St Catherine's School

2016 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A reference sheet is provided
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations
- Task Weighting – 40%

Total Marks – 70

Section I Pages 3 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 6 – 13

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hours and 45 minutes for this section

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Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 R is a point $(-2, -1)$ and S is a point $(1,\ 5)$. Find the coordinates of the point X which divides RS externally in the ratio $5 : 2$.

(A) $(1\,, 9)$

(B) $(3\,, 9)$

(C) $\left(\frac{9}{7}\,, \frac{27}{7}\right)$

(D) $\left(\frac{3}{7}\,, \frac{23}{7}\right)$

2 Find $\int \sin^2 4x\ dx$.

(A) $\frac{1}{2}\left(x - \frac{\sin 8x}{8}\right) + c$

(B) $\frac{1}{2}\left(x - \frac{\cos 8x}{8}\right) + c$

(C) $\frac{1}{2}(x - \sin 8x) + c$

(D) $x - \frac{\sin 8x}{8} + c$

3 Find the remainder when $P(x) = 2x^3 + x^2 - 13x + 6$ is divided by $(x - 1)$.

(A) 18

(B) 6

(C) 4

(D) -4

4 Evaluate $\cos(\tan^{-1}\frac{1}{2})$.

(A) $\frac{1}{\sqrt{5}}$

(B) $\frac{2}{\sqrt{5}}$

(C) $\frac{1}{2}$

(D) 2

5 What is the domain of the function $f(x) = 2\sin^{-1}(3x - 2)$?

(A) $-5 \leq x \leq 1$

(B) $-2 \leq x \leq 2$

(C) $\frac{1}{3} \leq x \leq 1$

(D) $\frac{2}{3} \leq x \leq \frac{3}{2}$

6 Seven people attend a dinner party. How many ways can they be arranged around a round table if two particular people must sit apart from each other?

(A) 480

(B) 240

(C) 720

(D) 120

7 How many solutions does the equation $\sin 2x = 4\cos x$ have for $0 \leq x \leq 2\pi$.

(A) 2

(B) 1

(C) 3

(D) 4

8 The polynomial equation $P(x) = 2x^3 + x^2 - 13x + 6$ has 3 roots α, β and γ.
Find $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma}$.

(A) $\frac{26}{3}$

(B) $\frac{13}{3}$

(C) $\frac{13}{6}$

(D) $\frac{13}{12}$

9 If the acute angle between the lines $y = 2x - 3$ and $mx - y - 1 = 0$ is $\frac{\pi}{4}$, find the value of m.

(A) $-\frac{1}{2}$

(B) -2

(C) $-\frac{1}{3}$

(D) -3

10 Evaluate $\int \frac{2}{16+9x^2}\, dx$.

(A) $\frac{3}{4}\tan^{-1}\frac{3x}{4} + c$

(B) $\frac{1}{12}\tan^{-1}\frac{4x}{3} + c$

(C) $\frac{1}{6}\tan^{-1}\frac{3x}{4} + c$

(D) $\frac{1}{6}\tan^{-1}\frac{4x}{3} + c$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hours and 45 minutes

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $u = 2x - 1$, to find **3**

$$\int \frac{x}{\sqrt{2x-1}}\, dx$$

(b) Consider the letters of the word CALCULATOR.

(i) How many different arrangements can be made if there are no restrictions? **1**

(ii) What is the probability that the letter C's are at either ends? **2**

(c) Isabella guesses at random the answers to each of 10 multiple choice questions. In each question there are 4 options, only one of which is correct.

(i) Find the probability that Isabella answers exactly 6 of the 10 questions correctly. Give your answer correct to 3 decimal places. **2**

(ii) Find the probability that Isabella answers at least two questions correctly. Give your answer correct to 3 decimal places. **2**

Question 11 continues on the following page

Question 11 (continued)

(d) (i) By sketching on the same set of axes the graphs of $y = \cos^{-1} x$ and $y = \frac{\pi}{4} + x$, explain why the equation $\cos^{-1} x - x - \frac{\pi}{4} = 0$ has only one real solution. **2**

(ii) Taking $x = 0.5$ as the first approximation to the solution of $\cos^{-1} x - x - \frac{\pi}{4} = 0$, use one application of Newton's method to find a better approximation. Give your answer correct to 2 decimal places. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A function is defined by $f(x) = 2 - \frac{2}{x+1}$.

(i) Show that the points of intersection of $f(x)$ and its inverse function $f^{-1}(x)$ are $(0, 0)$ and $(1, 1)$. **2**

(ii) Sketch the graph of $y = f(x)$ for domain $x \geq -1$. **2**

Clearly show any equations of asymptotes and intercepts on the coordinate axes.

(Use at least one third of the page)

(iii) On the same set of axes, sketch the graph of the inverse function $y = f^{-1}(x)$. **1**

Clearly show any equations of asymptotes, intercepts on the coordinate axes and points of intersection with $y = f(x)$.

