St Catherine's School

Year: 12
Subject: Extension 1 Mathematics
Time allowed: 2 hours plus 5 minutes reading time
Date: July 2006

Student number____________________

Directions to candidates:

- All questions are to be attempted.(Q.1 to Q.7)
- Questions 1-3 are in booklet A.
- Questions 4-7 are in booklet B
- Each question is worth 12 marks
- Marks may be deducted for careless or badly arranged work

Marks:

Q 1	
Q 2	
Q 3	
Q 4	
Q.5	
Q.6	
Q.7	
Total	

Question 1 (12)

a) Prove the trigonometric identity $\operatorname{cosec}\theta - 2\cot 2\theta\cos\theta = 2\sin\theta$ (3)

b) Solve for x: $\dfrac{x^2-5}{x} > 4$ (3)

c) Find $\displaystyle\int \frac{1}{\sqrt{25-9x^2}}\,dx$ (3)

d) Find the general solution of the equation $2\cos(4x+\frac{\pi}{3}) = \sqrt{2}$ (3)

Question 2 *(start a new page)* (12)

a) i) Write out the expansion of $(a+b)^n$ showing the first three terms, the general term, and the last term (1)

ii) Substituting appropriate values for a and b, show that $\sum_{k=0}^{n}(-1)^k \, {}^nC_k = 0$ (3)

b) Find the coefficient of x in the expansion of $(3x^2 - \frac{2}{x^3})^8$ (3)

c) i) Draw the graph of $y = \sin\frac{x}{2}$, $-2\pi \le x \le 2\pi$ (1)

ii) Use your graph to show that $\sin\frac{x}{2} + x + 1 = 0$ has only one solution. (1)

iii) Taking $x = -0.5$ as the first approximation to the solution, use one application of Newton's method to find a better approximation. (3)

Question 3 *(start a new page)* (12)

a) i) On the same set of axes, sketch the curve $f(x) = \log_e x$ and its inverse, $y = f^{-1}(x)$ (2)

ii) A (x,y) is a point on $y = \log_e x$
B (y,x) is a point on $y = f^{-1}(x)$ (1)
Plot A and B on your graph.

iii) Show that the distance AB is $\sqrt{2}\,|(x - \log_e x)|$ (2)

iv) Find the minimum length of AB (3)

b) Prove by Mathematical Induction that (4)

$$2\times2 + 3\times2^2 + 4\times2^3 .... + n\times2^{n-1} = (n-1)2^n \quad \text{for integer } n \ge 2$$

Question 4 *(start a new booklet)* (12)

a) Find $\int_0^{\frac{\pi}{4}} \cos^3 x \sin x\, dx$ (2)

b) Find $\int \frac{1}{(x^2+4)^{\frac{3}{2}}} dx$ using the substitution $x = 2\tan\theta$ (3)

c) A polynomial $P(x)$ is given by

$$P(x) = ax^3 + bx^2 + 10x - 8$$

Find a and b if $(x + 2)$ is a factor of $P(x)$
and the remainder when $P(x)$ is divided by $(x - 1)$ is 12 (3)

d) α, β and γ are the roots of the equation $2x^3 - 4x - 7 = 0$

i) $\alpha^2 + \beta^2 + \gamma^2$ (2)

ii) $(\alpha+1)(\beta+1)(\gamma+1)$ (2)

Question 5 *(start a new page)* (12)

a) A rabbit population on a small island grows at a rate proportional to the difference between the population P and 100, i.e.

$$\frac{dP}{dt} = k(100 - P) \quad \text{where } t \text{ is measured in months}$$

i) Show that $P = 100 - Ae^{-kt}$ satisfies this condition. (1)

ii) Initially the population is 6 rabbits, and after 2 months it has reached 20 rabbits.

