Question 1

(a) Find the general solution to the equation **2**

$$\tan\left(\theta + \frac{\pi}{4}\right) = 1.$$

(b) Graph $y = |x+1|$ and $y = 2x - 3$ on the same set of axes. **4**
Hence, or otherwise, solve the inequality

$$|x+1| < 2x - 3.$$

(c) Prove $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$. **3**

(d) For what values of m does the line $y = m(x+1)$ have no intersection with the parabola $y = 2x^2$? **3**

Question 2 **Begin a new page.**

(a) Find the value of x: **2**

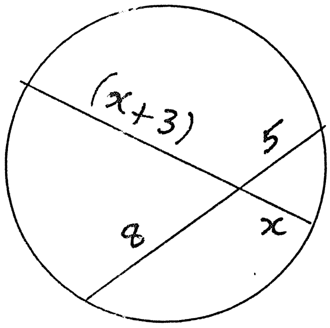


Diagram not to scale.

(b) Consider the polynomial $P(x) = x^3 + ax^2 + bx + 2$, which has factors $x + 1$ and $x - 2$. Find the values of a and b. **4**

(c) Evaluate exactly: **2**

$$\sin\left(2\cos^{-1}\frac{5}{13}\right).$$

(d) Differentiate with respect to x: **4**

$$\left(\tan^{-1}\left(\frac{x}{3}\right)\right)^2.$$

Hence find the exact value of $\displaystyle \frac{1}{\pi}\int_0^{\sqrt{3}} \frac{\tan^{-1}\left(\frac{x}{3}\right)}{x^2+9}\,dx$.

Question 3 **Begin a new page.**

(a) Find $\int_0^1 \frac{x\,dx}{\sqrt{x+1}}$, using the substitution $u = x+1$. **3**

(b) Consider the expression $\sin^{-1} x + \cos^{-1} x$ for $0 \le x \le 1$. **3**

Prove that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

Hence, or otherwise, evaluate $\int_0^1 \sin^{-1} x + \cos^{-1} x\,dx$.

(c) Consider the point $P(2p, p^2)$ on the parabola $x^2 = 4y$.

(i) Prove the normal at P is given by $x + py - 2p - p^3 = 0$. **2**

(ii) The normal meets the y-axis at Q. Find the coordinates of Q. **1**

(iii) Find the coordinates of M, the midpoint of PQ. **1**

(iv) Prove that the locus of M is a parabola. **2**

Question 4 **Begin a new page.**

(a)

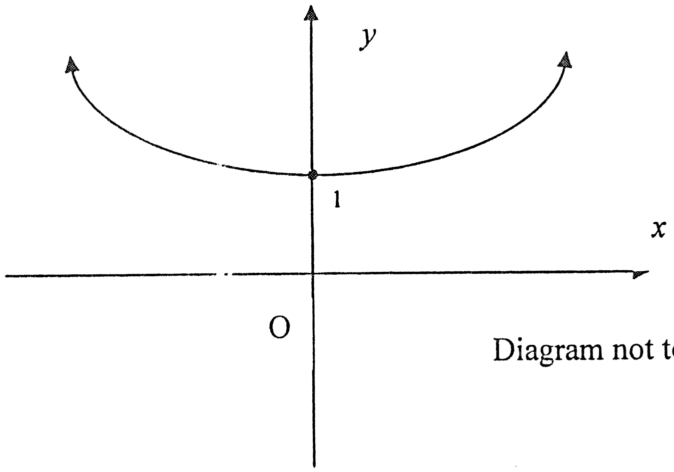


Diagram not to scale.

Consider the curve $y = \frac{e^{2x} + e^{-2x}}{2}$, shown above.

(i) Show that the curve is an even function . **1**

(ii) State the largest possible domain containing $x = 2$ such that the graph **1**

defines a function which has an inverse function.

(iii) State the domain of this inverse function. **1**

(iv) Show that the equation of this inverse function is **3**

$$y = \frac{\ln\left(x + \sqrt{x^2 - 1}\right)}{2}.$$

(b)

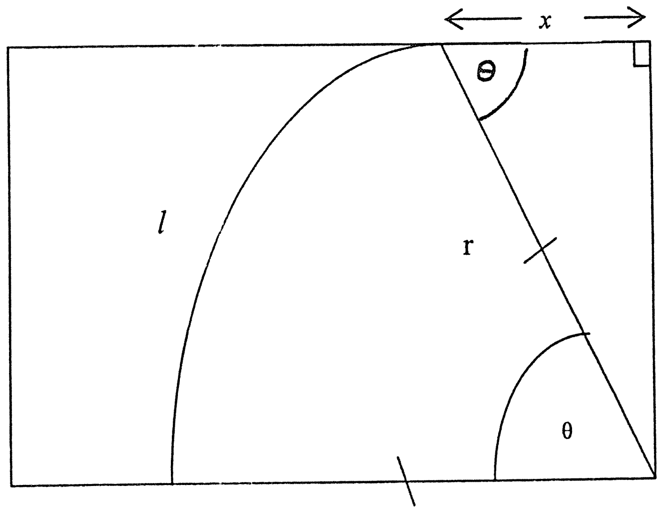


Diagram not to scale.

In the UNESCO World Heritage Site of Split, in Croatia, there are stormwater grates consisting of circular arcs of length l enclosed within a rectangle.