(iv) Find an expression for $f^{-1}(x)$ in terms of x and clearly state the restriction on its domain. **2**

(b) The polynomial $P(x)$ is given by $P(x) = x^3 + (k-1)x^2 + (1-k)x - 1$ for some real number k.

(i) Show that $x = 1$ is a root of the equation $P(x) = 0$. **1**

(ii) Given that $P(x) = (x-1)(x^2 + kx + 1)$ and $P(x) = 0$ has only one real root, find the possible value(s) of k. **2**

Question 12 continues on the following page

Question 12 (continued)

(c) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1} given by $v^2 = 32 + 8x - 4x^2$ and acceleration $\ddot{x}$ ms^{-2}.

(i) Show that the particle is moving in Simple Harmonic Motion. **1**

(ii) Find the centre and amplitude of the motion. **3**

(iii) Find the maximum speed of the particle. **1**

End of Question 12

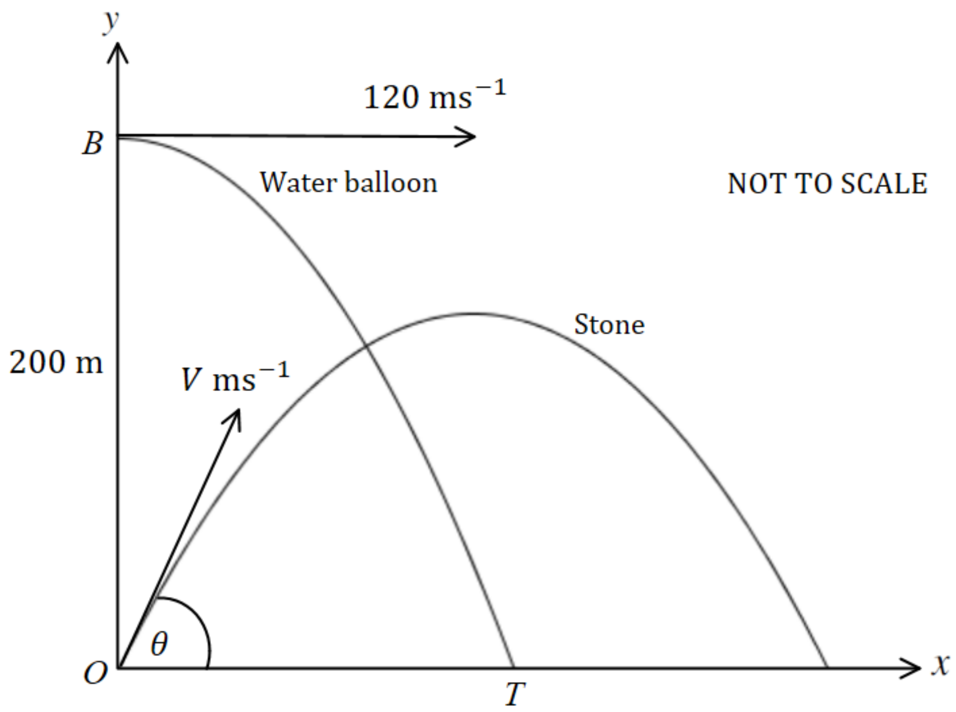
Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A water balloon is fired horizontally by a cannon from the point B with a velocity of 120 ms^{-1} to reach a target at T.

At the same time, a stone is launched from the point O with a velocity of V ms^{-1} and an angle of projection of θ in order to burst the water balloon in the air.

The point O is 200 metres directly below the point B and $\theta = \tan^{-1}\left(\frac{3}{4}\right)$.

Take the acceleration due to gravity as 10 ms^{-2}.



For the water balloon,

(i) Show that the equations of motion of the water balloon are given by **2**

$$x = 120t \quad \text{and} \quad y = -5t^2 + 200.$$

Question 13 continues on the following page

For the stone, assume that the equations of motion are given by

$x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$. (Do NOT prove this.)

(ii) Show that in order for the stone to successfully burst the water balloon in the air, it must be launched at a velocity of $150\ \text{ms}^{-1}$. **2**

(iii) How high above the ground does the collision occur? Give your answer correct to the nearest metre. **3**

(b) Find the exact value of $\sin\left[\cos^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{5}{12}\right)\right]$. **3**
Show all working.

(c) At time t years the number N of individuals in a population is given by $N = 5000 - 4250e^{-kt}$ for some $k > 0$.

(i) Find the initial population. **1**

(ii) Sketch the graph of N as a function of t showing clearly the initial and limiting populations. **2**

(iii) Find the value of k if $\frac{dN}{dt} = 250$ when N is three times the initial population. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) Prove by mathematical induction that **3**

$$\sum_{r=1}^{n} (r^2 + 1)r! = n(n+1)!$$

(b) In the binomial expansion of $\left(1 - \frac{a}{x}\right)^n$, the coefficient of x^{-4} and the coefficient of x^{-3} are in the ratio of $3 : 2$. **3**

Prove that $na - 3a + 6 = 0$.

(c) Consider the geometric series $1 + (1+x) + (1+x)^2 + \cdots + (1+x)^n$, where $x > 0$.

