Pymble Ladies' College
Mathematics Department

St Catherine's School

Year: 12
Subject: Extension I Mathematics
Time Allowed: 2 hours
(plus 5 mins reading time)
Date: August 2001

Exam number:________________

Directions to candidates:

- All questions are to be attempted.
- All questions are of equal value.
- All necessary working must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Each question attempted should be started on a **new booklet**.
- Approved calculators and geometrical instruments are required.
- This page is a cover sheet for **Section A**. Write a cover page for **Section B and C** and include your number.
- Hand in your work in **3 bundles**:
  - Section A Questions 1, 2 and 3.
  - Section B Questions. 4 and 5
  - Section C Questions. 6 and 7.

TEACHER'S USE ONLY Total Marks	
A	
B	
C	
TOTAL	

Question 1:

(a) Solve for x: $\frac{3}{x+5} \le 1$ (2 marks)

(b) A root of $e^x - x^2 = 0$ lies near $x = -0.5$. Use Newton's Method once to find a better approximation. (3 marks)

(c) Consider the function $f(x) = 2\sin^{-1}\frac{x}{2}$.

(i) Find the exact value of $f(\sqrt{2})$. (1 mark)

(ii) What is the domain and range of $f(x)$. (2 marks)

(iii) Sketch $f(x)$. (1 mark)

(iv) Find the equation of the tangent to the curve at $x = \sqrt{2}$ (3 marks)

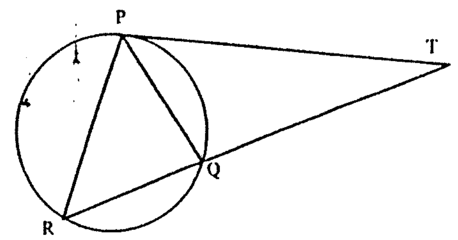
Question 2:

(a) Solve for x: $2|x-1| = 4x - 1$. (3 marks)

(b) If α, β, γ are the roots of $x^3 - 3x^2 - 2x + 1 = 0$, find:

(i) $\alpha+\beta+\gamma$ (1 mark) (ii) $\alpha\beta\gamma$ (1 mark)
(iii) $\alpha\beta+\alpha\gamma+\beta\gamma$ (1 mark) (iv) $\alpha^2+\beta^2+\gamma^2$ (2 marks)

(c) PT is a tangent to the circle PRQ. RQ is a secant intersecting the circle in Q and R. The line RQ intersects PT at T.



(3 marks)

Question 3:

(a) Find $\int x^2(x^3-5)^5\,dx$ using the substitution $u = x^3 - 5$ (3 marks)

(b) The angle between the lines $y = mx$ and $y = \frac{1}{2}x$ is 45° (3 marks)
Find two possible values of m.

(c) The rate at which a body cools in air is proportional to the difference between its temperature T and the temperature C of its surroundings. That is:

$\frac{dT}{dt} = -k(T - C)$ where t is the time in hours and k is a positive constant.

i) Show that $T = C + Ae^{-kt}$ is a solution to the differential equation above (where A is a real number). (2 marks)

A heated piece of metal is initially 90°C but cools to 70°C in one hour. Given the surroundings are 25°C find:

ii) the constants A and C (3 marks)
iii) the temperature of the metal after three hours (to the nearest degree). (1 mark)

Question 4:

(a) Find $\int \cos^2 2x\,dx$ (2 marks)

(b) A team of 4 is to be chosen from 5 boys and 6 girls. How many teams are possible if:

(i) there are no restrictions (1 mark)

(ii) the shortest girl must be included (1 mark)

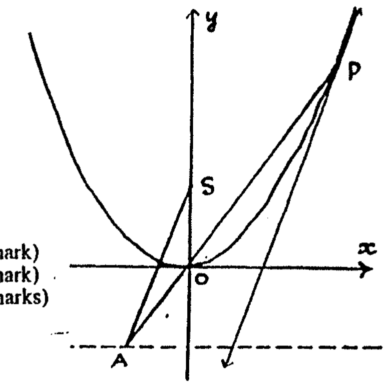
(c) Show by Mathematical Induction that the following statement: (4 marks)

$$(1\times 2)+(2\times 3)+(3\times 4)+\dots\dots\dots\dots+(n-1)(n) = \frac{(n-1)n(n+1)}{3}$$

is true for all integers $n \geq 2$.

(d) The diagram shows the parabola with parametric coordinates $x = 2ap$ and $y = ap^2$
P is a point on the parabola. S is the focus and A is a point on the directrix.

The straight line drawn from point P $(2ap\ ap^2)$ on the parabola through the vertex at O(0,0) intersects the directrix at A.



