Student Number:

Set:

Shore

Year 12
Mathematics Extension 1
Trial Examination
2010

General Instructions

- Reading time − 5 minutes
- Working time − 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this question paper
- All necessary working should be shown in every question

Note: Any time you have remaining should be spent revising your answers.

Total marks − 84

- Attempt Questions 1 − 7
- All questions are of equal value
- Start each question in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Total Marks − 84
Attempt Questions 1−7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) **Marks**

(a) The interval AB, where A is $(7,-4)$ and B is $(-3,-9)$, is divided internally in the ratio $2:3$ by the point $P(x,y)$. Find the values of x and y. **2**

(b) Find $\dfrac{d}{dx}\left(\sin^{-1}3x\right)$. **2**

(c) Consider the polynomial $P(x)=2x^3+x^2-13x+6$.

(i) Show that $(x-2)$ is a factor of $P(x)$. **1**

(ii) Hence, or otherwise, factorise $P(x)$ completely. **2**

(iii) Without the use of calculus, sketch the graph of $y=P(x)$ clearly labelling all intercepts with the coordinate axes. **2**

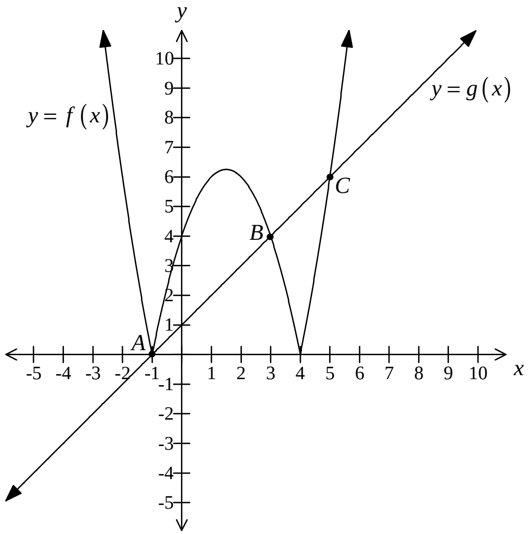
(d) Use the substitution $u=\ln x$ to evaluate $\displaystyle\int_e^{e^4}\frac{3}{x\ln x}\,dx$. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) A Rotary Club has 12 female and 25 male members. The club is to choose a representative team consisting of 2 women and 4 men to send to an international conference. In how many ways can this representative team be chosen? **2**

(b) Find the exact value of $\displaystyle\int_0^3 \frac{dx}{36+4x^2}$. **3**

(c) The diagram shows the functions $f(x) = \left|x^2 - 3x - 4\right|$ and $g(x) = x + 1$. **2**

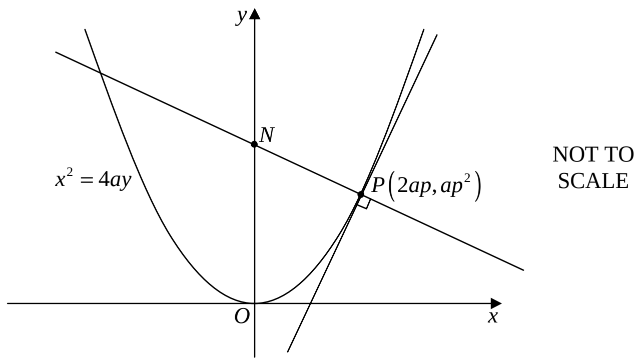


The three points of intersection of the functions $y = f(x)$ and $y = g(x)$ are $A(-1,0)$, $B(3,4)$ and $C(5,6)$.

Use the graph to find the values of x for which $\left|x^2 - 3x - 4\right| > x + 1$.

Question 2 continues

(d) The diagram shows the point $P\left(2ap, ap^2\right)$ which moves along the parabola $x^2 = 4ay$. The normal at P intersects the y-axis at N.



(i) Using calculus, show that the equation of the normal at P is given by **2**

$$x + py = ap^3 + 2ap.$$

(ii) Find the coordinates of N. **1**

(iii) Let M be the midpoint of PN. Show that the locus of the point M is given by $x^2 = a(y - a)$. **2**

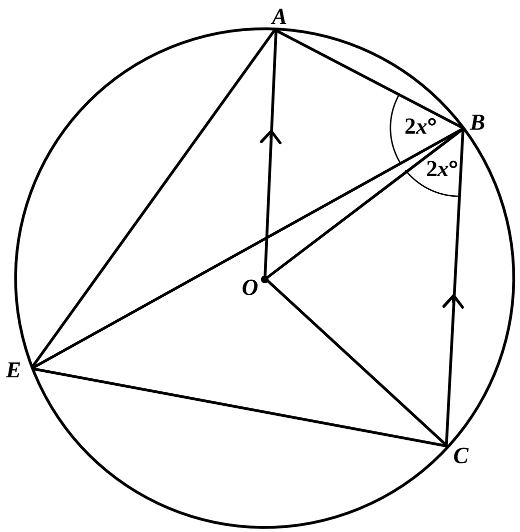
End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) By considering $f(x) = x - 3\sin x$, show that the curves $y = x$ and $y = 3\sin x$ meet at a point P whose x-coordinate is between $x = 2$ and $x = 3$. **1**

(ii) Use one application of Newton's method, starting at $x = 2$, to find an approximation to the x-coordinate of P. Give your answer correct to two decimal places. **2**

(b) In the diagram, $ABCE$ is a cyclic quadrilateral such that AO is parallel to BC. O is the centre of the circle and $\angle ABE = \angle OBC = 2x°$



Copy or trace the diagram into your writing booklet.