(i) Show by summation that **1**

$$1 + (1+x) + (1+x)^2 + \cdots + (1+x)^n = \frac{(1+x)^{n+1}}{x} - \frac{1}{x}$$

(ii) Hence, show that **2**

$$\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \cdots + \binom{r}{r} = \binom{n+1}{r+1}$$

Question 14 continues on the following page

Question 14 (continued)

(d) (i) From a point $A\ (-p, q)$, where $p > 0$ and $q > 0$, perpendiculars AP and AQ are drawn to meet the x and y axes at $P(-p, 0)$ and $Q(0, q)$ respectively. **2**

Show that the equation of PQ is given by $x = \frac{p}{q}y - p$.

(ii) Show that the condition for the line PQ to be a tangent to the parabola $y^2 = 4ax$ is $ap - q^2 = 0$. **3**

(iii) If the points $P(-p, 0)$ and $Q(0, q)$ move on the x and y axes respectively, such that PQ is a tangent to the parabola $y^2 = 4ax$, then the point $A(-p, q)$ traces out a curve. **1**

Find the locus of A.

End of Question 14

End of paper

Student Number: SOLUTION

2016 **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

Mathematics Extension 1

Multiple Choice Answer Sheet

Completely fill the response circle representing the most correct answer

	A	B	C	D
1.	○	●	○	○
2.	●	○	○	○
3.	○	○	○	●
4.	○	●	○	○
5.	○	○	●	○
6.	●	○	○	○
7.	●	○	○	○
8.	○	●	○	○
9.	○	○	○	●
10.	○	○	●	○

2016 Mathematics Extension 1 HSC Trial Examination

SOLUTIONS

Section 1

Question 1 — B

$R(-2,-1)$, $S(1,5)$, $m=5$, $n=-2$

$$x=\frac{mx_2+nx_1}{m+n} \qquad y=\frac{my_2+ny_1}{m+n}$$

$$=\frac{5(1)-2(-2)}{5+(-2)} \qquad =\frac{5(5)-2(-1)}{5+(-2)}$$

$$=\frac{9}{3} \qquad =\frac{27}{3}$$

$$=3 \qquad =9$$

$\therefore X(3,9)$

Question 2 — A

$$\cos 2\theta = 1-2\sin^2\theta$$
$$2\sin^2\theta = 1-\cos 2\theta$$
$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$
$$\sin^2 4x = \tfrac{1}{2}(1-\cos 8x)$$

$$\int \sin^2 4x \, dx = \tfrac{1}{2}\int 1-\cos 8x \, dx$$
$$= \tfrac{1}{2}\left(x-\tfrac{1}{8}\sin 8x\right)+c$$

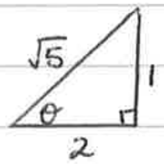
Question 3 — D

$$P(1) = 2(1)^3+(1)^2-13(1)+6$$
$$=-4$$

Question 4 – B

let $\theta = \tan^{-1}\frac{1}{2}$

$\tan\theta = \frac{1}{2}$



then $\cos(\tan^{-1}\frac{1}{2}) = \cos\theta$

$= \frac{2}{\sqrt{5}}$

Question 5 – C

$-1 \leqslant 3x-2 \leqslant 1$

$1 \leqslant 3x \leqslant 3$

$\frac{1}{3} \leqslant x \leqslant 1$

Question 6 – A

No. of ways to sit apart = Total ways − No. of ways to sit together

$= 6! - 5!2!$

$= 480$

Question 7 – A

$\sin 2x = 4\cos x$

$2\sin x\cos x = 4\cos x$

$0 = 2\sin x\cos x - 4\cos x$

$0 = 2\cos x(\sin x - 2)$

$0 = 2\cos x$ or $0 = \sin x - 2$

$\cos x = 0$ $\qquad$ $\sin x = 2$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad$ no solution

$\therefore x = \frac{\pi}{2}$ & $\frac{3\pi}{2}$ only i.e. two solutions

Question 8 — B

$$\alpha\beta + \beta\gamma + \alpha\gamma = -\frac{13}{2}$$

$$\alpha\beta\gamma = -\frac{6}{2}$$

$$= -3$$

then $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} = \frac{2(\beta\gamma + \alpha\gamma + \alpha\beta)}{\alpha\beta\gamma}$

$$= \frac{2(-\frac{13}{2})}{-3}$$

$$= \frac{13}{3}$$

Question 9 — D

$y = 2x - 3$ and $mx - y - 1 = 0$

$m_1 = 2$

$y = mx - 1$

$m_2 = m$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\frac{\pi}{4} = \left|\frac{2-m}{1+2m}\right|$$

$$1 = \left|\frac{2-m}{1+2m}\right|$$

$1 = \frac{2-m}{1+2m}$ or $-1 = \frac{2-m}{1+2m}$

$1 + 2m = 2 - m$

$3m = 1$

$m = \frac{1}{3}$

$-1 - 2m = 2 - m$

$-m = 3$

$m = -3$ ✓

Question 10 — C

$$\int \frac{2}{16 + 9x^2}\,dx = 2\int \frac{dx}{9\left(\frac{16}{9} + x^2\right)}$$

$$= \frac{2}{9}\int \frac{dx}{\left(\frac{4}{3}\right)^2 + x^2}$$

$$= \frac{2}{9} \times \frac{1}{\frac{4}{3}} \tan^{-1}\left(\frac{x}{\frac{4}{3}}\right) + c$$

$$= \frac{2}{9} \times \frac{3}{4} \tan^{-1}\left(\frac{3x}{4}\right) + c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{4}\right) + c$$

