THE SCOTS COLLEGE

Mathematics Extension 1

Trial Examination

9th August 2011

General Instructions:

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A page of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Begin a new booklet for each question

Total Marks – 84

- Attempt Questions 1-7
- All questions are of equal value

Question 1 *(12 marks) Start a SEPARATE writing booklet*

a) Find $\int \frac{1}{\sqrt{4-x^2}} dx$ *(1)*

b) Sketch the region of the plane defined by $y \geq |3x-2|$ *(2)*

c) State the domain and range of $y = sin^{-1}(x^2)$ *(2)*

d) Using the substitution $u = 2x^5 - 1$. Find $\int x^4 \, (2x^5-1)^3 dx$ *(3)*

e) The point $\mathrm{P}(3{,}6)$ divides the line segment joining $\mathrm{A}(1{,}2)$ and $\mathrm{B}(x, y)$ internally in the ratio 2:1. Find the coordinates of B. *(2)*

f) The acute angle between the lines $y = 2x - 3$ and $y = mx + 1$ is 30°. Find the two possible values of m. *(2)*

Question 2 *(12 marks) Start a SEPARATE writing booklet*

a) Find $\frac{d}{dx}\left(3sin^{-1}4x\right)$. *(2)*

b) A particle moves on the *x*-axis with velocity *v*. The particle is initially at rest at $x = 2$. Its acceleration is given by $\ddot{x} = x + 2$. Using the fact that $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, find the speed of the particle at $x = 4$. *(3)*

c) *(i)* Differentiate $e^{2x}(sinx + 2cosx)$ *(2)*

(ii) Hence or otherwise, find $\int e^{2x} cosx\, dx$ *(1)*

d) A curve has parametric equation $x = 2t$, $y = 2t^2$. Find the Cartesian equation for this curve *(2)*

e) Evaluate $\sum_{n=6}^{8}(3n - 1)$ *(1)*

f) Evaluate the limit of $\lim_{x\to 0}\frac{sin2x}{x}$ *(1)*

Question 3 *(12 marks) Start a SEPARATE writing booklet*

a) *(i)* Show that $e^x = sinx + 3$ has a root between $x = 1$ and $x = 2$. *(1)*

(ii) Starting with $x = 1.5$, use one application of Newton's method to find a better approximation for this root. Write your answer correct to three significant figures. *(2)*

b) A particle moves in a straight line and its position at time t is given by

$$x = 3cos\left(2t + \frac{\pi}{4}\right)$$

(i) Show that the particle is undergoing simple harmonic motion. *(2)*

(ii) Find the amplitude of the motion. *(1)*

(iii) When does the particle first reach its maximum speed after time $t = 0$? *(1)*

c) *(i)* Starting from the identity $\sin(\theta + 2\theta) = sin\theta cos2\theta + cos\theta sin2\theta$, and using the double angle formulae, prove the identity

$$sin3\theta = 3sin\theta - 4sin^3\theta$$ *(2)*

(ii) Hence find the general solution for the equation

$$sin3\theta = sin\theta$$ *(3)*

Question 4 *(12 marks) Start a SEPARATE writing booklet*

a) The polynomial $P(x) = 2x^3 - 5x^2 + kx + 40$ has roots α, β, γ.

(i) Find the value of $\alpha + \beta + \gamma$. *(1)*

(ii) Find the value of $\alpha\beta\gamma$. *(1)*

(iii) It is known that two of the roots are equal in magnitude but opposite in sign. Find the third root and hence find the value of *k*. *(2)*

b) Solve $\dfrac{5x}{x-2} \leq 3$ *(3)*

c) A grain silo dispenses grain at a constant rate of $7m^3$ per minute. As the grain falls it forms a cone shape such that the height of the cone is twice its radius.

(i) Show that $\frac{dV}{dh} = \frac{\pi}{4}h^2$ *(2)*

(ii) Find the rate at which the height is changing when its height is 2*m*. *(3)*

Question 5 *(12 marks) Start a SEPARATE writing booklet*

a) $\int_0^{\frac{\pi}{4}} sin^2\,(4x)\,dx$ *(3)*

b) *(i)* Sketch the curve $y = 3cos^{-1}\left(\frac{x}{2}\right)$ for $-2 \leq x \leq 2$. *(2)*

(ii) On the same set of axes, sketch $y = 2x^2 - 2$. *(1)*

(iii) State how many roots the equation $3cos^{-1}\left(\frac{x}{2}\right) - 2x^2 + 2 = 0$ has in the domain $-2 \leq x \leq 2$. *(1)*

c) Consider the function $f(x) = 4tan^{-1}x$.