Find values for A and k (3)

iii) What is the expected number of rabbits on the island in the long term (that is, as t becomes very large)? (1)

b) i) Show that the curve $y = \sin x$ and the line $y = \frac{2x}{\pi}$ intersect at the origin and at $(\frac{\pi}{2}, 1)$ (1)

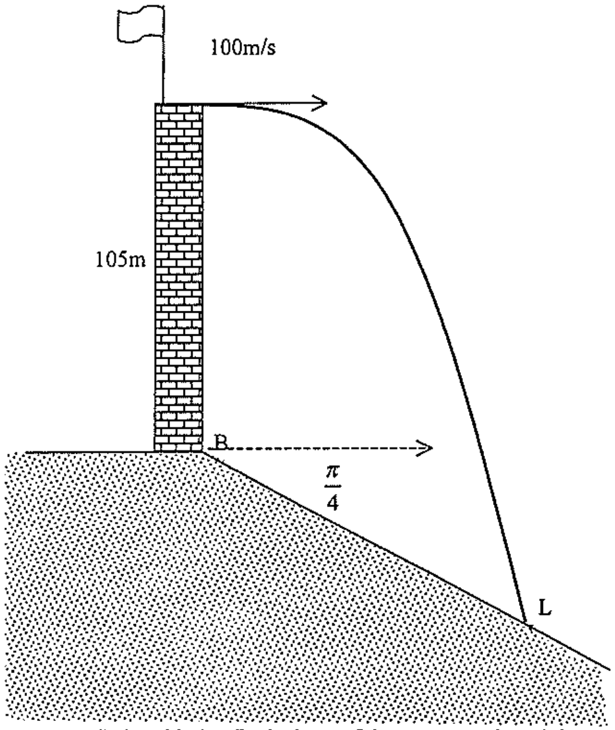
ii) The region enclosed by the curve $y = \sin x$ and the line $y = \frac{2x}{\pi}$ is rotated about the x-axis to form a solid. Calculate the volume of the solid. (3)

c) Graph the curve $y = 2\sin^{-1}\frac{x}{3}$ showing clearly its domain and range (3)

Question 6 *(start a new page)* (12)

a) A bullet is fired horizontally with a velocity of 100 m/s from the top of a tower 105 m high. The tower is at the top of a hill, which slopes downwards at an angle of depression of $\frac{\pi}{4}$

The bullet lands at L



i) Considering B, the base of the tower, as the origin, and using the acceleration due to gravity as $-10m/s^2$, show that the expressions for the x- and y- co-ordinates of the position of the bullet at time t sec are

$x = 100t$ and $y = 105 - 5t^2$ (2)

ii) Show that the equation of the line BL is $y = -x$ (1)

iii) Find the time taken for the bullet to hit the ground at L. (2)

iv) Find the distance BL to the nearest metre. (1)

Question 6 *(continued)*

b) i) Show that if $f(x)$ is an odd function defined for all x, then $f(0) = 0$ (1)

ii) An odd polynomial $P(x)$ of degree 5 has a double zero at $x = 2$, and $P(1) = -12$ (2)

What is the leading term of $P(x)$?

c) i) Find n if ${}^nC_{14} = {}^nC_{12}$ (1)

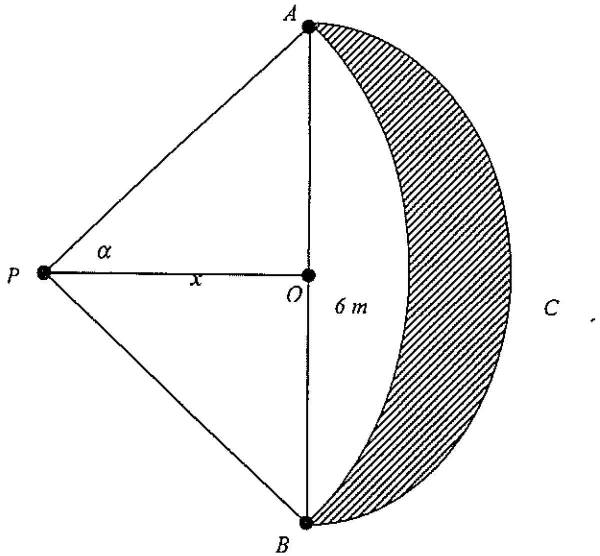
ii) Simplify $\frac{{}^nC_r}{{}^nC_{r-1}}$ (2)

Question 7 *(start a new page)* (12)

a) By considering the term in x^n on both sides of the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$, show that

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots\ldots({}^nC_n)^2 = {}^{2n}C_n$$ (3)

b)



A semicircle ACB has diameter AB 6 m long. O is the midpoint of AB. OP $\perp$ AB.
An arc of another circle, centre P, passes through A and B.