End of Section 1

Section 2

Question 11

(a) $u = 2x - 1$

$x = \frac{1}{2}(u+1)$

$\frac{du}{dx} = 2$

$dx = \frac{1}{2}\,du$

then $\int \frac{x}{\sqrt{2x-1}}\,dx = \frac{1}{2} \times \frac{1}{2} \int \frac{u+1}{\sqrt{u}}\,du$

$= \frac{1}{4} \int \frac{u}{\sqrt{u}} + \frac{1}{\sqrt{u}}\,du$

$= \frac{1}{4} \int u^{\frac{1}{2}} + u^{-\frac{1}{2}}\,du$

$= \frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right) + c$

$= \frac{1}{6}(2x-1)^{\frac{3}{2}} + \frac{1}{2}(2x-1)^{\frac{1}{2}} + c$

- 1 mark for correct substitution of dx and x with du and u.
- 2 marks for correct integration in terms of u.
- 3 marks for converting final answer in terms of x & adding constant c.

(b) (i) $\frac{10!}{2!\,2!\,2!}$ since there are 2 C's, L's and A's

$= 453\,600$ arrangements

- 1 mark for correct answer

(ii) C _ _ _ _ _ _ _ _ C

Place C's on either ends = 1

Arrange 8 remaining letters = $\frac{8!}{2!2!}$

then no. of ways of arranging C's on either ends $= 1 \times \frac{8!}{2!2!}$

$= 10080$

$\therefore$ P(C's on either ends) $= \frac{10080}{453600}$ using part (i)

$= \frac{1}{45}$

- 1 mark for correct no. of ways of arranging C's and 8 remaining letters
- 2 marks for correctly accounting for repetitions of A and L.

(c) Let X = number of questions answered correctly

(i) $P(X=6) = {}^{10}C_6 \left(\frac{1}{4}\right)^6 \left(\frac{3}{4}\right)^4$

$= 0.016$ (3 decimal places)

- 1 mark for showing substantial effort to write correct expression
- 2 marks for correct probability

(ii) $P(X \geq 2) = 1 - [P(X=1) + P(X=0)]$

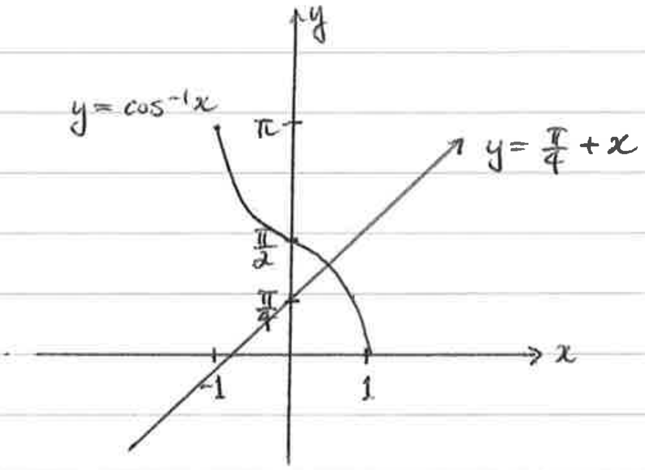
$= 1 - \left[{}^{10}C_1 \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^9 + \left(\frac{3}{4}\right)^{10}\right]$

$= 0.75597\ldots$

$= 0.756$ (3 decimal places)

- 1 mark for correct evaluation of probability of $X=0$ or $X=1$
- 2 marks for correct probability

(d) (i)



From the sketch, there is only one intersection point of $y = \cos^{-1}x$ and $y = \frac{\pi}{4} + x$ i.e. $\cos^{-1}x = \frac{\pi}{4} + x$

$\therefore$ $\cos^{-1}x - \frac{\pi}{4} - x = 0$ has one real solution.

- 1 mark for correctly sketching both equations.
- 2 marks for correct conclusion.