(i) State the range of the function $y = f(x)$. *(1)*

(ii) Sketch the graph of $y = f(x)$. *(2)*

(iii) Find the equation of the tangent to the curve $y = f(x)$ at $x = \sqrt{3}$. *(2)*

Question 6 *(12 marks) Start a SEPARATE writing booklet*

a) Wilhemy's Law states that the rate of transformation of a substance in a chemical reaction is related to its concentration according to;

$$\frac{dE}{dt} = k(E - c)$$

where *E* is the amount of substance transformed, and *c* is the initial concentration of the substance.

(i) By integration show that $E = Ae^{kt} + c$ is a solution to the given rate of change, where *A* is a constant. *(2)*

(ii) Initially none of the substance is transformed. If the initial concentration is 8.3 and the amount transformed after 3 minutes is 2.9. Find how much of the substance will be transformed after 5 minutes, to 2 significant figures. *(2)*

b) Consider the curve , $f(x) = 3 + \frac{1}{2x - 5}$, $x \neq \frac{5}{2}$

The region enclosed by the curve of *f(x)* , the *x*-axis, and the lines *x* = 3 and *x* = *a* , is revolved through 360° about the *x*-axis. Let *V* be the volume of the solid formed.

Given that $V = \pi\left(\frac{28}{3} + 3ln3\right)$, find the value of *a*. *(3)*

c) A cylinder has height *H* and radius *R*. Point *X* is at one end of the cylinder, on the bottom and on the circumference. Point *Y* is directly opposite, halfway up the cylinder. Length *XY*=*D*.

(i) Show that the volume of the cylinder is given by

$$V = \frac{\pi H}{16}(4D^2 - H^2)$$ *(2)*

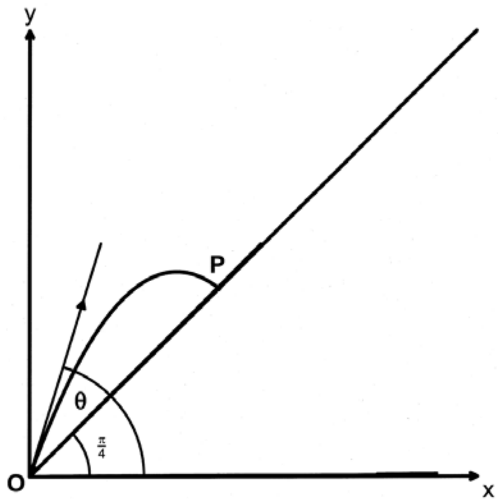
(ii) Find the maximum volume of the cylinder in terms of *D* if *D* is fixed. *(3)*

Question 7 *(12 marks) Start a SEPARATE writing booklet*

a) Use mathematical induction to prove that *(4)*

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+2}{2n+2} \text{ for all } n \geq 1.$$

b) A tennis ball is hit with a velocity of $10m/s$. Initially it is at *O*. P lies on an inclined plane. The inclined plane *OP* makes an angle of $\frac{\pi}{4}$ to the horizon.



The tennis ball is projected at an angle of θ to the horizontal and $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

The tennis ball has position at any time *t* given by

$x = 10tcos\theta$ and $y = -5t^2 + 10tsin\theta$ (*Do not derive these equations*)

(i) If $OP = R$ meters and the tennis ball lands at *P*. Show that

$$x = y = \frac{R}{\sqrt{2}}$$ *(1)*

(ii) Show that

$$R = 20\sqrt{2}(cos\theta sin\theta - cos^2\theta)$$ *(3)*

(iii) Find the maximum value of *R*. *(4)*

END OF EXAM

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE : $\ln x \equiv \log_e x, \quad x > 0$

