-1}(1/2)\right)$

(c) $h = r$ since semivertical angle $= 45^\circ$

$l^2 = h^2 + r^2$

$l^2 = 2r^2$

$r = l/\sqrt{2}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3$

$= \frac{1}{3}\pi\left(\frac{l}{\sqrt{2}}\right)^3$

$\frac{dV}{dl} = \frac{\pi l^2}{2^{3/2}}$

$\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt}$

$10 = \frac{\pi l^2}{2^{3/2}} \times \frac{dl}{dt}$

$\frac{dl}{dt} = \frac{10 \times 2^{3/2}}{\pi l^2}$

$r = 7 \quad l = 7\sqrt{2}$

$\therefore \frac{dl}{dt} = \frac{10 \times 2^{3/2}}{\pi(7\sqrt{2})^2} = \frac{10\sqrt{2}}{49\pi}$ cm/min

$= 0.092$ cm/min

Q.5.

(a) $y = 2\cos^{-1}(4x-1)$

$\frac{y}{2} = \cos^{-1}(4x-1)$

(i) Domain : $-1 \le 4x-1 \le 1$

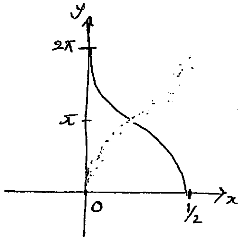
$0 \le 4x \le 2$

$0 \le x \le \frac{1}{2}$

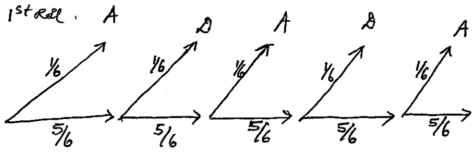
Range : $0 \le \frac{y}{2} \le \pi$

$0 \le y \le 2\pi$

(ii)



(b) 1st Roll.



(i) P (Aaron winning in the 2nd Roll) = $\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}$

(ii) P (Aaron winning) = $\frac{1}{6} + \frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} + \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} + \cdots$

$a = \frac{1}{6} \qquad r = \left(\frac{5}{6}\right)^2$

$S = \frac{a}{1-r}$

$= \frac{1/6}{1-\left(\frac{5}{6}\right)^2} = \frac{1/6}{\frac{36-25}{36}} = \frac{1/6}{11/36}$

$= \frac{1}{6} \times \frac{36}{11}$

$= \frac{6}{11}$

Q. 5(c) Method 1

$2x^3 - 7x^2 + 7x - 2 = 0$

Let α, β, γ be the roots.

$\therefore \frac{\beta}{\alpha} = \frac{\gamma}{\beta} = r$ [G.P].

$\beta^2 = \alpha\gamma$ — ①

Also $\alpha + \beta + \gamma = \frac{7}{2}$ — ②

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{7}{2}$ — ③

$\alpha\beta\gamma = 1$ — ④

$\alpha\beta + \beta^2 + \beta\gamma = \frac{7}{2}$ [sub ① in ③

$\beta(\alpha+\beta+\gamma) = \frac{7}{2}$

$\beta\left(\frac{7}{2}\right) = \frac{7}{2}$ [subbing ②.

$\beta = 1$ ✓

$\therefore \alpha\gamma = 1$ [sub $\beta = 1$ in ④

$\gamma = \frac{1}{\alpha}$.

use this info. in ②

$\alpha + 1 + \frac{1}{\alpha} = \frac{7}{2}$.

$2\alpha^2 - 5\alpha + 2 = 0$.

$(2\alpha - 1)(\alpha - 2) = 0$

$\alpha = \frac{1}{2}$ or $\alpha = 2$

When $\alpha = \frac{1}{2}$ $\gamma = 2$

$\alpha = 2$ $\gamma = \frac{1}{2}$.

Answer : $\frac{1}{2}, 1, 2$

or $2, 1, \frac{1}{2}$.

Method 2.

Let the roots be

$\frac{\alpha}{r}, \alpha, r\alpha$.

$\frac{\alpha}{r} + \alpha + r\alpha = \frac{7}{2}$ — ①

$\frac{\alpha}{r}\alpha + \frac{\alpha}{r}\times r\alpha + \alpha r\alpha = \frac{7}{2}$ — ②

$\frac{\alpha}{r} \times \alpha \times r\alpha = 1$. — ③.