(i) Prove that $\angle AEB = x°$. **2**

(ii) Prove that $\angle BCE = 3x°$. **2**

Question 3 continues

(c) Nick takes a bottle of milk from the refrigerator for baby Ed. To heat the bottle, Nick puts it in a saucepan of continuously boiling water.

Let y°C be the temperature of the milk at time t minutes after the baby's bottle is placed in the boiling water. The temperature of the milk increases such that $\frac{dy}{dt} = a(100 - y)$ where a is a positive constant.

The milk's temperature when the bottle is placed into the boiling water is 5°C.

(i) Verify that $y = 100 - 95e^{-at}$ satisfies the differential equation. **1**

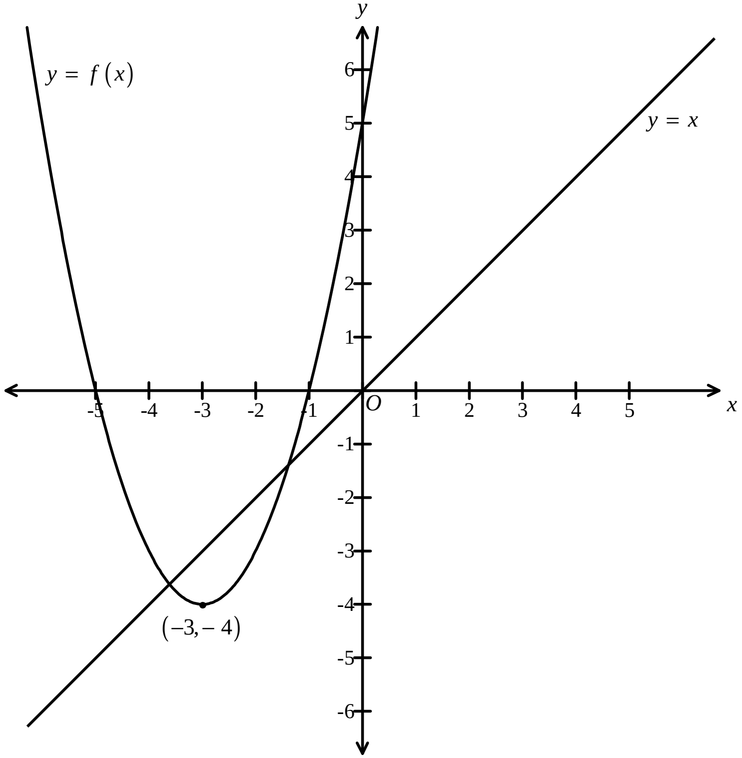
(ii) After two minutes, the temperature of the milk is measured to be 18°C. Find the value of a. **2**

(iii) Ed can be given the bottle safely when the temperature of the milk is no more than 39°C. What is the maximum length of time that Nick can leave the bottle in the boiling water so that it can be given to the baby safely? Answer correct to the nearest minute. **2**

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) The diagram below is drawn to scale and shows the graphs of $f(x) = (x+3)^2 - 4$ and $y = x$.



(i) State the largest domain of $f(x)$ which includes the value $x = 0$ for which the function has an inverse function $f^{-1}(x)$. **1**

(ii) Copy or trace the diagram into your writing booklet. On the same set of axes, sketch the graph of the inverse function $y = f^{-1}(x)$ in part (i). Clearly label all important features of $y = f^{-1}(x)$. **2**

(iii) Find an expression for $y = f^{-1}(x)$ in terms of x. **2**

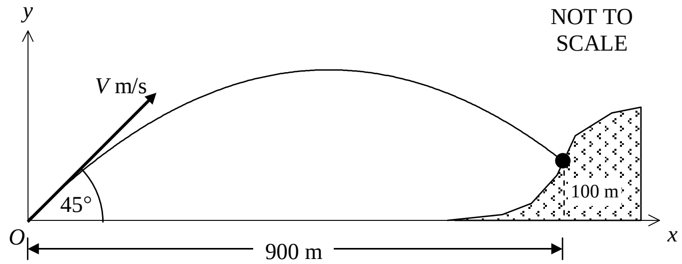
Question 4 continues

(b) There are ten people at a dinner party. They will sit at a round table.