Scots Yr 12 Ext 1 Trial HSC 2011

SOLUTIONS

ANSWER BOOKLET

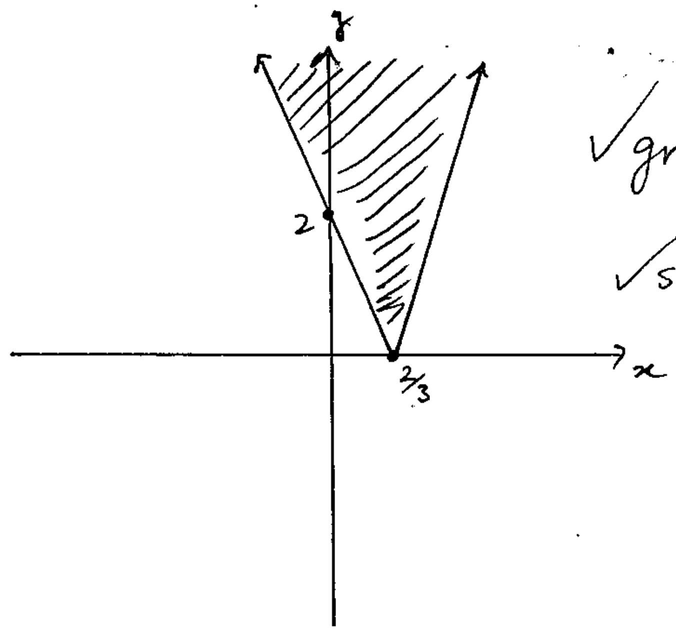
Name: ____________

Teacher: ____________

Question No. 1

a) $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + C$ ✓solution

b) $y \geq |3x - 2|$



✓graph

✓shading

test $(0,0)$

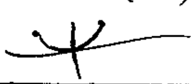
$0 \geq |0-2|$

$0 \geq 2$ Not true

c) Note:

$y = \sin^{-1}(x)$

$y = \sin^{-1}(x^2)$



$\therefore\ y = \sin^{-1}(x^2)$

domain : $-1 \leq x \leq 1$ ✓

Range : $0 \leq y \leq \frac{\pi}{2}$ ✓

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 1

$u = 2x^5 - 1$

$\frac{du}{dx} = 10x^4$

$du = 10x^4 \, dx$

d) $\int x^4 (2x^5 - 1)^3 \, dx$

$\frac{1}{10} \int 10x^4 (2x^5 - 1)^3 \, dx$ ✓ correct substitution

$\frac{1}{10} \int u^3 \, du$

$\frac{1}{10} \left[\frac{u^4}{4} + c \right]$ ✓ integral

$\frac{u^4}{40} + c$

$\frac{(2x^5 - 1)^4}{40} + c$

✓ solution

Note: +C not put down loss of 1 mark.

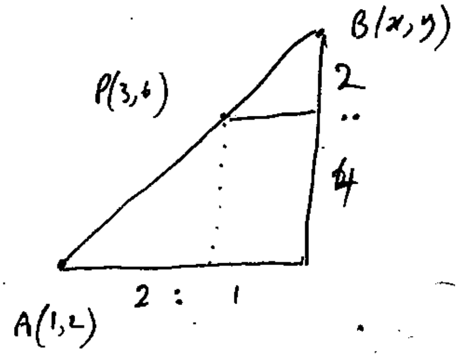
ANSWER BOOKLET

Name: ____________

Teacher: ____________

Question No. 1

e) $P(3,6)$



$\boxed{B(4,8)}$ ✓ correct method ✓ solution

f) $y = 2x - 3$, $y = mx + 1$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan 30 = \left|\frac{2-m}{1+2m}\right|$$

✓ formula with values & two possible outcomes

$\therefore$

$$-\frac{1}{\sqrt{3}} = \frac{2-m}{1+2m} \quad \text{OR} \quad \frac{1}{\sqrt{3}} = \frac{2-m}{1+2m}$$

$$m = \frac{-1-2\sqrt{3}}{2-\sqrt{3}} \qquad m = \frac{2\sqrt{3}-1}{2+\sqrt{3}}$$

$$m = -5\sqrt{3} - 8 \qquad m = 5\sqrt{3} - 8$$

✓ solution

Question No. 2

$$\frac{d}{dx}\left(3\sin^{-1}(4x)\right)$$

let

$y = 3\sin^{-1}(4x)$ find $\frac{dy}{dx}$

let $u = 4x$ $\frac{du}{dx} = 4$

$y = 3\sin^{-1}(u)$

$$\frac{dy}{du} = \frac{3}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{12}{\sqrt{1-16x^2}}$$ ✓ ✓

ANSWER BOOKLET

Name: ______________________

Teacher: ______________________

Question No. 2

b) $t=0,\ x=2,\ v=0$

$\ddot{x} = x+2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x+2$

$\frac{1}{2}v^2 = \frac{x^2}{2} + 2x + C$ ✓ using substitution & finding c.

when $x=2,\ v=0$

$0 = \frac{4}{2} + 4 + C$

$0 = 6 + C$

$-6 = c$

$\therefore$

$\frac{1}{2}v^2 = \frac{x^2}{2} + 2x - 6$ ✓ attempting to solve for v.

$\therefore$ at $x=4$

$\frac{1}{2}v^2 = \frac{16}{2} + 8 - 6$

$\frac{1}{2}v^2 = 10$

$v = \pm\sqrt{20}$

$\therefore\ V = \pm\sqrt{20}$ ✓ solution

speed is $\sqrt{20}$ when $x=4$.