(i) Find the equation of line PO (1 mark)
(ii) Show that the coordinates of A are $\left(\frac{-2a}{p}, -a\right)$ (1 mark)
(iii) Prove that AS is parallel to the tangent at P. (2 marks)

Question 5:

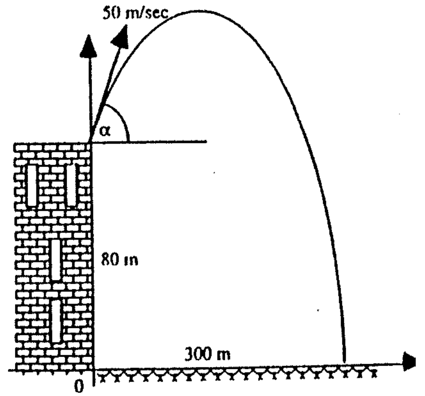
(a) Eight different coloured beads are arranged so that two particular colours are next to each other. In how many ways can they be arranged in:
(You may leave your answer in factorial notation)

(i) a line (1 mark)
(ii) a circle (1 mark)
(iii) a necklace (1 mark)

(b) Find the exact value of (2 marks)

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}}$$

(c) Kelly throws a stone at an angle of elevation of α from the top of a tower 80 m high at an initial velocity of 50 m/sec, as in the diagram.



The acceleration due to gravity is assumed to be 10m/sec². Take the origin to be the base of the tower.

(i) Given that $\frac{d^2x}{dt^2} = 0$ and $\frac{d^2y}{dt^2} = -10$ show that

$x = 50t\cos\alpha$ and

$y = -5t^2 + 50t\sin\alpha + 80,$

where x and y are the horizontal and vertical displacements of the stone in metres from the origin at time t seconds after throwing. (3 marks)

(ii) Kelly wants the stone to land in the sea 300 metres from the base of the tower.

Question 6:

(a) Differentiate $f(x) = \ln(\tan^3 x)$ (2 marks)

(b) Consider the function $f(x) = \dfrac{x^2}{x^2 - 2}$

(i) Give the equations of any horizontal and vertical asymptotes. (2 marks)

(ii) Find the x and y intercepts if they exist. (1 mark)

(iii) Given that this curve has only one stationary point and it is a local maximum, find its coordinates. (2 marks)

(iv) Sketch the curve, indicating on your sketch all important features. (2 marks)

(c) Show that the equation $2x^3 - 5x^2 + 3x - 2 = 0$ has only one real root. (3 marks)

Question 7:

(a) The displacement x metres of a particle from the origin is in simple harmonic motion and is given by $x = 5\cos \pi t$, where the time t is in seconds.

(i) What is the period of the oscillation? (1 mark)

(ii) What is the speed v of the particle as it moves through the origin? (2 marks)

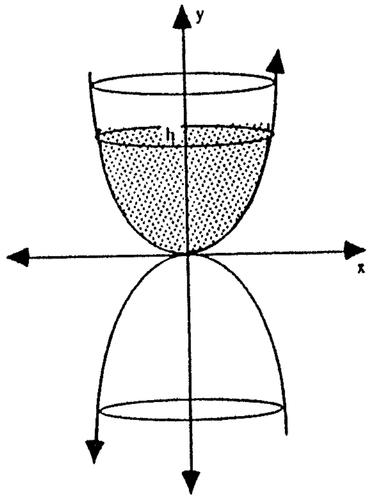
(b) Show that $(x-1)(x-2)$ is a factor of $P(x) = x^n(2^m - 1) + x^m(1 - 2^n) + 2^n - 2^m$ where m, n are positive integers. (2 marks)

Question 7 (continued):

(c) An eggtimer has the same shape as the curve $y = x^3$ rotated about the y axis. The top half of the eggtimer is filled with sand to a depth of h units.

(i) Show that the volume V of sand needed is given by $V = \dfrac{3\pi}{5}\sqrt[3]{h^5}$. (3 marks)

The rate with which the sand falls into the bottom of the timer is found to be proportional to the height h of the sand in the top of the eggtimer.
(ie $\dfrac{dV}{dt} = kh$ where k is a real number)



(ii) Find the exact rate at which the height of the sand in the top of the eggtimer is falling when $h = \dfrac{27}{8}$ cm if the sand is flowing through the neck at 1 cm^3 / minute when $h = 5$ cm

(4 marks)