(i) How many different seating arrangements are possible? **1**

(ii) Two of the people at the dinner party, Bill and Hillary, are divorced and refuse to sit next to each other. How many different seating arrangements are possible if Bill and Hillary do not sit next each other? **2**

(c)



A cannon ball is fired with a projection speed of V m/s from a point on the ground. The angle of projection to the horizontal is $45°$ as shown.

Assume the equations of motion of the cannon ball are

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -10$$

referred to the coordinate axes as shown.

(i) Let (x, y) be the position of the cannon ball at time t seconds after it is fired, and before it hits the hill. **2**

It is known that $x = \dfrac{Vt}{\sqrt{2}}$.

Starting with $\ddot{y} = -10$ show that $y = \dfrac{Vt}{\sqrt{2}} - 5t^2$.

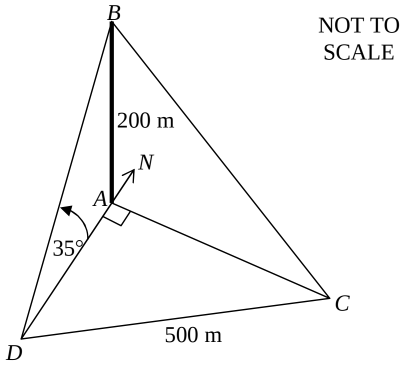
(ii) The ball needs to hit a target on the ground 900 metres away on a hill, 100 metres above the horizontal. **2**

Find the initial speed, V, correct to the nearest m/s.

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) **3**



The diagram shows a vertical spire AB, 200 metres high, standing on level ground. The top of the tower B is observed at an angle of elevation of $35°$ from a point D on the ground due south of the tower. The point C is 500 metres distant from D and due east of the tower.

Calculate the angle of elevation of the top of the tower from the point C. Give your answer correct to the nearest degree.

(b) (i) Use mathematical induction to prove that, for integers $n \geq 1$, **3**

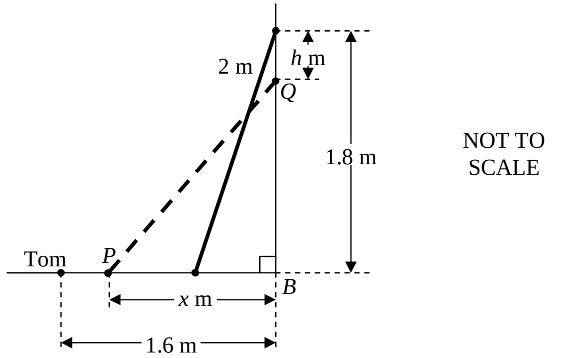
$$1 + x + x^2 + x^3 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} \quad \text{where} \quad x \neq 1.$$

(ii) Hence, or otherwise, find the largest integer, n, such that **2**

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^n < 10^9.$$

Question 5 continues

(c) The diagram shows a ladder PQ, 2 metres in length, leaning against a wall such that the top of the ladder, Q, initially reaches 1.8 metres up the wall. The base of the ladder, P, is x metres from the base of the wall, B.



The ladder begins to slide down the wall at the rate of 0.5 metres per minute such that the top of the ladder is h metres below its original position after t minutes.

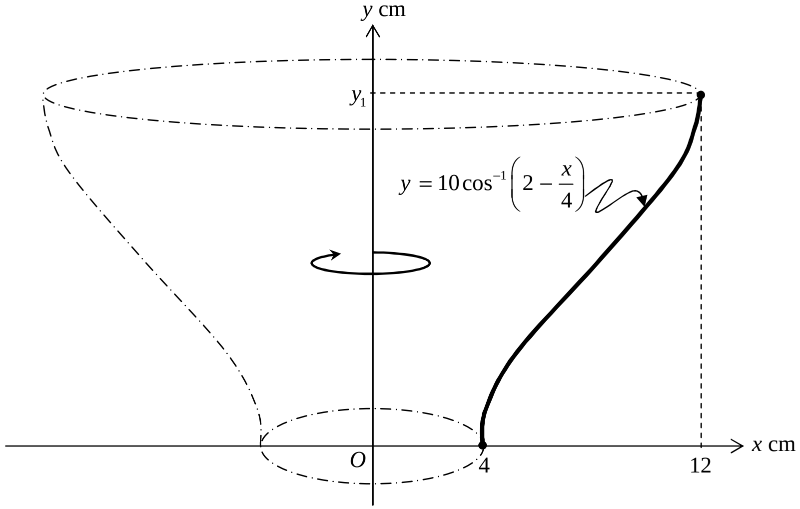
(i) Show that t minutes after the ladder begins to slide down the wall, $h = 1.8 - \sqrt{4 - x^2}$. **1**

(ii) Tom is standing on the ground 1.6 metres from the base of the wall in a direct line with the ladder. At what rate does base of the ladder hit Tom? **3**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) The diagram below shows a vase formed by rotating the function $y = 10\cos^{-1}\left(2 - \frac{x}{4}\right)$ around the y-axis for its entire natural range. All measurements are in centimetres.