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 2

c) i) $e^{2x}(\sin x + 2\cos x)$

$u \times v$

$u = e^{2x}$ $u' = 2e^{2x}$

$v = \sin x + \cos x$ $v' = \cos x - 2\sin x$

✓ attempts product rule

$\frac{d}{dx} = 2e^{2x}(\sin x + 2\cos x) + e^{2x}(\cos x - 2\sin x)$

$= 4e^{2x}\cos x + e^{2x}\cos x$

$= 5e^{2x}\cos x$ ✓ soltion

ii) $\int e^{2x}\cos x \, dx$

$\frac{1}{5}\int 5e^{2x}\cos x \, dx$

$\frac{1}{5}\left[e^{2x}(\sin x + 2\cos x)\right] + c$

$\frac{1}{5}e^{2x}(\sin x + 2\cos x) + c$ ✓ solution

ANSWER BOOKLET

Name: ______________________

Teacher: ______________________

Question No. 2

d) $x = 2t$, $y = 2t^2$

$\frac{x}{2} = t$ ✓solving for t

$\therefore$

$y = 2\left(\frac{x}{2}\right)^2$ ✓equating

$= \frac{2x^2}{4}$

$y = \frac{x^2}{2}$

e) $\sum_{n=6}^{8} (3n-1) = [3\times6-1] + [3\times7-1] + [3\times8-1]$

$= 17 + 20 + 23$

$= 60$

ANSWER BOOKLET

Name: ______________________

Teacher: ______________________

Question No. 2

f)

$$\lim_{x \to 0} \frac{\sin 2x}{x}$$

$$\lim_{x \to 0} \frac{2\sin 2x}{2x}$$

$$\lim_{x \to 0} 2 \times \frac{\sin 2x}{2x}$$

$$\lim_{x \to 0} 2 \times \lim_{x \to 0} \frac{\sin 2x}{2x}$$

$$2 \times 1$$

$$2$$

ANSWER BOOKLET

Name: ______________

Teacher: ______________

Question No. 3

a) i) $e^x - \sin x - 3 = ?$

at $x = 1$

$e^1 - \sin(1) - 3 = -1.12$

at $x = 2$

$e^2 - \sin(2) - 3 = 3.48$

$\therefore$ change of sign there exists a root between $x = 1$ & $x = 2$.

ii) Newtons' $\boxed{x = a - \frac{f(a)}{f'(a)}}$

$f(x) = e^x - \sin x - 3$

$f(1.5) = 0.484$

$f'(1.5) = 4.41095$

$f'(x) = e^x - \cos x$

at $a = 1.5$

$x = 1.5 - \frac{0.484}{4.41095}$

$x = 1.5 - 0.1098$

$x = 1.39$

ANSWER BOOKLET

Name: ____________

Teacher: ____________

Question No. 3

b i) $x = 3\cos\left(2t + \frac{\pi}{4}\right)$ where period is π

$\therefore\ P = \frac{2\pi}{n}$

$n = 2$

$\dot{x} = -6\sin\left(2t + \frac{\pi}{4}\right)$

$\ddot{x} = -12\cos\left(2t + \frac{\pi}{4}\right)$

$\ddot{x} = -4 \times 3\cos\left(2t + \frac{\pi}{4}\right)$

$\ddot{x} = -4x$

$\therefore$ this is in the form

$\ddot{x} = -n^2 x$

$\therefore$ particle is undergoing SHM.

b ii) amplitude is 3.

b iii) $\dot{x} = -6\sin\left(2t + \frac{\pi}{4}\right)$

$-6 = -6\sin\left(2t + \frac{\pi}{4}\right)$

$+1 = \sin\left(2t + \frac{\pi}{4}\right)$

$\therefore$

$2t + \frac{\pi}{4} = \frac{\pi}{2}$

$t = \frac{\pi}{8}$

$\therefore$ at time $\frac{\pi}{8}$ particle first reaches its max speed.

ANSWER BOOKLET

Name: __________

Teacher: __________

Question No. 3

6 i) $\sin 3\theta = \sin\theta\cos 2\theta + \cos\theta\sin 2\theta$

$= \sin\theta\left[1-2\sin^2\theta\right] + \cos\theta\left[2\sin\theta\cos\theta\right]$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta\left(1-\sin^2\theta\right)$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta - 2\sin^2\theta$.