$\alpha^3 = 1$

$\alpha = 1$

sub $\alpha = 1$ in ①

$\frac{1}{r} + 1 + r = \frac{7}{2}$

$2r^2 - 5r + 2 = 0$

$(2r-1)(r-2) = 0$

$r = \frac{1}{2}$ or $r = 2$.

If $r = \frac{1}{2}$ the roots are $2, 1, \frac{1}{2}$.

If $r = 2$ the roots are $\frac{1}{2}, 1, 2$.

5(d).

$\cos(2\sin x) = \sin(2\cos x)$

$\cos(2\sin x) = \cos(\pi/2 - 2\cos x)$

$2\sin x = 2n\pi \pm \left(\frac{\pi}{2} - 2\cos x\right)$ general solution

$2\sin x = 2n\pi \pm \frac{\pi}{2} \mp 2\cos x$

$2\sin x \pm 2\cos x = 2n\pi \pm \frac{\pi}{2}$

$\sin x \pm \cos x = n\pi \pm \frac{\pi}{4}$

$\sqrt{2}\sin\left(x \pm \frac{\pi}{4}\right) = n\pi \pm \frac{\pi}{4}$

$\sin\left(x \pm \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(n\pi \pm \frac{\pi}{4}\right)$

Q 6

(a) (i) $x^2 = 4ay$ $P(2at, at^2)$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

at $x = 2at$ gradient $= \frac{2at}{2a} = t$ for the tangent at P.

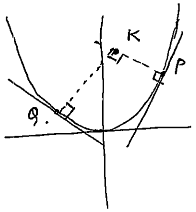
gradient of Normal is $-\frac{1}{t}$

∴ equation of Normal (PK) is

$y - at^2 = -\frac{1}{t}(x - 2at)$

$y - at^2 = -\frac{x}{t} + 2a$

$y = at^2 + 2a - \frac{x}{t}$ — ①



(ii) Let the point Q be $(2aq, aq^2)$

∴ The gradient of Normal at Q is $-\frac{1}{q}$.

This normal is perpendicular to PK

∴ $-\frac{1}{q} \times -\frac{1}{t} = -1$

$\Rightarrow q = -\frac{1}{t}$

∴ The coordinates of Q are $(2a(-\frac{1}{t}), a(-\frac{1}{t})^2)$

$(-\frac{2a}{t}, \frac{a}{t^2})$.

(iii) Find equation QK and solve QK & PK simultaneously.

Gradient of $PK = -\frac{1}{t}$

Gradient of $QK = t$ ($\because$ it ⊥ to PK).

Equation of QR is

$y - \frac{a}{t^2} = t(x + \frac{2a}{t})$

$y = tx + 2a + \frac{a}{t^2}$. — ②

Continued on next page:-

Q. 6 (a) cont.:-

(iii) solving ① & ②

$at^2 + 2a - \frac{x}{t} = tx + 2a + \frac{a}{t^2}$

$at^4 + 2at^2 - xt = xt^3 + 2at^2 + a$

$xt^3 + xt = at^4 - a$

$x(t^3+t) = a(t^4-1)$.

$x = a\frac{(t^2+1)(t^2-1)(t+1)}{t(t^2+1)}$

$= \frac{a(t^2-1)}{t}$.

$x = a(t - \frac{1}{t})$ ✓

Substituting this in ②

$y = t\,a(t - \frac{1}{t}) + 2a + \frac{a}{t^2}$

$= at^2 - a + 2a + \frac{a}{t^2}$

$y = at^2 + a + \frac{a}{t^2}$

Q.6.