(i) Show that the height of the vase, y_1, is 10π centimetres. **1**

(ii) Show that the volume, V cubic centimetres, is given by **2**

$$V = 16\pi\int_0^{10\pi}\left(4 - 4\cos\frac{y}{10} + \cos^2\frac{y}{10}\right)dy$$

(iii) Use the identity $\cos 2A = 2\cos^2 A - 1$ to show that **2**

$$\int \cos^2\frac{y}{10}\,dy = \frac{1}{2}\left(5\sin\frac{y}{5} + y\right) + C.$$

(iv) Hence, or otherwise, find the volume of the vase correct to the nearest cubic centimetre. **2**

Question 6 continues

(b) A particle is moving along the x-axis such that $x = 4\sin^2 t - 1$ where t is the time in seconds and x is measured in metres.

(i) Show that the particle is moving in simple harmonic motion by showing that $\ddot{x} = -4(x-1)$. **2**

(ii) Find the period of the motion. **1**

(iii) When does the particle first reach its maximum speed? **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) A particle is moving in a straight line. At time t seconds it has displacement x metres to the right of a fixed point O on the line. The velocity v ms^{-1} given by $v = \sin x \cos x$. The particle starts $\frac{\pi}{4}$ metres to the right of O.

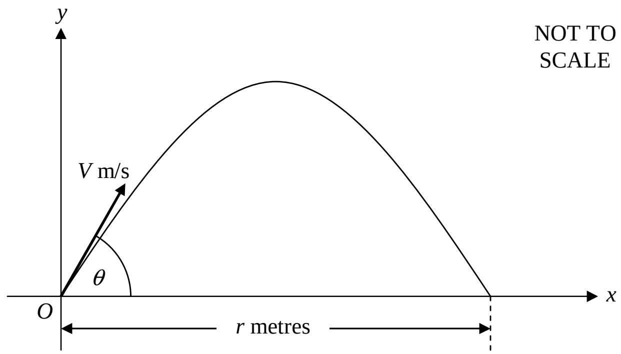
(i) Show that $\dfrac{d}{dx}\ln(\tan x) = \dfrac{1}{\sin x \cos x}$. **1**

(ii) Hence show that the displacement of the particle is given by $x = \tan^{-1}(e^t)$. **3**

(iii) Find the limiting position of the particle and sketch the graph of x against t. **2**

Question 7 continues

(b) A projectile is fired with initial speed V ms^{-1} at an angle of θ degrees, $0° < \theta < 90°$, from a point O on level ground. It strikes the ground r metres from the point of projection.



The equations of motion are

$$x = Vt\cos\theta$$
$$y = Vt\sin\theta - \frac{gt^2}{2}$$

where t is the time in seconds after projection and g is the constant of gravity. (Do NOT prove these equations of motion.)

(i) When the projectile strikes the ground, prove that **2**

$$\tan^2\theta - \left(\frac{2V^2}{gr}\right)\tan\theta + 1 = 0.$$

(ii) Prove that the quadratic equation in part (i) has two real and distinct values of $\tan\theta$, when $r < \dfrac{V^2}{g}$. **2**

(iii) Let $\tan\alpha$ and $\tan\beta$ be the roots of the quadratic equation in part (i). By considering the product of these roots show that the corresponding angles of projection, α and β are complementary. **2**

End of Paper

YEAR 12 MATHS EXT1 TRIAL 2010 II

QUESTION 1

a) $(7,-4)\quad(-3,-9)$

$2 : 3$

$P = \left(\dfrac{7\times3+-3\times2}{2+3}, \dfrac{-4\times3+-9\times2}{2+3}\right)$

$= (3, -6)$

b) $\dfrac{d}{dx}(\sin^{-1}3x) = \dfrac{1}{\sqrt{1-(3x)^2}}\times3$

$= \dfrac{3}{\sqrt{1-9x^2}}$

c) i) $P(2) = 2(2)^3 + 2^2 - 13(2) + 6$

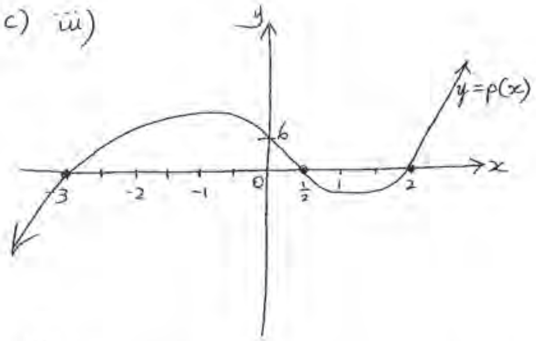
$= 0$

$\therefore x-2$ is a factor.