(ii) $f(x) = \cos^{-1}x - x - \frac{\pi}{4}$

$f'(x) = -\frac{1}{\sqrt{1-x^2}} - 1$

$$f(0.5) = \cos^{-1}0.5 - 0.5 - \frac{\pi}{4}$$
$$= \frac{\pi}{3} - 0.5 - \frac{\pi}{4}$$
$$= \frac{\pi}{12} - 0.5$$

$$f'(0.5) = \frac{-1}{\sqrt{1-0.5^2}} - 1$$
$$= \frac{-1}{\sqrt{\frac{3}{4}}} - 1$$
$$= \frac{-1}{\frac{\sqrt{3}}{2}} - 1$$
$$= -\frac{2}{\sqrt{3}} - 1$$

then $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$= 0.5 - \dfrac{\frac{\pi}{12} - 0.5}{-\frac{2}{\sqrt{3}} - 1}$

$= 0.3894\ldots$

$\therefore \; x = 0.39$ (2 decimal places)

- 1 mark for correct value of $f(0.5)$
- 2 marks for correct differentiation of $f(x)$ & value of $f'(0.5)$
- 3 marks for correct answer

End of Question 11

Question 12

(a) (i) $f(x)$ & $f^{-1}(x)$ intersect with $y=x$

then find the intersection point of $y=x$ and $y=f(x)$

i.e. $2 - \dfrac{2}{x+1} = x$

$$2(x+1) - 2 = x(x+1)$$
$$2x + 2 - 2 = x^2 + x$$
$$0 = x^2 - x$$
$$0 = x(x-1)$$

$\therefore$ $x=0$ or $x=1$

and $y=0$ or $y=1$ since $y=x$

$\therefore$ intersection points are $(0,0)$ and $(1,1)$

- 1 mark for showing point of intersection occurs for $x=f(x)$.
- 2 marks for clearly solving for two intersection points.

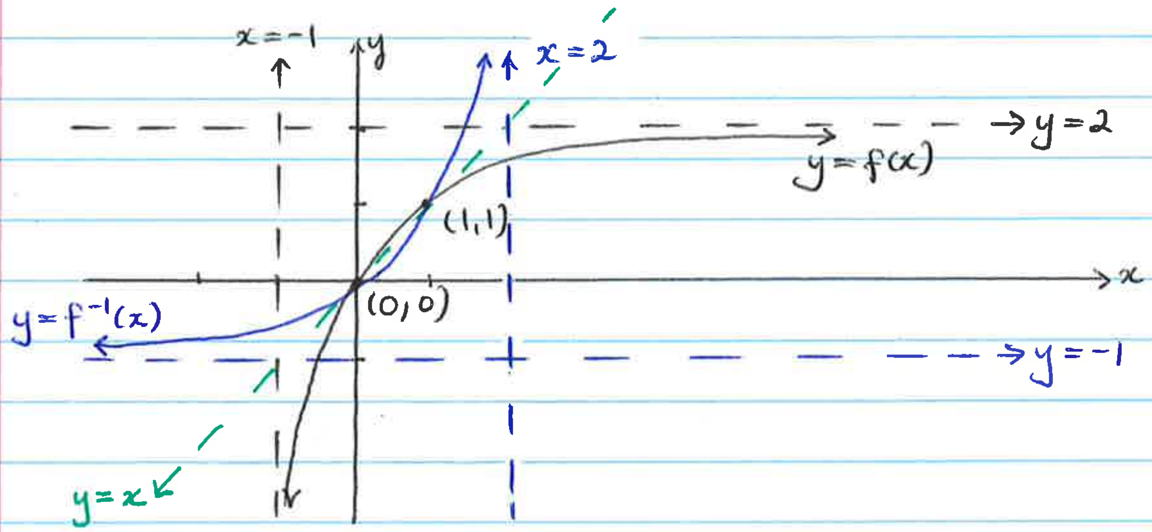
(ii) vertical asymptote : $x=1$

horizontal asymptote : $y=2$

intercept : $(0,0)$

limits : as $x \to \infty$, $y \to 2^-$

as $x \to -1$, $y \to -\infty$



- 1 mark for correct vertical and horizontal asymptotes
- 2 mark for correct shape of the curve and passing through the intercept $(0,0)$.

(iii) see solution in (ii)

vertical asymptote : $x=2$
horizontal asymptote : $y=-1$

- 1 mark for correct inverse function graph including correct asymptotes, intersection points and shape of the curve.

(iv) $f^{-1}(x)$: $x = 2 - \frac{2}{y+1}$

$$\frac{2}{y+1} = 2 - x$$

$$y+1 = \frac{2}{2-x}$$

$$\therefore \quad y = \frac{2}{2-x} - 1 \quad \text{where } x < 2$$

$$\text{or} \quad y = \frac{x}{2-x}$$

- 1 mark for swapping x and y variables
- 2 marks for correctly finding the expression for $f^{-1}(x)$.

(b) (i) $P(1) = 1^3 + (k-1)\cdot 1^2 + (1-k)\cdot 1 - 1$

$= 1 + k - 1 + 1 - k - 1$

$= 0$

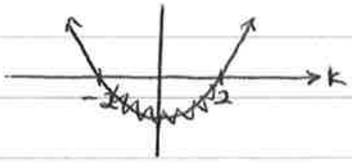
$\therefore$ Since $P(1) = 0$, $x=1$ is a root

- 1 mark for correct substitution of $x=1$ into $P(x)$ and showing $P(1) = 0$.

(ii) If $P(x)$ has only one solution and since $x=1$ is a solution, as shown in (i), then x^2+kx+1 cannot have any real solutions, i.e. $\Delta<0$ for x^2+kx+1.

now $\Delta = k^2-4$
$= (k-2)(k+2)$

then $(k-2)(k+2)<0$



$\therefore \quad -2<k<2$ for only one solution to exist for $P(x)=0$.