$= 3\sin\theta - 4\sin^3\theta$

ii) $\sin 3\theta = \sin\theta$

$3\sin\theta - 4\sin^3\theta = \sin\theta$

$2\sin\theta = 4\sin^3\theta$, $\sin\theta = 0$

$\frac{2}{4} = \sin^2\theta$

$\pm\frac{1}{\sqrt{2}} = \sin\theta$

$\therefore$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4},$

General solutions $\theta = \left(\frac{\pi}{4} \pm n\pi\right)$ or $\left(\frac{3\pi}{4} \pm n\pi\right)$ where $(n = 0, 1, 2, \ldots)$

or $n\pi$

ANSWER BOOKLET

Name: ____________

Teacher: ____________

Question No. 4

$P(x) = ax^3 + bx^2 + cx + d$

$\alpha+\beta+\gamma = \frac{-b}{a}$

$\alpha\beta+\gamma\alpha+\beta\gamma = \frac{c}{a}$

$\alpha\beta\gamma = \frac{-d}{a}$

a) i) $\alpha+\beta+\gamma = \frac{5}{2}$

ii) $\alpha\beta\gamma = \frac{-40}{2} = -20$

iii) let $\alpha = -\beta$

$\therefore$

$-\alpha^2\gamma = -20 \quad \cdots (1)$

$\gamma = \frac{5}{2} \quad \cdots (2)$

$-\alpha^2\left(\frac{5}{2}\right) = -20$

$5\alpha^2 = 40$

$\alpha^2 = 8$

$\alpha = \pm\sqrt{8}$

$\therefore$

$\alpha = \sqrt{8}, \ \beta = -\sqrt{8}, \ \gamma = \frac{5}{2}$

$\alpha\beta + \gamma\alpha + \beta\gamma = \frac{k}{2}$

$-8 + \sqrt{8}\left(\frac{5}{2}\right) - \sqrt{8}\left(\frac{5}{2}\right) = \frac{k}{2}$

$-8 = \frac{k}{2}$

$-16 = k$

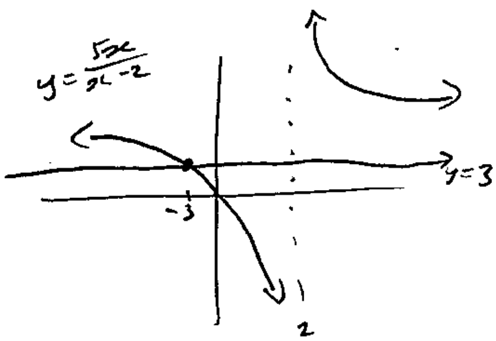
ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 4

b) $\dfrac{5x}{x-2} \le 3 \qquad x \neq 2$



$5x(x-2) \le 3(x-2)^2$

$5x^2-10x \le 3(x^2-4x+4)$

$5x^2-10x \le 3x^2-12x+12$

$2x^2+2x-12 \le 0$

$2(x^2+x-6) \le 0$

$2(x+3)(x-2) \le 0$

$\therefore$

$-3 \le x < 2$

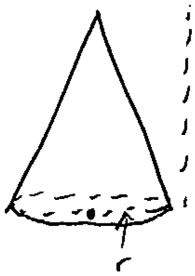
ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 6

6) i



$2r = h$

$r = \frac{h}{2}$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$

$V = \frac{\pi}{12} h^3$

$\frac{dV}{dh} = \frac{3\pi}{12} h^2 = \frac{\pi}{4} h^2$

ii) $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$\frac{dV}{dt} = 7$

$\frac{dh}{dV} = \frac{4}{\pi h^2}$

$= 7 \times \frac{4}{\pi h^2}$

$\frac{dh}{dt} = \frac{28}{\pi h^2}$

when $h = 2$

$\frac{dh}{dt} = \frac{28}{\pi \times 4}$

$= \frac{7}{\pi}$

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 5

$\cos 2A = 1 - 2\sin^2 A$

$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$

a) $$\int_0^{\frac{\pi}{4}} \sin^2(4x)\,dx$$

$$\int_0^{\frac{\pi}{4}} \frac{1}{2} - \frac{1}{2}\cos(8x)\,dx$$

$$= \left[\frac{x}{2} - \frac{1}{16}\sin(8x)\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{8} - \frac{1}{16}\sin(2\pi)\right] - \left[0 - 0\right]$$