ii) $P(x) = (x-2)(2x^2+5x-3)$ $\quad 2x \;\; 1$, $\; x \;\; 3$

$= (x-2)(2x-1)(x+3)$

1c) iii)



d) $\displaystyle\int_e^{e^4}\frac{3}{x\ln x}dx = \int_1^4 \frac{3}{u}du$

$u = \log_e x$

$\dfrac{du}{dx} = \dfrac{1}{x}$

$du = \dfrac{1}{x}dx$

Limits

$x=e$, $u=\log_e e = 1$ $\quad$ $x=e^4$, $u=\log_e e^4 = 4$

$= \Big[3\log_e u\Big]_1^4$

$= 3\log_e 4 - 3\log_e 1$

$= 3\ln 4$

QUESTION 2

a) ways $= {}^{12}C_2 \times {}^{25}C_4$

$= 834\,900$

b) $\displaystyle\int_0^3 \frac{dx}{36+4x^2} = \frac{1}{4}\int_0^3\frac{1}{9+x^2}dx$

$= \dfrac{1}{4}\left[\dfrac{1}{3}\tan^{-1}\dfrac{x}{3}\right]_0^3$

$= \dfrac{1}{4}\left[\dfrac{1}{3}\tan^{-1}\dfrac{3}{3} - \dfrac{1}{3}\tan^{-1}0\right]$

$= \dfrac{1}{4}\times\left(\dfrac{1}{3}\times\dfrac{\pi}{4} - 0\right)$

$= \dfrac{\pi}{48}$

c) $x<3$ or $x>5$, $x\neq-1$

OR alternatively

$x<-1$ or $-1<x<3$ or $x>5$

d) i) $y = \dfrac{x^2}{4a}$

$\dfrac{dy}{dx} = \dfrac{x}{2a}$

when $x = 2ap$

$m_{tang} = \dfrac{2ap}{2a}$

$= p$

$\therefore m_{norm} = -\dfrac{1}{p}$

Eqn of normal

$y - ap^2 = -\dfrac{1}{p}(x-2ap)$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$

ii) y-intercept let $x=0$

$py = ap^3 + 2ap$

$\therefore y = ap^2 + 2a$

$\therefore N$ is $(0, ap^2+2a)$

iii) $M = \left(\dfrac{2ap+0}{2}, \dfrac{ap^2+2a+ap^2}{2}\right)$

$= \left(ap, \dfrac{2(ap^2+a)}{2}\right)$

$= (ap, ap^2+a)$

5

d iii) cont'd...

$x = ap \Rightarrow p = \frac{x}{a}$

$y = ap^2 + a$

$\therefore \ y = a\left(\frac{x}{a}\right)^2 + a$

$y = \frac{ax^2}{a^2} + a$

locus $y = \frac{x^2}{a} + a$

$ay = x^2 + a^2$

$x^2 = ay - a^2$

$x^2 = a(y-a)$

6

QUESTION 3

a) i) $f(x) = x - 3\sin x$

$f(2) = 2 - 3\sin 2$

$= -0.727\ldots$

$f(3) = 3 - 3\sin 3$

$= 2.5766\ldots$

Since $f(2)$ and $f(3)$ are of different sign, then root lies between $x=2$ and $x=3$.

ii) $f'(x) = 1 - 3\cos x$

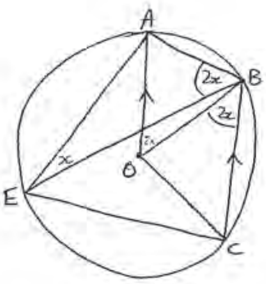
better approx $x_1 = 2 - \frac{2 - 3\sin 2}{1 - 3\cos 2}$

$= 2.3237\ldots$

$\therefore$ better approx is $\underline{\underline{2.32}}$

7

3b)



i) $\angle AOB = \angle CBO$ (alternate $\angle$'s $AO \parallel BC$)

$= 2x$

$\angle AEB = \frac{1}{2}\angle AOB$ (angle at centre of circle is twice angle at circumference standing on same arc)

$= \frac{1}{2}(2x)$

$\underline{\underline{= x}}$

ii) $\angle EAB = 180^\circ - \angle AEB - \angle ABE$ (angle sum $\triangle AEB$)

$= 180 - x - 2x$

$= 180 - 3x$

$\angle BCE = 180 - \angle EAB$ (opposite angle of cyclic quadrilateral ABCE are supplementary)

$= 180 - (180 - 3x)$

$\underline{\underline{= 3x}}$

8

(3) c) $y = 100 - 95e^{-at}$ ——— (1)

$\frac{dy}{dt} = -95 \times -ae^{-at}$

$= a(95e^{-at})$

But from (1) $95e^{-at} = 100 - y$

$\therefore \frac{dy}{dt} = a(100 - y)$

ii) When $t = 2$, $y = 18$

$18 = 100 - 95e^{-a(2)}$

$-82 = -95e^{-2a}$

$\frac{82}{95} = e^{-2a}$

$\log_e\left(\frac{82}{95}\right) = -2a$

$\underline{\underline{a = -\frac{1}{2}\ln\left(\frac{82}{95}\right)}}$

iii) need $y = 39$.