i) Show that if OP = x m, then $\sin\alpha = \dfrac{3}{\sqrt{x^2+9}}$ (1)

ii) Show that the shaded portion S expressed as a function of x is (4)

$$S = \frac{9\pi}{2} + 3x - (x^2+9)\tan^{-1}\left(\frac{3}{x}\right)$$

iii) The point P moves to the left at 0.1 m/min Find the rate of change of the area S when $x = 3$ m (3)

iv) Explain what happens to the shape of shaded area S as $x \to \infty$ (1)

SOLUTIONS

STUDENT NUMBER ______________

COURSE NAME: St Cath's Maths Ext 1 Trial 2006

SECTION ______________

QUESTION: 1–7.

Q1:

a) $\operatorname{cosec}\theta - 2\cot 2\theta \cos\theta = 2\sin\theta$

$LHS = \dfrac{1}{\sin\theta} - \dfrac{2\cos 2\theta}{\sin 2\theta}\cdot\cos\theta$ ✓

$= \dfrac{1}{\sin\theta} - \dfrac{\cancel{2}(\cos^2\theta - \sin^2\theta)}{\cancel{2}\sin\theta\,\cancel{\cos\theta}}\cdot\cancel{\cos\theta}$ ✓

$= \dfrac{1-\cos^2\theta+\sin^2\theta}{\sin\theta}$ ✓

OTHER WAYS POSSIBLE

$= \dfrac{2\sin^2\theta}{\sin\theta}$ (3)

$= 2\sin\theta$

$= RHS$ as req.

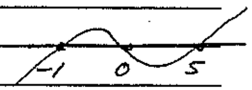
b) $\dfrac{x^2-5}{x} > 4$

$x(x^2-5) > 4x^2$ ✓

$x^3 - 4x^2 - 5x > 0$

$x(x^2-4x-5) > 0$

$x(x-5)(x+1) > 0$ ✓



$-1 < x < 0, \quad x > 5$ ✓✓ (3)

c) $\displaystyle\int \frac{1}{\sqrt{25-9x^2}}\,dx = \int \frac{1}{\sqrt{9(\frac{25}{9}-x^2)}} = \frac{1}{3}\int\frac{1}{\sqrt{\frac{25}{9}-x^2}}$ 1 ✓

$= \dfrac{1}{3}\sin^{-1}\left(\dfrac{3x}{5}\right) + c$ (3)

1 1

d) $2\cos\left(4x+\frac{\pi}{3}\right) = \sqrt{2}$

$\cos\left(4x+\frac{\pi}{3}\right) = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

$4x+\frac{\pi}{3} = 2\pi n \pm \frac{\pi}{4}$

$= \dfrac{\pi n}{2} - \dfrac{\pi}{48}$ or $\dfrac{\pi n}{2} - \dfrac{7\pi}{48}$

Q2

a) i) $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-1} b^2 + \cdots$ (1)

$\cdots \; {}^nC_k a^{n-k} b^k \; \cdots \cdots + {}^nC_n b^k.$

✓

ii) let $a = 1$; $b = -1$ ✓

Then ${}^nC_0 + {}^nC_1(-1) + {}^nC_2 \cdots \cdots {}^nC_k(-1)^k \cdots {}^nC_n(-1)^n$ ✓

$= (1-1)^n$

$\therefore \sum_{k=0}^{n} {}^nC_k (-1)^k = 0$ as req.

b) $\left(3x^2 + \frac{2}{x^3}\right)^8 = \left(3x^2 + 2x^{-3}\right)^8$ ✓

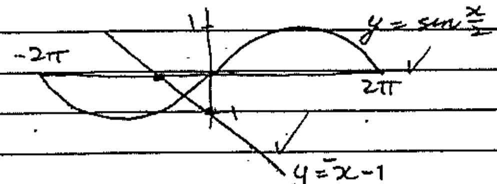
$\Rightarrow {}^nC_k (3x^2)^{8-k} (-2x^{-3})^k$ term x^1

$\therefore 2(8-k) - 3k = 1$ ✓

$16 - 5k = 1 \quad \therefore k = 3$

$\therefore$ Coeff is $-{}^8C_3\, 3^5\, 2^3$ ✓

c)