YR 12 EXTENSION 1 AUG 2001 ANSWERS

1. a) $\frac{3}{x+5} \le 1$

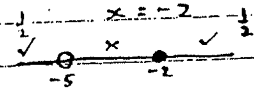
$x \ne -5$

Solve eqtn to find critical pts

$3/x+5 = 1$

$3 = x+5$

$x = -2$



∴ $\boxed{x<-5,\ x\ge -2}$ ✓✓ 1

b) let $f(x) = e^x - x^2$ ✓ 1

$f'(x) = e^x - 2x$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= -0.5 - \frac{f(-0.5)}{f'(0.5)}$ ✓ 1

$= -0.5 - \frac{e^{-0.5} - (0.5)^2}{e^{-0.5} - 2(-0.5)^2}$

$= -0.5 - \frac{0.35653\ldots}{1.60653\ldots}$

$= -0.72196\ldots$ ✓ 1

c) i) $f(\sqrt2) = 2\sin^{-1}\frac{\sqrt2}{2}$ ½

$= 2\sin^{-1}\frac{1}{\sqrt2}$

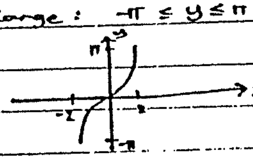
$= 2(\pi/4)$

$= \frac{\pi}{2}$ ½ ✓

ii) Domain: $-2 \le x \le 2$ ✓ 1

Range: $-\pi \le y \le \pi$ ✓ 1

iii)



✓ 1

iv) If $x = \sqrt2$, $f(\sqrt2) = \pi/2$ from (i) → P.T.O.

Grad $f'(x) = 2\left(\frac{2}{\sqrt{4-x^2}}\right)$ ✓ 1

$f'(\sqrt2) = 2\,\frac{2}{\sqrt{4-\sqrt2^2}} = \frac{4}{\sqrt2} = \frac{4\sqrt2}{\sqrt2\sqrt2} = 2\sqrt2$ ✓ 1

Eqtn $y - y_1 = m(x - x_1)$

$y - \pi/2 = \sqrt2(x - \sqrt2)$ ✓ 1

$y = \sqrt2 x - 2 + \pi/2$

Q2

a) $2|x-1| = 4x-1$

$2(x-1) = 4x-1$ or $-2(x-1) = 4x-1$

$2x-2 = 4x-1$ | $-2x+2 = 4x-1$

$-1 = 2x$ | $3 = 6x$

$x = -1/2$ ✓ | $x = 1/2$ ✓

CHECK NO | YES

∴ $\boxed{x = 1/2}$ ✓ (3)

b) $x^3 - 3x^2 - 2x + 1 = 0$

i) $\alpha+\beta+\gamma = -b/a = \frac{-(-3)}{1} = 3$ ✓ (1)

ii) $\alpha\beta\gamma = -d/a = -1$ ✓ (1)

iii) $\alpha\beta+\beta\gamma+\alpha\gamma = c/a = -2$ ✓ (1)

iv) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$ ✓ (2) hard to give part marks.

$= 3^2 - 2(-2)$

$= 13$ ✓

c) i) In △PRT, △QPT

∠PTQ is common ✓

∠QRP,

∠TPQ = ∠TRP (∠ in altern. segment) ✓

∠PQT = ∠RPT (∠ sum △)

∴ △PRT ||| △QPT (equiangular) ✓ 3

ii) $\frac{PT}{RT} = \frac{QT}{PT}$ (corresp sides, △PRT ||| △QPT) ✓ (1)

∴ $PT^2 = QT \times RT$

Q3

a) $\int x^2(x^3-5)^5\,dx$ | $u = x^3 - 5$, $\frac{du}{dx} = 3x^2$, $dx = \frac{du}{3x^2}$ ✓

$= \int x^2 u^5 \cdot \frac{du}{3x^2}$ ✓

$= \frac13 \int u^5\,du$

$= \frac13 \frac{u^6}{6} + c$

$= \frac{1}{18}(x^3-5)^6 + c$ ✓

−½ for incorrect resubst'n for u

−½ for $\cancel{\frac{1}{18}(x^3-5) + C}$ ✓ 1 reinstate

b) $\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

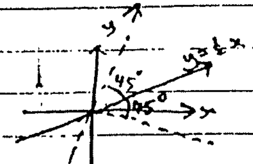
$\tan 45 = \frac{\frac12 - m_2}{1 + \frac12 m_2}$ ✓