- 1 mark for showing that one solution exist when $\Delta<0$ for x^2+kx+1.
- 2 mark for correct values of k.

(c) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
$= \frac{d}{dx}\left[\frac{1}{2}(32+8x-4x^2)\right]$
$= \frac{d}{dx}(16+4x-2x^2)$
$= 4-4x$
$= -4(x-1)$

$\therefore \ddot{x} = -2^2(x-1)$ is in the form $\ddot{x}=-n^2(x-c)$
hence it's performing a SHM.

- 1 mark for correctly & clearly showing SHM.

(ii) from (i), $n=2$ and $c=1$
$\therefore$ centre of motion is $x=1$

at endpoints, $v=0$
then $0 = 32+8x-4x^2$

$0 = x^2 - 2x - 8$

$0 = (x-4)(x+2)$

then $x = 4$ and $x = -2$ are the endpoints of the motion

then $a = \dfrac{4 + |-2|}{2}$

$= 3$

$\therefore$ amplitude is 3 units

- 1 mark for correct centre of motion
- 2 marks for correctly finding the endpoints of the motion
- 3 marks for correct amplitude.

(iii) maximum speed occurs at centre of motion

when $x = 1$, $v^2 = 32 - 8(1) - 4(1)^2$

$= 36$

$\therefore$ $|v| = 6 \text{ ms}^{-1}$ is the max. speed.

- 1 mark for correct maximum speed

End of Question 12

Question 13

(a) (i)

$\ddot{x} = 0$ and $\ddot{y} = -10$

$\dot{x} = \int 0 \, dt$

$= c_1$

at $t=0$, $\dot{x} = V\cos\alpha$

$= 120\cos 0$

$= 120$

$= c_1$

$\dot{x} = 120$

$x = \int 120 \, dt$

$= 120t + c_2$

at $t=0$, $x = 0$

$= c_2$

$\therefore \ x = 120t$

$\dot{y} = \int -10 \, dt$

$= -10t + c_3$

at $t=0$, $\dot{y} = V\sin\alpha$

$= 120\sin 0$

$= 0$

$= c_3$

$\dot{y} = -10t$

$y = \int -10t \, dt$

$= -5t^2 + c_4$

at $t=0$, $y = 200$

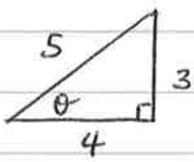
$= c_4$

$\therefore \ y = -5t^2 + 200$

- 1 mark for correctly & clearly deriving vertical equation motion
- 2 marks for correctly & clearly deriving horizontal equation motion

(ii) since $\theta = \tan^{-1}\left(\frac{3}{4}\right)$

$\tan\theta = \frac{3}{4}$



then $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$

for stone to burst water balloon $x_{WB} = x_S$

i.e. $120t = Vt\cos\theta$

$120t = Vt \times \frac{4}{5}$

$120 = \frac{4}{5}V$

$\therefore \quad V = 150 \ ms^{-1}$

- 1 mark for correctly showing that stone bursts water balloon when $x_{WB} = x_S$
- 2 marks for correctly and clearly showing that $V = 150\ ms^{-1}$.

(iii) collision occurs when $y_{WB} = y_S$

i.e. $-5t^2 + 200 = -5t^2 + 120t\sin\theta$

$$200 = 120t\sin\theta$$

$$200 = 120t \times \frac{3}{5} \quad \text{using result from (ii)}$$

$$200 = 90t$$

$$t = \frac{20}{9}$$

when $t = \frac{20}{9}$, $y_{WB} = -5\left(\frac{20}{9}\right)^2 + 200$

$$= 175.308\ldots$$

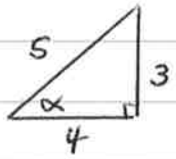
$$= 175\ m \quad \text{(nearest metre)}$$

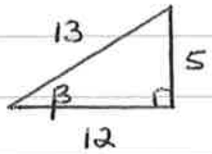
$\therefore$ collision occurs 175 m above the ground.

- 1 mark for showing substantial effort to solve for correct time t.
- 2 marks for correctly solving for time t of collision
- 3 marks for correct height of collision.