(i) Write expressions for l and x in terms of θ. **1**

(ii) Hence show that $l\cos\theta - x\theta = 0$. **2**

(iii) If $l = 30$ cm and $x = 14$ cm, use one application of Newton's method to find a better approximation for the value of θ. Begin with $\theta = 1.047$. **3**

Question 5 **Begin a new page.**

(a) The volume V cm ^{3} of a truncated spherical wooden knob of radius r cm, being manufactured for a door handle, is given by **4**

$$V = \frac{2\pi r^3}{3} + \pi r^2 c - \frac{\pi c^3}{3} \quad \text{where c is a constant.}$$

If the knob is being sanded such that the radius is decreasing by 2 cm/s, find the rate at which the volume is decreasing when the radius is 7 cm. Take $c = 3$.

(b) The path of a projectile fired from the origin, O, is given by

$$x = Vt\cos\theta$$
$$y = Vt\sin\theta - 5t^2$$

where V is the initial speed in m/s and θ is the angle of projection and t is in seconds.

(i) Find the maximum height reached by the particle in terms of V and θ. **3**

(ii) Find the range in terms of V and θ. **3**

(iii) Prove that the range is maximum when θ = 45°. **2**

Question 6 Begin a new page.

(a) In ideal conditions a student will have learned 100% of the content for an HSC subject before they sit the HSC exam. If P% represents the percentage of the content learned after t weeks of study then the rate at which the student is learning is given by $\frac{dP}{dt} = 25 - 0.25P$.

(i) Show that $P = 100 + Ce^{-0.25t}$ is a solution to $\frac{dP}{dt} = 25 - 0.25P$. **2**

Jemima initially knows nothing when she begins to study. She thinks that she'll do well enough if she knows 95% of the content by the time she sits the Maths exam.

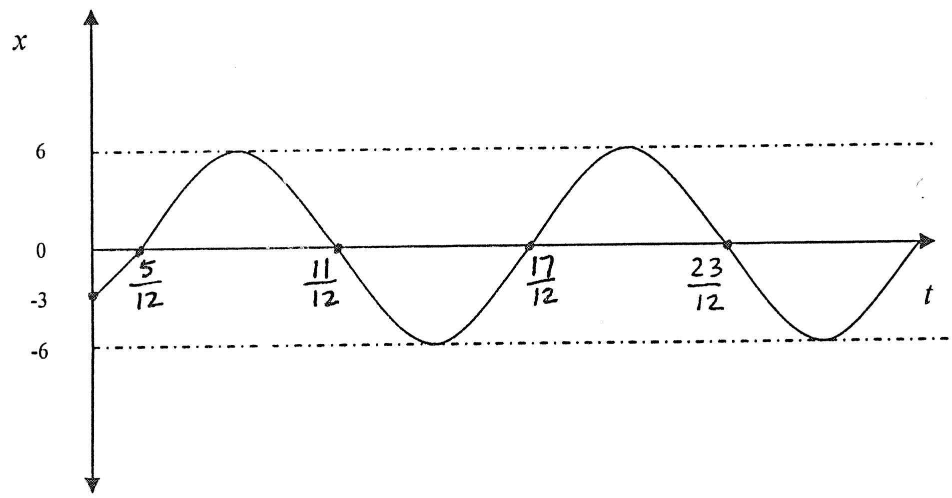
(ii) Find the value of C. **1**

(iii) If the HSC Maths exam is in 12 weeks from the time she begins to study, determine whether or not Jemima will reach her goal. **2**

(iv) How long will it take for Jemima to learn 100% of the content? **1**

(v) Sketch the graph of P versus t. **1**

(b) A student violinist practises her bowing technique and bows back and forth about the centre of the bow, O, according to the equation $x = A\cos(nt + \varepsilon)$, where x is the displacement in cm from O at time t seconds and A, n and ε are constants. The graph of displacement versus time is sketched below.



(i) Find the values of A, n and ε, where A>0, n>0 and $0 < \varepsilon < \pi$. **3**

(ii) Prove that $\ddot{x} = -4\pi^2 x$. **2**

Question 7 **Begin a new page.**

(a)

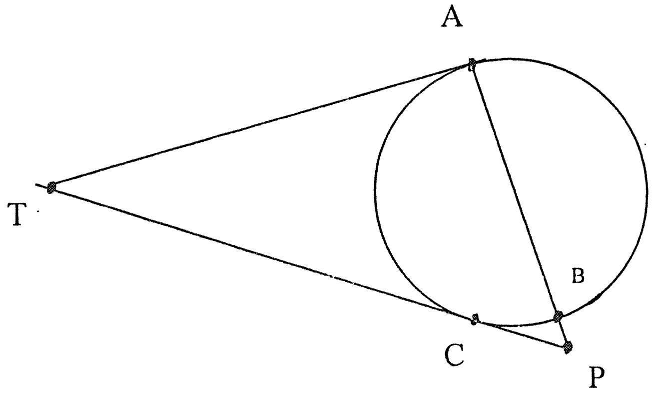


Diagram not to scale.

AB is a diameter of the circle ABC. The tangents at A and C meet at T. The lines TC and AB are produced to meet at P.

Copy the diagram. Join AC and CB.

(i) Prove that $\angle CAT = 90^o - \angle BCP$. **2**

(ii) Hence or otherwise prove $\angle ATC = 2\angle BCP$. **3**

(b) (i) Prove the identity $\dfrac{\cos y - \cos(y+2q)}{2\sin q} = \sin(y+q)$. **3**

(ii) Use Mathematical Induction and the result of part (i) above to prove **4**

$$\sin q + \sin 3q + \sin 5q + \ldots + \sin(2N-1)q = \frac{1-\cos 2Nq}{2\sin q}, \quad N = 1,2,3,\ldots$$

End of paper!