$$= \left[\frac{\pi}{8} - 0\right] - 0$$

$$= \frac{\pi}{8}$$

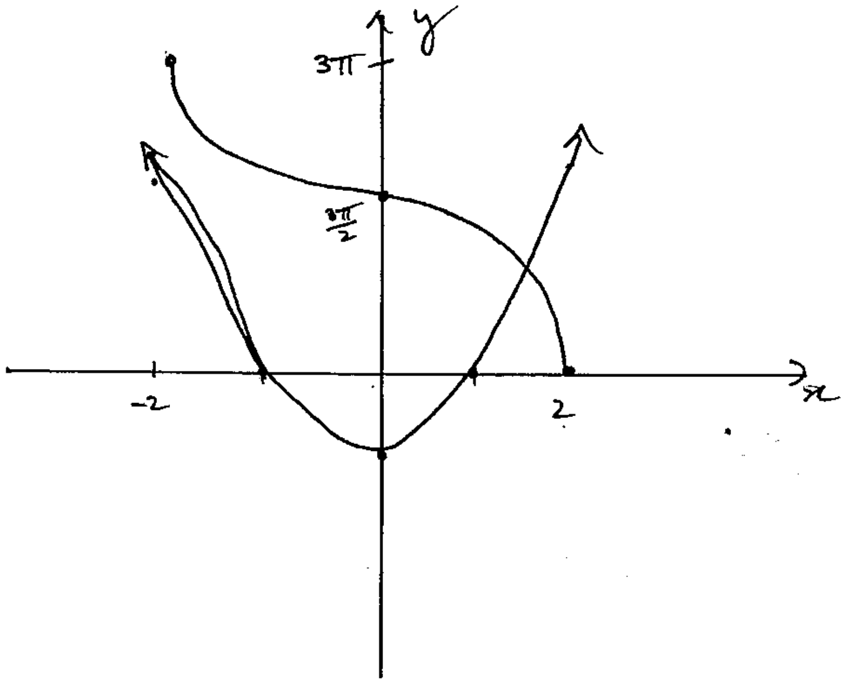
ANSWER BOOKLET

Name: ______________________

Teacher: ______________________

Question No. 5

b)



$$\boxed{\begin{aligned} y &= 3\cos^{-1}\left(\tfrac{x}{2}\right) \\ \tfrac{y}{3} &= \cos^{-1}\left(\tfrac{x}{2}\right) \\ x &= 2\cos\left(\tfrac{1}{3}y\right) \end{aligned}}$$

ii) $y = 2x^2 - 2$

at

x	-2	-1	0	1	2
y	6	0	-2	0	6

iii) on domain

$-2 \le x \le 2$

only 1 solution

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

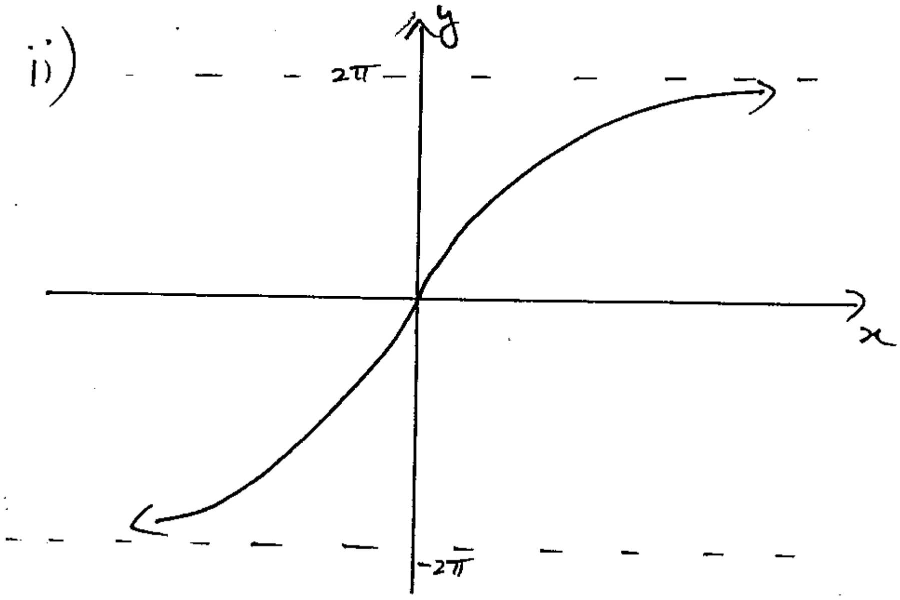
Question No. 5

c) i) range of $y = f(x)$

$-4 \times \frac{\pi}{2} \le y \le 4 \times \frac{\pi}{2}$

$\boxed{-2\pi < y < 2\pi}$

ii)



iii) $y = 4\tan^{-1}(x)$

$\frac{dy}{dx} = \frac{4}{1+x^2}$

at $x = \sqrt{3}$

$\frac{dy}{dx} = \frac{4}{4} = 1$

at $x = \sqrt{3}$

$y = 4\tan^{-1}(\sqrt{3})$

$= \frac{4\pi}{3}$

Tangent

$y - \frac{4\pi}{3} = 1(x - \sqrt{3})$

$3x - 3y - 3\sqrt{3} + 4\pi = 0$

ANSWER BOOKLET

Name: ____________________

Teacher: ____________________

Question No. 6

a) i $$\frac{dE}{dt} = k(E - c)$$

$$\frac{dt}{dE} = \frac{1}{k} \times \frac{1}{(E-c)}$$

$$t = \frac{1}{k}\int \frac{1}{E-c} dE$$

$$t = \frac{1}{k}\ln(E-c) + D$$

$$kt - D = \ln(E-c)$$

$$e^{kt-D} = E - c$$

$$\frac{e^{kt}}{e^{D}} = E - c \qquad \left(\text{let } \frac{1}{e^{D}} = A\right)$$