$1(1 + \frac12 m_2) = \frac12 - m_2$

$\frac32 m_2 = -\frac12$

$3m_2 = -1$

$m_2 = -1/3$ ✓ ∴ $m_1 = 3$ ✓



Other'm would be 45° the other way i.e. ⊥ 90° from $m = -\frac13$

−½ for $\frac{M_1 + M_2}{1 + M_1 M_2}$ or variants

3c

$\frac{dT}{dt} = -k(T-c)$

i) $T = c + Ae^{-kt}$ ... $Ae^{-kt} = T - c$ ①

$\frac{dT}{dt} = 0 + -kAe^{-kt}$

$\frac{dT}{dt} = -k(T-c)$ from ①

need to explain that $Ae^{-kt} = T-C$ somehow −1 if they don't

ii) When $t=0$, $T=90$. Also $c=25$

$90 = 25 + Ae^{0}$

$65 = A$

$\therefore T = c + 65e^{-kt}$

When $t=1$, $T=70$

$70 = 25 + 65e^{-k}$

$45 = 65e^{-k}$

$\frac{45}{65} = e^{-k}$

$\ln \frac{45}{65} = -k$

$\therefore k = 0.3677\ldots$

so $T = 25 + 65e^{-0.3677\ldots t}$

$-\frac{1}{2}$ for $t = 60$

(iii) After

$T = 25 + 65e^{-0.3677\ldots \times 3}$

$= 46.568$ or $47°$ (nearest °)

1 for correct subs^n and calc. 0 otherwise

4a. $\int \cos^2 2x\,dx$

$\cos 2A = 2\cos^2 A - 1$

$\therefore \cos^2 A = \frac{1}{2}(\cos 2A + 1)$

$\frac{1}{2}$ for $\cos^2 A = \frac{1}{2}(\ldots$ etc) only

$= \frac{1}{2}\int \cos 4x + 1\,dx$

$= \frac{1}{2}(\frac{1}{4}\sin 4x + x) + c$

$= \frac{1}{8}\sin 4x + \frac{1}{2}x + c$

$-\frac{1}{2}$ for $\left[-\frac{1}{8}\sin 4x\right]$

$1\frac{1}{2}$ for leaving off $\frac{1}{2}$ from $\frac{1}{2}(\cos 4x + 1)$ with rest correct

b. i) ${}^{11}C_4 = 330$

ii) ${}^{10}C_3 = 120$

4 c)

Let $S(n)$: $(0\times1) + (1\times2) + (2\times3) + \ldots + (n-1)n = \frac{(n-1)n(n+1)}{3}$

1/ SHOW TRUE FOR S(2)

LHS = $(0\times1) + (1\times2)$ $= 2$

RHS = $\frac{(2-1)2(2+1)}{3}$ $= 2$ = LHS $\therefore$ true for S(2)

2/ ASSUME TRUE FOR S(k)

i.e. $(0\times1) + (1\times2) + (2\times3) + \ldots + (k-1)k = \frac{(k-1)k(k+1)}{3}$ * $\frac{1}{2}$

3/ PROVE TRUE FOR S(k+1)

i.e. $(0\times1) + (1\times2) + (2\times3) + \ldots + (k-1)k + k(k+1) = \frac{k(k+1)(k+2)}{3}$ $\frac{1}{2}$

LHS = $(0\times1) + (1\times2) + (2\times3) + \ldots + (k-1)k + k(k+1)$

$= \frac{(k-1)k(k+1)}{3} + k(k+1)$ from ASSUMPTION * $\frac{1}{2}$

$= \frac{(k-1)k(k+1)}{3} + \frac{3k(k+1)}{3}$

$= \frac{k(k+1)[(k-1)+3]}{3}$

$= \frac{k(k+1)(k+2)}{3}$

= RHS

$\therefore$ if true for S(k) then true for S(k+1) * $\frac{1}{2}$

Since true for S(2) then by the principle of mathematical induction, S(n) is true for all positive integers $n \geq 2$.

4d) i) $S(0,a)$ $A(x_a, -a)$

EQTN LINE PO: $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$\frac{y-0}{x-0} = \frac{ap^2-0}{2ap-0}$

$y = \frac{ap^2}{2ap}x$

$y = \frac{p}{2}x$

accept $2y - px = 0$ and $2ap^2 - 2ap^2 = 0$

ii) At A, $y = -a$

$-a = \frac{p}{2}x$ (A lies on PO)

$\therefore x = \frac{-2a}{p}$

$\therefore A\left(\frac{-2a}{p}, -a\right)$

iii) $M_{AS} = \frac{-a-a}{-2a/p - 0}$

$= \frac{-2a}{-2a}\cdot p$

$= p$

$M_{\text{tangent at } P} = \frac{dy}{dx}$

$= \frac{dy}{dp}\cdot\frac{dp}{dx}$

$= 2ap \cdot \frac{1}{2a}$

$= p$

AS // tangent at P (equal gradients)

(1 ... showing ... directly)