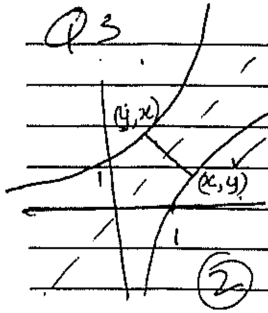
(2)

$P(x) = \sin\frac{x}{2} + x + 1$ ✓ $\quad P(-0.5) = 0.0205$ ✓

$P'(x) = \frac{1}{2}\cos\frac{x}{2} + 1 \quad P'(-.5) = 1.48$

$x_2 = -0.5 - \frac{0.2}{1.48} \ldots = -.67$

Q3



(2)

$D = \sqrt{(x-y)^2 + (y-x)^2}$ ✓

$= \sqrt{2(x-y)^2}$

$= \sqrt{2}\,|x-y|$

$= \sqrt{2}\,|x - \log_e x|$ ✓

$\frac{dD}{dx} = \sqrt{2}\left(1 - \frac{1}{x}\right)$ ✓

max/min $\frac{dD}{dx} = 0 \quad \therefore \quad 1 - \frac{1}{x} = 0 \quad \therefore x = 1$

$\frac{d^2D}{dx^2} = \sqrt{2}\,(x^{-2}) > 0$ for all x $\therefore$ min D ✓

$AB = \sqrt{2}\,(1 - \log_e 1) = \sqrt{2}$ min length.

b) Test for $n = 2$.

$LHS = 2 \times 2 \qquad RHS = 1 \times 2^2$ ✓

$= 4 \qquad\qquad = 4 \quad \therefore$ true for $n = 2$.

Assume true for $n = k$

$2\times 2 + 3\times 2^2 \cdots\cdots k \times 2^{k-1} = (k-1)2^k$. ½

Prove for $n = k+1$

$2\times 2 + 3\times 2^2 \cdots\cdots (k+1)2^k = k \times 2^{k+1}$

$LHS = (k-1)2^k + (k+1)2^k$

$= 2k \times 2^k$ ✓✓

$= k\,2^{k+1} = RHS$

$\therefore$ True for $k+1$ if true for k. ½

Q4

a) $\int_0^{\frac{\pi}{4}} \cos^3 x \sin x \, dx = \left[\frac{-1}{4}\cos^4 x\right]_0^{\frac{\pi}{4}}$

$= \frac{-1}{4}\left(\frac{1}{\sqrt{2}}\right)^4 - \frac{-1}{4}(1)^4$

$= \frac{-1}{16} + \frac{1}{4} = \frac{3}{16}$ a

b) $\int \frac{1}{(x^2+4)^{3/2}} dx$ $\quad x = 2\tan\theta$

$\frac{dx}{d\theta} = 2 \sec^2\theta$

$\int \frac{1}{(4\tan^2\theta+4)^{3/2}} 2\sec^2\theta \, d\theta$

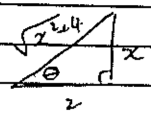
$= \int \frac{1}{4^{3/2}(\sec^2\theta)^{3/2}} \cdot 2\sec^2\theta \, d\theta$

$= \int \frac{1}{\cancel{8}\sec^3\theta} \cdot \cancel{2}\sec^2\theta \, d\theta$

$= \frac{1}{4}\int \frac{1}{\sec\theta} d\theta$

$\tan\theta = \frac{x}{2}$

$= \frac{1}{4}\int \cos\theta \, d\theta$



$= \frac{1}{4}\sin\theta + c$ $\quad 2\frac{1}{2}$

$= \frac{1}{4} \cdot \frac{x}{\sqrt{x^2+4}} + c.$

$= \frac{x}{4\sqrt{x^2+4}} + c.$

4c) $P(x) = ax^3 + bx^2 + 10x - 8.$

$P(-2) = 0 \quad \therefore \quad -8a + 4b - 20 - 8 = 0$

$P(1) = 12 \qquad a + b + 10 - 8 = {}^-12$

$\therefore -8a + 4b = 28 \qquad a + b = -14$

$-2a + b = 7$

$3a = -21$

$\therefore a = -7, \ b = -7.$

d) $\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{-4}{2} = -2$

$\alpha\beta\gamma = \frac{7}{2}.$

i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= 0^2 - 2\times -2 = 4$

ii) $(\alpha+1)(\beta+1)(\gamma+1) = \alpha\beta\gamma + (\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha+\beta+\gamma) + 1$