$39 = 100 - 95e^{\frac{1}{2}\ln\left(\frac{82}{95}\right)t}$

$-61 = -95e^{\frac{1}{2}\ln\left(\frac{82}{95}\right)t}$

$\frac{61}{95} = e^{\frac{1}{2}\ln\left(\frac{82}{95}\right)t}$

$\log_e\left(\frac{61}{95}\right) = \frac{1}{2}\ln\left(\frac{82}{95}\right)t$

9

$$t = \frac{\ln\left(\frac{61}{95}\right)}{\frac{1}{2}\ln\left(\frac{82}{95}\right)}$$

$t = 6.02\ldots$

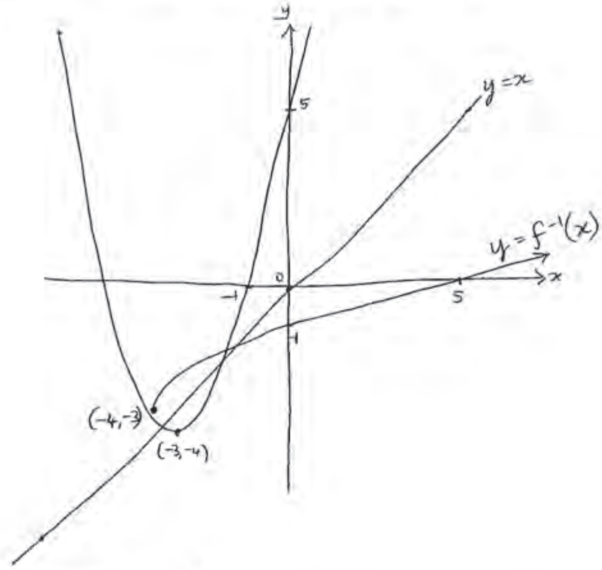
$\therefore$ Can leave bottle in for max 6 minutes

10

QUESTION 4

a) i) domain : real x, such that $x \geqslant -3$

ii)



iii) swap x and y

$x = (y+3)^2 - 4$

$x + 4 = (y+3)^2$

$\pm\sqrt{x+4} = y+3$

$y = -3 \pm \sqrt{x+4}$

but range is $y \geqslant -3$

$\therefore f^{-1}(x) = -3 + \sqrt{x+4}$.

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4b) i) $(10-1)! = 9!$

$= 362880$

ii) If Bill & Hill together:

9 items to place in circle $= 8!$

2 arrangements amongst themselves

$\therefore 8! \times 2 = 80640$ ways together

$\therefore$ Separated $= 362880 - 80640$

$= 282240$

12

4 c) i) $\ddot{y} = -10$

$\dot{y} = -10t + c_1$

When $t = 0$

$\sin 45° = \frac{\dot{y}}{V}$

$\frac{1}{\sqrt{2}} = \frac{\dot{y}}{V}$

$\dot{y} = \frac{V}{\sqrt{2}}$

$\therefore \frac{V}{\sqrt{2}} = -10(0) + c_1$

$c_1 = \frac{V}{\sqrt{2}}$

$\therefore \dot{y} = -10t + \frac{V}{\sqrt{2}}$

$y = -5t^2 + \frac{Vt}{\sqrt{2}} + c_2$

When $t = 0$, $y = 0$

$\therefore c_2 = 0$

$y = \frac{Vt}{\sqrt{2}} - 5t^2$

ii) When $x = 900$

$900 = \frac{Vt}{\sqrt{2}}$

$\frac{900\sqrt{2}}{V} = t$

4)c) ii) cont'd...

when $t = \frac{900\sqrt{2}}{V}$, $y = 100$

$$100 = \frac{\cancel{V}}{\sqrt{2}} \cdot \frac{900\sqrt{2}}{\cancel{V}} - 5\left(\frac{900\sqrt{2}}{V}\right)^2$$

$$100 = 900 - \frac{8100\,000}{V^2}$$

$$\frac{8100\,000}{V^2} = 800$$

$$\frac{8100\,000}{800} = V^2$$

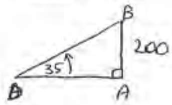
$$V^2 = 10125$$

$$V = \pm 100.62\ldots$$

$\therefore$ Initial speed 101 m/s

QUESTION 5

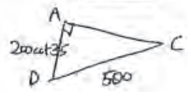
a) In ΔABD



$$\tan 35° = \frac{200}{AD}$$

$$AD = 200\cot 35°$$

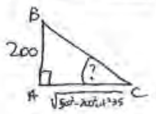
In ΔADC



$$AC^2 = 500^2 - (200\cot 35)^2$$

$$AC = \sqrt{500^2 - 200^2\cot^2 35°}$$

In ΔBAC



$$\tan C = \frac{200}{\sqrt{500^2 - 200^2\cot^2 35}}$$

$$C = 25.982\ldots$$

$\therefore$ Angle of elevation of B from C is 26°

b) over page...

QUESTION 5

a) i) Show true for $n=1$

$LHS = 1 + x$

$$RHS = \frac{x^2 - 1}{x - 1}$$

$$= \frac{(x+1)(x-1)}{x-1}$$

$$= x + 1$$

$\therefore$ LHS = RHS

$\therefore$ true for $n=1$.