(b) let $\alpha = \cos^{-1}\frac{4}{5}$ and $\beta = \tan^{-1}\frac{5}{12}$

$\cos\alpha = \frac{4}{5}$ $\qquad$ $\tan\beta = \frac{5}{12}$





$0 < \alpha < \frac{\pi}{2}$ $\qquad$ $0 < \beta < \frac{\pi}{2}$

then $\sin\left(\cos^{-1}\frac{4}{5} - \tan^{-1}\frac{5}{12}\right) = \sin(\alpha - \beta)$

$$= \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$= \frac{3}{15} \times \frac{12}{13} - \frac{5}{13} \times \frac{4}{5}$$

$$= \frac{16}{65}$$

- 1 mark for correct two diagrams representing the given trigonometric ratios.
- 2 marks for correct substitution of $\sin\alpha\cos\beta$
- 3 marks for correct substitution of $\sin\beta\cos\alpha$ and correct answer

(c) (i) when $t=0$, $N = 5000 - 4250e^{-k(0)}$

$= 5000 - 4250e^{0}$

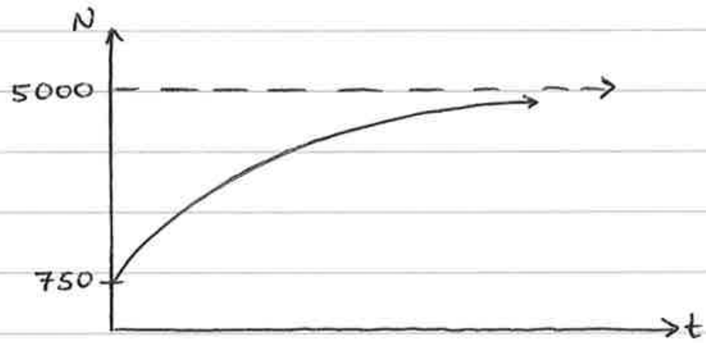
$= 750$

- 1 mark for correct answer

(ii) as $t \to \infty$, $e^{-kt} \to 0$

$4250e^{-kt} \to 0$

$5000 - 4250e^{-kt} \to 5000$



- 1 mark for correct limiting population
- 2 marks for clearly showing the initial population and correct shape of the curve.

(iii) $N = 5000 - 4250e^{-kt}$

$\frac{dN}{dt} = k \times 4250e^{-kt}$

$= k(5000 - N)$

when $\frac{dN}{dt} = 250$, $N = 3 \times 750$

$= 2250$

then $250 = k(5000 - 2250)$

$$250 = 2750k$$

$$k = \frac{250}{2750}$$

$$\therefore \quad k = \frac{1}{11}$$

- 1 mark for correctly showing that $\frac{dN}{dt} = k(5000 - N)$
- 2 marks for correctly solving for the value of k.

End of Question 13

Question 14

(a) $$\sum_{r=1}^{n} (r^2+1)r! = n(n+1)!$$

i.e. $(1^2+1)1! + (2^2+1)2! + (3^2+1)3! + \ldots + (n^2+1)n! = n(n+1)!$

Step 1. Show that the statement is true for $n=1$

$$\begin{aligned} LHS &= (1^2+1)1! \\ &= 2 \times 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} RHS &= 1(1+1)! \\ &= 1 \times 2! \\ &= 2 \end{aligned}$$

since LHS = RHS

the statement is true for $n=1$

Step 2. Assume that the statement is true for $n=k$

i.e. $(1^2+1)1! + (2^2+1)2! + \ldots + (k^2+1)k! = k(k+1)!$

Step 3. Show that the statement is true for $n=k+1$

i.e. $(1^2+1)1! + (2^2+1)2! + \ldots + (k^2+1)k! + [(k+1)^2+1](k+1)!$
$= (k+1)(k+2)!$

$$\begin{aligned} LHS &= k(k+1)! + [(k+1)^2+1](k+1)! \quad \text{using the assumption} \\ &= (k+1)![k + (k+1)^2+1] \\ &= (k+1)!(k + k^2 + 2k + 1 + 1) \\ &= (k+1)!(k^2+3k+2) \\ &= (k+1)!(k+2)(k+1) \\ &= (k+2)!(k+1) \\ &= RHS \end{aligned}$$

$\therefore$ the statement is true for $n=k+1$ if it's true for $n=k$

$\therefore$ the statement is true by mathematical induction

- 1 mark for correctly and clearly showing step 1
- 2 marks for substantial attempt to use the induction assumption with minor errors.
- 3 marks for correctly proving step 3 completely

(b) for $(1-\frac{a}{x})^n$:

$$T_n = \binom{n}{r}\left(-\frac{a}{x}\right)^r$$
$$= \binom{n}{r}(-a)^r x^{-r}$$

for x^{-4} : $r=4$, $T_4 = \binom{n}{4}(-a)^4 x^{-4}$
$$= \frac{n!\,a^4}{4!\,(n-4)!}x^{-4}$$

for x^{-3} : $r=3$, $T_3 = \binom{n}{3}(-a)^3 x^{-3}$
$$= \frac{-n!\,a^3}{3!\,(n-3)!}x^{-3}$$

then $\dfrac{\text{coefficient of } x^{-4}}{\text{coefficient of } x^{-3}} = \dfrac{3}{2}$

$$\frac{\dfrac{\cancel{n!}\,a^{\cancel{4}}}{4!\,(n-4)!}}{\dfrac{-\cancel{n!}\,\cancel{a^3}}{3!\,(n-3)!}} = \frac{3}{2}$$

$$\frac{a^4 n!\,3!\,(n-3)!}{-a^3 n!\,4!\,(n-4)!} = \frac{3}{2}$$

$$\frac{a^{\cancel{4}}\cancel{n!}\,\cancel{3!}\,(n-3)\cancel{(n-4)!}}{-\cancel{a^3}\cancel{n!}\,4\cdot\cancel{3!}\,\cancel{(n-4)!}} = \frac{3}{2}$$

$$\frac{-a(n-3)}{4} = \frac{3}{2}$$

$$-2a(n-3) = 12$$

$$-a(n-3) = 6$$

$$-na+3a = 6$$

$$na - 3a + 6 = 0 \quad \text{as required}$$

- 1 mark for correct expressions of T_4 and T_3
- 2 marks for correctly forming an equation relating the coefficients & its ratio
- 3 marks for clearly & completely deriving the given expression