$= \frac{7}{2} + -2 + 0 + 1$

$= \frac{5}{2}$

Q5 $\frac{dP}{dt} = k(100-P)$

(i) if $P = 100 - Ae^{-kt}$

LHS $= \frac{dP}{dt} = kAe^{-kt}$, RHS $= k(100-P) = k(100-100+Ae^{-kt})$
$= k\cdot Ae^{-kt}$

$\therefore P = 100 - Ae^{-kt}$ satisfies $\frac{dP}{dt} = k(100-P)$ 1

(ii) at $t=0$, $P=6$

$6 = 100 - Ae^{0}$ $\therefore A = 94$ 1

at $t=2$, $P=20$

$20 = 100 - 94e^{-2k}$

$\therefore e^{-2k} = \frac{-80}{-94}$ 1

$k = -\frac{1}{2}\ln\left(\frac{80}{94}\right) \doteqdot +0.0806$ 1

(iii) as $t \to \infty$, $e^{-kt} \to 0$ $\therefore P \to 100$.
expected no is 100. 1

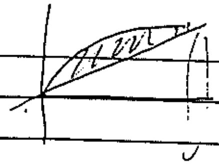
b) $y = \sin x$ $y = \frac{2x}{\pi}$.

at $(0,0)$

$0 = \sin 0$ True. $0 = \frac{2\times 0}{\pi}$ True $\therefore (0,0)$ is int pt. $\frac{1}{2}$

at $(\frac{\pi}{2}, 1)$

$1 = \sin\frac{\pi}{2}$ true $1 = \frac{2\times\frac{\pi}{2}}{\pi}$ true $\therefore (\frac{\pi}{2}, 1)$ is int pt $\frac{1}{2}$

b ii) 

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$V = \pi\int y^2\,dx$

OR $V = \pi\int \sin^2 x - \left(\frac{2x}{\pi}\right)^2 dx$

$V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ − cone

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)\,dx$ − cone

$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$ − cone

$= \frac{\pi}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0-0)\right]$ − cone $V = \frac{1}{3}\pi r^2 h$

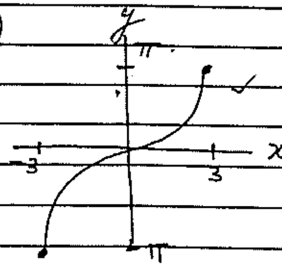
$= \frac{\pi^2}{4} - \frac{1}{3}\times\pi\times 1^2\times\frac{\pi}{2}$.

$= \frac{\pi^2}{4} - \frac{\pi^2}{6} = \frac{\pi^2}{12}$

$= \frac{\pi^2}{4} - \frac{4}{\pi}\cdot\frac{\pi^3}{24}$

$= \frac{\pi^2}{4} - \frac{\pi^2}{6}$

$= \frac{\pi^2}{12}$

c) 

$-1 \le \frac{x}{3} \le 1$

D: $-3 \le x \le 3$

R: $-\pi \le y \le \pi$

Question 6.

a) i) $\ddot{x} = 0$ $\ddot{y} = -10$

$\dot{x} = c_1$ $\dot{y} = -10t + c_2$

at $t=0$, $\dot{x} = 100$ and $\dot{y} = 0$

$\therefore$ $c_1 = 100$ and $c_2 = 0$.

$\therefore$ $\dot{x} = 100$ $\dot{y} = -10t$

$x = 100t + c_3$ $y = -5t^2 + c_4$

at $t=0$, $x=0$ and $y=105$

$\therefore$ $x = 100t$ $y = -5t^2 + 105$

ii) Gradient of BL : $\tan(-\frac{\pi}{4}) = -1$

BL passes through $(0,0)$

$\therefore$ eq is $y - 0 = -1(x-0)$

$y = -x$

iii) $y = -x$ & $x = 100t$, $y = -5t^2 + 105$

$\therefore$ $-5t^2 + 105 = -100t$

$5t^2 - 100t + 105 = 0$

$t^2 - 20t - 21 = 0$

$(t-21)(t+1) = 0$

$\therefore$ $t = 21$ or $t = -1$ ignore -ve

$\therefore$ takes 21 sec to hit ground.