Assume true for $n=k$

i.e. $1 + x + x^2 + \ldots + x^k = \frac{x^{k+1} - 1}{x - 1}$

Prove true for $n=k+1$ if true for $n=k$

RTP: $1 + x + x^2 + \ldots + x^k + x^{k+1} = \frac{x^{k+1+1} - 1}{x - 1} = \frac{x^{k+2} - 1}{x - 1}$

$LHS = \frac{x^{k+1} - 1}{x - 1} + x^{k+1}$ by assumption

$$= \frac{x^{k+1} - 1 + (x-1)x^{k+1}}{x - 1}$$

$$= \frac{x^{k+1} - 1 + x^{k+2} - x^{k+1}}{x - 1}$$

$= \frac{x^{k+2} - 1}{x - 1}$ = RHS as required.

5a) ii) $\underbrace{1 + 2 + 2^2 + 2^3 + \ldots + 2^n}$ $< 10^9$

$$\frac{2^{n+1} - 1}{2 - 1} < 10^9$$

$$2^{n+1} - 1 < 10^9$$

$$2^{n+1} < 10^9 + 1$$

crit pts

$$2^{n+1} = 10^9 + 1$$

$$\log_e 2^{n+1} = \log_e(10^9 + 1)$$

$$(n+1)\log_e 2 = \log_e(10^9 + 1)$$

$$n + 1 = \frac{\ln(10^9 + 1)}{\ln 2}$$

$$n = \frac{\ln(10^9 + 1)}{\ln 2} - 1$$

$$= 28.897\ldots$$

28.897...

$\therefore$ largest value of n is 28

5) c) i) Pythag

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$$x^2+(1\cdot8-h)^2=2^2$$
$$(1\cdot8-h)^2=4-x^2$$
$$1\cdot8-h=\sqrt{4-x^2} \quad \text{note } h>0$$
$$\underline{h=1\cdot8-\sqrt{4-x^2}}$$

ii) given $\frac{dh}{dt}=0.5$

Need $\frac{dx}{dt}=\frac{dx}{dh}\times\frac{dh}{dt}$

Now, $h=1\cdot8-(4-x^2)^{\frac{1}{2}}$

$$\therefore \frac{dh}{dx}=-\tfrac{1}{2}(4-x^2)^{-\frac{1}{2}}\times-2x$$
$$=\frac{x}{\sqrt{4-x^2}}$$
$$\therefore \frac{dx}{dh}=\frac{\sqrt{4-x^2}}{x}$$

So, $\frac{dx}{dt}=\frac{\sqrt{4-x^2}}{x}\times0\cdot5$

When $x=1\cdot6$,
$$\frac{dx}{dt}=\frac{\sqrt{4-1\cdot6^2}}{1\cdot6}\times0\cdot5$$
$$=0\cdot375$$

$\therefore$ Ladder hits Tom at $0\cdot375$ m/min

QUESTION 6

18

a) i) $\frac{y}{10}=\cos^{-1}\left(2-\frac{x}{4}\right)$

range of function: $0\leqslant\frac{y}{10}\leqslant\pi$

$\therefore\ 0\leqslant y\leqslant10\pi$

$\therefore\ y_1=10\pi$

OR
When $x=12$
$y=10\cos^{-1}\left(2-\frac{12}{4}\right)$
$=10\cos^{-1}(-1)$
$=10\pi$

ii) $V=\pi\int_0^{10\pi}x^2\,dy$

If $y=10\cos^{-1}\left(2-\frac{x}{4}\right)$

Then $\frac{y}{10}=\cos^{-1}\left(2-\frac{x}{4}\right)$
$$\cos\frac{y}{10}=2-\frac{x}{4}$$
$$\frac{x}{4}=2-\cos\frac{y}{10}$$
$$x=8-4\cos\frac{y}{10}$$
$$\therefore V=\pi\int_0^{10\pi}\left(8-4\cos\frac{y}{10}\right)^2dy$$
$$=\pi\int_0^{10\pi}\left(64-64\cos\frac{y}{10}+16\cos^2\frac{y}{10}\right)dy$$
$$=16\pi\int_0^{10\pi}\left(4-4\cos\frac{y}{10}+\cos^2\frac{y}{10}\right)dy$$

iii)

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$$\cos2A=2\cos^2A-1$$
If $A=\frac{y}{10}$