$$Ae^{kt} = E - c$$

$$\therefore E = Ae^{kt} + c$$

ANSWER BOOKLET

Name: ______________

Teacher: ______________

Question No. 6

a) ii) $E = Ae^{kt} + c$

at $t=0$ $E=0$ $c=8.3$

$0 = Ae^{0} + 8.3$

$A = -8.3$

$\therefore$

$E = -8.3e^{kt} + 8.3$

at $t=3$ $E=2.9$ $c=8.3$

$2.9 = -8.3e^{3k} + 8.3$

$\frac{-5.4}{-8.3} = e^{3k}$

$\frac{54}{83} = e^{3k}$

$k = \frac{1}{3}\ln\left(\frac{54}{83}\right)$

$\therefore$

$E = -8.3e^{kt} + 8.3$

at $t=5$

$E = -8.3e^{5k} + 8.3$

$= 4.245$

$= 4.2$ (2 sig fig)

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Question No. 6

b)

$$V = \pi \int_3^a \left(3 + \frac{1}{2x-5}\right)^2 dx$$

$$= \pi \int_3^a 9 + \frac{6}{2x-5} + (2x-5)^{-2} \, dx$$

$$= \pi \left[9x + 3\ln(2x-5) + -\frac{1}{2}(2x-5)^{-1} \right]_3^a$$

$$= \pi \left[9x + 3\ln(2x-5) - \frac{1}{4x-10} \right]_3^a$$

$$= \pi \left[\left(9a + 3\ln(2a-5) - \frac{1}{4a-10}\right) - \left(27 + 0 - \frac{1}{2}\right) \right]$$

$$= \pi \left[9a - 27 + \frac{1}{2} - \frac{1}{4a-10} + 3\ln(2a-5) \right]$$

$$\therefore \quad 2a - 5 = 3$$

$$a = 4$$

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Question No. 6

b) $\therefore$ if $a = 4$

$$9 \times 4 - 27 + \frac{1}{2} - \frac{1}{16 - 10} =$$

$$9\frac{1}{2} - \frac{1}{6} =$$

$$\frac{28}{3} = \text{ as required.}$$

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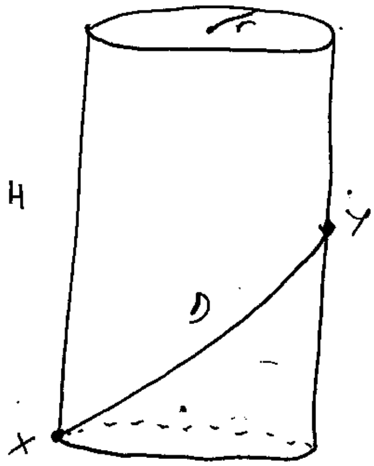
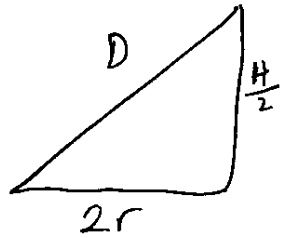
Question No. 6

a) i)

$$D^2 = \frac{H^2}{4} + 4r^2$$

$$r^2 = \frac{D^2}{4} - \frac{H^2}{16}$$





$$V = \pi r^2 H$$

$$V = \pi\left(\frac{D^2}{4} - \frac{H^2}{16}\right)H$$

$$V = \frac{\pi H}{16}\left(4D^2 - H^2\right)$$

---

ii)

$$V = -\frac{\pi}{16}H^3 + \frac{\pi D^2}{4}H$$

$$\frac{dV}{dH} = -\frac{3\pi}{16}H^2 + \frac{\pi D^2}{4}$$

at $\frac{dV}{dH} = 0$ is max or min

cubic is of form

$y = -ax^3 + bx$

$\therefore$ take +ve solution

$$0 = -\frac{3\pi}{16}H^2 + \frac{\pi D^2}{4}$$

$$H^2 = \frac{4D^2}{3}$$

$$H = \pm\sqrt{\frac{4D^2}{3}}$$

only +ve solution

$$H = \frac{2D}{\sqrt{3}}$$

ANSWER BOOKLET

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Question No. 7

a) Test for $n=1$

$$\left(1-\frac{1}{2^2}\right) = \frac{1+2}{2+2}$$

$$\frac{3}{4} = \frac{3}{4}$$

$\therefore$ True for $n=1$

We assume it is true for $n=k$ $(k \in \mathbb{N})$

i.e.