$BL = \sqrt{(2100)^2 + (-2100)^2}$

~~$= \sqrt{2} \times 2100$~~

$\doteqdot 2970$ m

b) i) $f(x)$ is odd so $f(-x) = -f(x)$

defined at 0 so $f(0) = -f(0)$

$2f(0) = 0$

$\therefore$ $f(0) = 0$.

ii) $P(x) = A(x-2)^2(x+2)^2 x$

$P(1) = -12$ so $A(-1)^2(3)^2 1 = -12$

$9A = -12$

$A = -4/3$

$\therefore$ leading term is $-\frac{4}{3}x^5$

c) i) ${}^nC_{14} = {}^nC_{12}$

so $r = 14$

$n - r = 12$ $\therefore$ $n = 26$

ii) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n!}{r!(n-r)!} \div \dfrac{n!}{(n-r+1)!(r-1)!}$

$= \dfrac{(n-r+1)\cancel{(n-r)!}\cancel{(r-1)!}}{r\cancel{(r-1)!}\cancel{(n-r)!}}$

$= \dfrac{n-r+1}{r}$

Q7

a) LHS $= (1+x)^n(1+x)^n$

$= \left({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 \cdots {}^nC_r x^r \cdots {}^nC_n x^n\right)\left({}^nC_0 + {}^nC_1 x \cdots\right)$

Term in x^n:

${}^nC_0 \cdot {}^nC_n x^n + {}^nC_1\, {}^nC_{n-1} x^n + {}^nC_2\, {}^nC_{n-2} x^n \cdots$

$\cdots\ {}^nC_n\, {}^nC_0 x^n.$

But ${}^nC_r = {}^nC_{n-r}$ so ${}^nC_0 = {}^nC_n$ etc. (1)

(1) Coeff of x^n is

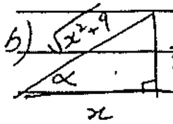
$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 \cdots\cdots ({}^nC_n)^2$

On RHS

term in x^n is ${}^{2n}C_n x^n$ (1)

equating co-eff on both sides

$\therefore ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 \cdots ({}^nC_n)^2 = {}^{2n}C_n$

b) 

by Pyth Rule, hypot is $\sqrt{x^2+9}$

$\sin\alpha = \dfrac{3}{\sqrt{x^2+9}}$ 1

ii) Shaded area =

semicircle − (sector − triangle)

$= \dfrac{\pi r^2}{2} - \frac{1}{2}r^2\theta + \frac{1}{2}\times 6\times x$

$= \dfrac{9\pi}{2} + 3x - \frac{1}{2}\times\left(\sqrt{x^2+9}\right)^2 \cdot 2\alpha$

$\tan\alpha = \dfrac{3}{x}$

$= \dfrac{9\pi}{2} + 3x - (x^2+9)\tan^{-1}\left(\dfrac{3}{x}\right)$

$\alpha = \tan^{-1}\dfrac{x}{3}$.

(iii) Find $\dfrac{dS}{dt}$ by finding $\dfrac{dS}{dx}\times\dfrac{dx}{dt}$.

$\dfrac{dS}{dx} = 3 - \left[(x^2+9)\left(\dfrac{1}{1+\left(\frac{3}{x}\right)^2}\right)\cdot -3x^{-2} + \tan^{-1}\dfrac{3}{x}\cdot 2x\right]$

$= 3 - \left[(x^2+9)\dfrac{x^2}{x^2+9}\cdot\dfrac{-3}{x^2} + 2x\tan^{-1}\dfrac{3}{x}\right)$

$= 3 + 3 - 2x\tan^{-1}\dfrac{3}{x}$: 2

$= 6 - 2x\tan^{-1}\dfrac{3}{x}$.

$\dfrac{dS}{dt} = \dfrac{dS}{dx}\cdot\dfrac{dx}{dt}$

$= \left(6 - 2x\tan^{-1}\dfrac{3}{x}\right)(0\cdot1)$ 1

at $x = 3$ $= \dfrac{6 - 6\tan^{-1}1}{10} = \dfrac{6 - \frac{3\pi}{2}}{10}$

$= \dfrac{12-3\pi}{20}$.

(IV) as $x \to \infty$, arc $\to AB$, $\therefore$ S app semicircle.