Then $\cos\left(\frac{2y}{10}\right)=2\cos^2\frac{y}{10}-1$
$$\cos\left(\frac{y}{5}\right)+1=2\cos^2\frac{y}{10}$$
$$\cos^2\frac{y}{10}=\tfrac{1}{2}\left(\cos\left(\frac{y}{5}\right)+1\right)$$
$$\therefore\int\cos^2\frac{y}{10}\,dy=\tfrac{1}{2}\int\left(\cos\frac{y}{5}+1\right)dy$$
$$=\tfrac{1}{2}\left[5\sin\frac{y}{5}+y\right]+C$$

iv) $V=16\pi\left[4y-40\sin\frac{y}{10}+\frac{1}{2}\left(5\sin\frac{y}{5}+y\right)\right]_0^{10\pi}$
$$=16\pi\left(40\pi-40\sin\frac{10\pi}{10}+\tfrac{1}{2}\left(5\sin\frac{10\pi}{5}+10\pi\right)-0\right)$$
$$=16\pi\left(40\pi-0+\tfrac{1}{2}(10\pi)\right)$$
$$=16\pi\times45\pi$$
$$=\underline{\underline{720\pi^2\ \text{cm}^3}}$$

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6 b) i) $x=4\sin^2t-1$
$$\dot{x}=4\times2\sin t\cos t$$
$$=4\sin2t$$
$$\ddot{x}=8\cos2t$$
$$=8(1-2\sin^2t)$$
$$=8-16\sin^2t$$
$$=-4(4\sin^2t-2)$$
$$=-4(4\sin^2t-1-1)$$
$$=\underline{\underline{-4(x-1)}}$$

ii) period $=\frac{2\pi}{2}$ $\leftarrow$ $n^2=4$, $\therefore n=2$
$$=\underline{\pi}$$

iii) max speed when $x=1$ (centre of motion)
$$4\sin^2t-1=1$$
$$\sin^2t=\tfrac{1}{2}$$
$$\sin t=\pm\tfrac{1}{\sqrt2}$$

First max speed after $\frac{\pi}{4}$ seconds

QUESTION 7

21

a) i) $\frac{d}{dx}\ln(\tan x) = \frac{1}{\tan x} \times \sec^2 x$

$= \frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x}$

$= \frac{1}{\sin x \cos x}$

ii) $\frac{dx}{dt} = \sin x \cos x$

$\frac{dt}{dx} = \frac{1}{\sin x \cos x}$

$t = \int \frac{1}{\sin x \cos x}\, dx$

$t = \ln(\tan x) + c_1$

when $t = 0$, $x = \frac{\pi}{4}$

$0 = \ln(\tan \frac{\pi}{4}) + c_1$

$0 = \log_e 1 + c_1$

$\therefore c_1 = 0$.

$\therefore t = \ln(\tan x)$

$e^t = \tan x$

$\therefore x = \tan^{-1}(e^t)$

7a) iii) $x = \tan^{-1}(e^t)$

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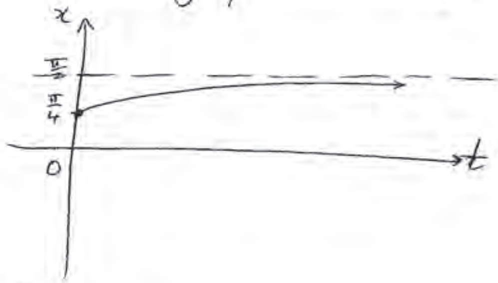
range of function $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$\therefore$ limiting position is $\frac{\pi}{2}$.



7b) i) Cartesian eqn

$t = \frac{x}{V\cos\theta}$

$\therefore y = V\sin\theta \cdot \frac{x}{V\cos\theta} - \frac{g}{2} \times \frac{x^2}{V^2\cos^2\theta}$

$y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$

$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2}$

$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2V^2}$

7b) i) cont'd...

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when $x = r$, $y = 0$

$0 = r\tan\theta - \frac{gr^2(1+\tan^2\theta)}{2V^2}$

$0 = 2V^2 r\tan\theta - gr^2 - gr^2\tan^2\theta$

$gr^2\tan^2\theta - 2V^2 r\tan\theta + gr^2 = 0$

Divide all terms by gr^2

$\tan^2\theta - \frac{2V^2}{gr}\tan\theta + 1 = 0$.

ii) $\Delta = \left(\frac{2V^2}{gr}\right)^2 - 4 \times 1 \times 1$

$= \frac{4V^4}{g^2r^2} - 4$

Real & distinct roots when $\Delta > 0$

$\frac{4V^4}{g^2r^2} - 4 > 0$.

$\frac{4V^4}{g^2r^2} > 4$

$4V^4 > 4g^2r^2$ (note $g^2 > 0$, $r^2 > 0$)

$V^4 > g^2r^2$

cont'd...

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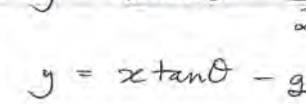
7b ii) $\therefore r^2 < \frac{V^4}{g^2}$

crit pts $r = \pm\sqrt{\frac{V^4}{g^2}}$

$= \pm\frac{V^2}{g}$ but $r > 0$ as it is distance



$\therefore r < \frac{V^2}{g}$

7 b) iii) $\tan\alpha \tan\beta = 1$

$\tan\alpha = \frac{1}{\tan\beta}$

$\tan\alpha = \cot\beta$

Complementary ratios

$\therefore \alpha + \beta = 90°$