(c) (i) $a = 1$

$r = 1 + x > 0$ since $x > 0$

$N = n + 1$

using $S_N = \dfrac{a(r^N - 1)}{r-1}$

$$= \frac{1\left[(1+x)^{n+1} - 1\right]}{1+x-1}$$

$$= \frac{(1+x)^{n+1} - 1}{x}$$

$$\therefore S_N = \frac{(1+x)^{n+1}}{x} - \frac{1}{x} \quad \text{as required}$$

- 1 mark for clear proof including explicitly stating there are $(n+1)$ terms

(ii) from part (i)

$$1 + (1+x) + (1+x)^2 + \ldots + (1+x)^r + \ldots + (1+x)^{n-2} + (1+x)^{n-1} + (1+x)^n = \frac{(1+x)^{n-1}}{x} - \frac{1}{x}$$

the coefficient of x^r on LHS is

$$\binom{r}{r} + \ldots + \binom{n-2}{r} + \binom{n-1}{r} + \binom{n}{r}$$

the coefficient of x^r on RHS :

for $$\frac{(1+x)^{n+1}}{x} = \frac{\binom{n+1}{0}x}{x} + \ldots + \frac{\binom{n+1}{r}x^r}{x} + \frac{\binom{n+1}{r+1}x^{r+1}}{x} + \ldots$$

$$= \ldots + \binom{n+1}{r}x^{r-1} + \binom{n+1}{r+1}x^r + \ldots$$

so coefficient of x^r is $\binom{n+1}{r+1}$

for $\frac{1}{x}$ there is no term in x^r

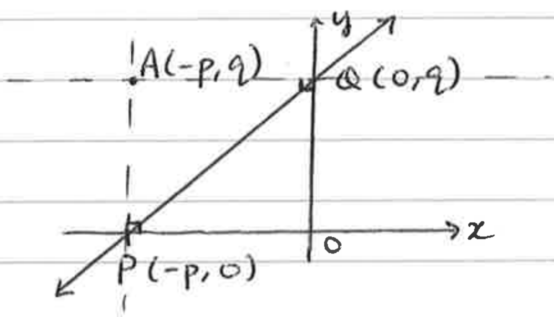
$\therefore$ coefficient of x^r on RHS is $\binom{n+1}{r+1}$

then equating coefficients of x^r on LHS and RHS gives :

$$\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \ldots + \binom{r}{r} = \binom{n+1}{r+1}$$ as required

- 1 mark for clearly deriving the coefficient of x^r on LHS
- 2 marks for clearly deriving the coefficient of x^r on RHS

(d) (i) $m = \frac{q}{p}$



y-intercept $= q$

then $y = \frac{q}{p}x + q$

$y - q = \frac{q}{p}x$

$yp - pq = qx$

$\therefore \quad x = \frac{p}{q}y - p$ as required

- 1 mark for correctly deriving the gradient and y-intercept
- 2 marks for correctly and clearly deriving the equation of the line

(ii) If PQ is a tangent to the parabola then there is only one intersection point of PQ and parabola.

then solving simultaneously for intersection point substitute $x = \frac{p}{q}y - p$ into $y^2 = 4ax$

$$y^2 = 4a\left(\frac{p}{q}y - p\right)$$
$$y^2 = \frac{4ap}{q}y - 4ap$$
$$0 = y^2 - \frac{4ap}{q}y + 4ap$$

for one intersection point there is only one solution to the above equation i.e. $\Delta = 0$

then $\Delta = \left(\frac{-4ap}{q}\right)^2 - 4(4ap)(1)$

$$= \frac{16a^2p^2}{q^2} - 16ap$$

then $0 = \frac{16a^2p^2}{q^2} - 16ap$

$$0 = 16a^2p^2 - 16apq^2$$
$$0 = 16ap(ap - q^2)$$

then $16ap = 0$ or $ap - q^2 = 0$

$\therefore ap - q^2 = 0$ as required.

- 1 mark for correct substitution of the two equations to get a quadratic equation
- 2 marks for correct expression of the discriminant
- 3 marks for correctly and clearly deriving the required expression

(iii) since for A : $x = -p$ and $y = q$

substitute $p = -x$ & $q = y$ into $ap - q^2 = 0$

then $-ax - y^2 = 0$

$\therefore$ $y^2 = -ax$ is the locus of A.

- 1 mark for correct locus of A

End of Question 14

End of Paper