(b) (i) $T = 20^\circ + 75e^{-kt}$

$\frac{dT}{dt} = 75e^{-kt} \times -k$

$= -k\,75e^{-kt}$

$= -k(T-20)$

(ii) $T = 20^\circ + 75e^{-kt}$

$75^\circ = 20^\circ + 75e^{-10k}$

$\frac{55}{75} = e^{-10k}$

$\ln \frac{11}{15} = -10k$

$k = -\frac{1}{10}\ln\left(\frac{11}{15}\right)$

$= 0.03$

(iii) $60^\circ = 20^\circ + 75e^{-kt}$

$\frac{40}{75} = e^{-kt}$

$\ln\left(\frac{40}{75}\right) = -kt$

$t = -\frac{1}{k}\ln\left(\frac{40}{75}\right)$

$= \frac{\ln(40/75)}{\frac{1}{10}\ln(11/15)}$

$= 20.26$ minutes

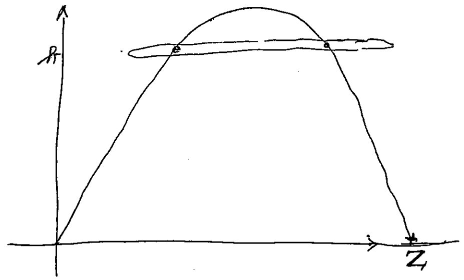
Lawrie should arrive on or before 10:40:17 a.m.

Q.7

(a)

(i) $x = Vt\cos\theta$

$y = -\frac{1}{2}gt^2 + Vt\sin\theta$



$y = 0$ when the projectile returns to ground at Z

$\Rightarrow Vt\sin\theta = \frac{1}{2}gt^2$

$t(V\sin\theta - \frac{1}{2}gt) = 0$

$\therefore t = 0$

OR $V\sin\theta - \frac{1}{2}gt = 0$

$\therefore t = \frac{2V\sin\theta}{g}$

And $Z = x = V\left(\frac{2V\sin\theta}{g}\right)\cos\theta$

$= \frac{V^2\, 2\sin\theta\cos\theta}{g}$

$= \frac{V^2\sin 2\theta}{g}$

(ii) when $y = h$

$h = -\frac{1}{2}gt^2 + Vt\sin\theta$

$2h = -gt^2 + 2Vt\sin\theta$

$gt^2 - 2Vt\sin\theta + 2h = 0$ [Quadratic in t].

$t = \frac{2V\sin\theta \pm \sqrt{4V^2\sin^2\theta - 4g\,2h}}{2g}$ — ①

to pass through the clouds twice t must have two distinct values.

$\therefore 4V^2\sin^2\theta - 4g\,2h > 0$

$V^2\sin^2\theta - 2gh > 0$

$V^2\sin^2\theta > 2gh.$

$h < \frac{V^2\sin^2\theta}{2g}$

Q.7 cont......
(a)

(iii) From equation (1) in part (ii)

$$t = \frac{2v\sin\theta \pm \sqrt{4v^2\sin^2\theta - 4g2h}}{2g}$$

$$t = \frac{v\sin\theta \pm \sqrt{v^2\sin^2\theta - 2gh}}{g}$$

$$t = \frac{320 \times \sqrt{3}/2 \pm \sqrt{320 \times 3/4 - 2 \times 10 \times 3000}}{10}$$

$$= \frac{277.13 \pm 129.6}{10}$$

$t_1 = 14.8$ $\qquad$ $t_2 = 40.67$

(iv) $x_1 = 320 \times 14.8 \times \cos 60$

$x_2 = 320 \times 40.67 \times \cos 60$

$x_2 - x_1 = 320 \times \cos 60 (40.67 - 14.8)$

$= 4,139.2 \ m$

Q.7 (b) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 32x^3 - 48x^2 + 16x^2$

Integrating $\frac{1}{2}v^2 = 8x^4 - 16x^3 + 8x^2 + c$

$v^2 = 16x^4 - 32x^3 + 16x^2 + C$

$x = 3, V = 24$ $\qquad$ $24^2 = 16 \times 3^4 - 32 \times 3^3 + 16 \times 3^2 + C$

$C = 0$

$\therefore v^2 = 16x^4 - 32x^3 + 16x^2$

$v^2 = 16x^2(x^2 - 2x + 1)$

$v^2 = 16x^2(x-1)^2$ ✓

(ii) $\therefore v = \pm 4x(x-1)$ $\qquad$ [since $x = 3, V = 24$].

$v = 4x(x-1)$

$\frac{dx}{dt} = v \Rightarrow \frac{dt}{dx} = \frac{1}{v}$ $\qquad$ $\int \frac{dt}{dx}dx = \int \frac{1}{4x(x-1)}dx$ ↗

$t = \int \frac{1}{4x(x-1)}dx$.

$\therefore \int \frac{1}{x(x-1)}dx = 4t$ ✓