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+2}$$

Now prove it is true for $n=k+1$

$$\underbrace{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)}_{\text{from assumption}}\left(1-\frac{1}{(k+2)^2}\right) = \frac{k+3}{2k+4}$$

$$\frac{k+2}{2k+2}$$

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Question No. 7

a) continued

$$\left(\frac{k+2}{2k+2}\right)\left(1-\frac{1}{(k+2)^2}\right) = \frac{k+3}{2k+4}$$

$$\frac{k+2}{2k+2} - \frac{k+2}{(2k+2)(k+2)^2}$$

✓ progress

$$\frac{(k+2)(k+2)^2-(k+2)}{(2k+2)(k+2)^2}$$

$$\frac{[k+2][(k+2)^2-1]}{2(k+2)^2(k+1)}$$

$$\frac{(k+2)^2-1}{2(k+2)(k+1)}$$

$$\frac{k^2+4k+4-1}{(2k+4)(k+1)}$$

$$\frac{(k+1)(k+3)}{(k+1)(2k+4)}$$

$$\frac{k+3}{2k+4} = RHS$$

$\therefore$ true for $n=k+1$

$\therefore$ Because it is true for $n=1$ it is true for $n=2$ and by mathematical induction is true for $n=3, 4$ etc

✓ solution.

ANSWER BOOKLET

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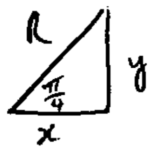
Question No. 7

b i)



iscosoles

$\therefore x = y$

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}} = \frac{y}{R}$

$\frac{R}{\sqrt{2}} = y$

$\therefore$

$x = y = \frac{R}{\sqrt{2}}$

ii)

$x = y$

$10t\cos\theta = -5t^2 + 10t\sin\theta$

$\therefore$

$5t^2 - 10t\sin\theta + 10t\cos\theta = 0$ ✓ setup

$5t(t - 2\sin\theta + 2\cos\theta) = 0$

$\therefore t = 0$
or
$t - 2\sin\theta + 2\cos\theta = 0$

$\therefore$

$t = 2\sin\theta - 2\cos\theta$ ✓ solve for t

$\therefore$

$x = 10t\cos\theta$

when $t = 2\sin\theta - 2\cos\theta$

$x = 10(2\sin\theta - 2\cos\theta)\cos\theta$

but

$x = \frac{R}{\sqrt{2}}$ ✓ substitute in for x

$\frac{R}{\sqrt{2}} = 20(\sin\theta\cos\theta - \cos^2\theta)$

$R = 20\sqrt{2}(\cos\theta\sin\theta - \cos^2\theta)$

As required

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Question No. 7

iii)

$$R = 20\sqrt{2}\left(\underbrace{\cos\theta\sin\theta}_{\text{Product Rule}} - \underbrace{\cos^2\theta}_{\text{Product Rule}}\right)$$

$$\frac{dR}{d\theta} =$$

$\frac{dR}{d\theta} = 0$ for max.

$$0 = 20\sqrt{2}(\cos 2\theta + \sin 2\theta) \quad \checkmark \frac{dR}{d\theta}$$

$\therefore$

$$\cos 2\theta + \sin 2\theta = 0$$

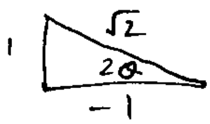
$$\sin 2\theta = -\cos 2\theta$$

$$\tan 2\theta = -1$$

S	A
T	C

2nd or 4th
but θ must be in
domain $\frac{\pi}{4} < \theta \leq \frac{\pi}{2}$
$\frac{\pi}{2} < 2\theta < \pi$

$\therefore$ 2nd



$$\therefore \cos 2\theta = \frac{-1}{\sqrt{2}} \quad \checkmark \text{solution}$$

$$\sin 2\theta = \frac{1}{\sqrt{2}}$$

$$R = 20\sqrt{2}(\cos\theta\sin\theta - \cos^2\theta)$$

Note:

$$\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta = \frac{1}{2\sqrt{2}}$$

$$\cos^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta = \frac{1}{2} - \frac{1}{2\sqrt{2}}$$

$$R = 20\sqrt{2}\left(\frac{1}{2\sqrt{2}} - \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)\right)$$

$$= 20\sqrt{2}\left(\frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)$$

$$= 20\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) \quad \checkmark \text{sub in value}$$

$$= 20 - 10\sqrt{2}$$

Maximum value of R is $(20 - 10\sqrt{2})$ $\checkmark$ exact value.