Trial Higher School Certificate Examination

2001

Mathematics
Extension 1

Time Allowed: Two hours
(Plus 5 minutes reading time)

Directions to Candidates

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- A table of standard integrals is attached.
- Board approved calculators may be used.
- Start each question on a new page.
- Return your answers in one bundle with a cover sheet.

This is a Trial paper only and does not necessarily reflect either the content or format of the final Higher School Certificate Examination in this subject.

Question 1 – (12 Marks)   Marks

a) (i) On the same set of axes draw the graphs of   3

$$y = |x + 2| \quad \text{and} \quad y = 2 - x.$$

(ii) Hence, or otherwise, solve $|x + 2| < 2 - x$

b) Solve: $x \geq \dfrac{4}{x}$ 3

c) Write down the general solution of the equation

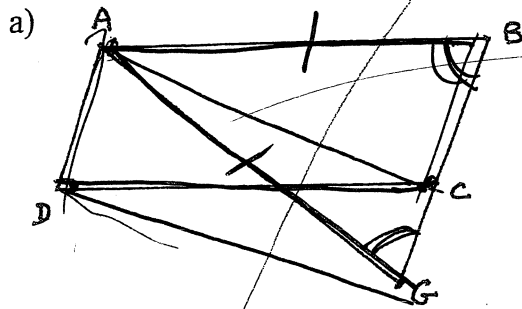$$2 \sin \theta = \sqrt{3}$$ 2

d) Evaluate $\displaystyle\int_0^{\sqrt{2}} \frac{1}{\sqrt{4 - x^2}} \, dx$ 2

e) An approximate solution to the equation $\cos x = x$ is 0.7.   2
Using one application of Newton's method find a better approximation
(correct to 2 decimal places).

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Question 2 – (12 Marks)

Marks

a)

Given that $AB = AG$ and
$ABCD$ is a parallelogram prove:

(i) $ACGD$ is cyclic

(ii)   and hence that $A\hat{C}D = A\hat{G}D$

3

b)   Evaluate: $\displaystyle\int_0^3 \sqrt{9 - x^2}\ dx$ using the substitution $x = 3\sin\theta$ for $0 \le \theta \le \dfrac{\pi}{2}$.

4

c)   A particle moves in a straight line so that its acceleration as a function of displacement is given by:

$$\frac{d^2x}{dt^2} = 1 - 4x$$

Initially: $x = 1.25\,\text{cm}, \quad v = 0$.

5

(i)   Show that the motion is simple harmonic.

(ii)   Find the amplitude. Write down the period and position of the centre.

(iii)   Find the velocity when $t = 0.2$ seconds.

Question 3 – (12 Marks)

Marks

a) (i) The letters of the word PERSEVERE are arranged in a row. How many different arrangements are possible?

3

  (ii) Out of all the different arrangements in part (i) one is chosen at random. Find the probability that this particular arrangement will have all the E's together and all the R's together in another group.

b) (i) Prove that $\dfrac{d}{dx}(x \log x) = \log x + 1$

5

  (ii) A particle moves according to $\ddot{x} = 1 + \log x$. Initially it is stationary at $x = 1$. Find v^2 as a function of x.

  (iii) Explain why v is always positive for $t > 0$ and find v when $x = e^2$.

c) Prove, using Mathematical Induction, that $\cos(x + n\pi) = (-1)^n \cos x$ for all positive integers $n \geq 1$.

4

Question 4 – (12 Marks)

Marks

a) Differentiate

2

    (i) $y = \cos^{-1} 2x$

    (ii) $y = \log_e \sqrt{2x+1}$

b)   (i)   Show that: $\sin x - \cos x = \sqrt{2} \sin\left(x - \dfrac{\pi}{4}\right)$

6

    (ii)   Hence sketch the graph of $y = \sin x - \cos x$ for $0 \le x \le 2\pi$.

    (iii)   Show that $x = \dfrac{\pi}{2}$ is a solution to $\sin x - \cos x = 1$ and hence, with the aid of your sketch, solve $\sin x - \cos x > 1$ for $0 \le x \le 2\pi$.

c)   A plane is flying at a constant height h, and with constant speed. An observer at P sighted the plane due east at an angle of elevation of $45°$. Soon after it was sighted again in a north-easterly direction at an angle of elevation of $60°$.

4
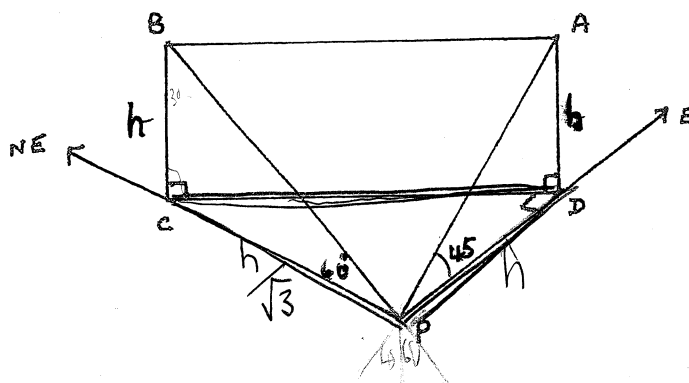
    (i)   Write down expressions for PC and PD in terms of h.

    (ii)   Show that $CD^2 = \dfrac{1}{3} h^2 (4 - \sqrt{6})$

Question 5 – (12 Marks)

Marks

a) Prove that $\sin^{-1}\dfrac{1}{\sqrt{5}} + \sin^{-1}\dfrac{1}{\sqrt{10}} = \dfrac{\pi}{4}$.

3

b) A committee of 3 men and 4 women must be chosen from a group of 5 men and 7 women. How many ways can this be done?

2

c) For the graph of $y = f(x)$ where $f(x) = \dfrac{6}{3+x^2}$

7

   (i) Show that $f(x)$ is an even function.

   (ii) Find the coordinates of the stationary point and determine its nature.

   (iii) Sketch the graph showing all essential features.

   (iv) Find the exact area between the graph, the x axis and the ordinates $x = -1$ and $x = 1$.

Question 6 – (12 Marks)

Marks

a) Sand is poured into a conical heap at a constant rate of $0.6m^3$/sec so that the height of the heap is always equal to twice the radius of the base. When the heap is 5m high, how fast is the height increasing?

3

b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

5

    (i) If PQ passes through $(2a, 0)$ show that $pq = p + q$.

    (ii) hence, find the locus of M, the midpoint of PQ.

c) When a polynomial $P(x)$ is divided by $(x-4)(x-3)$ explain why the remainder is of the form $ax + b$.

4

If the polynomial $P(x)$ is divided by $x - 3$ and $x - 4$ the remainders are 3 and 4 respectively. Find the remainder when $P(x)$ is divided by $(x-4)(x-3)$.

Question 7 – (12 Marks)

Marks

a)   A projectile is fired with initial velocity Vm/s at an angle of projection θ **6**
from a point O on horizontal ground. After 2 seconds it just passes over a
10 metre high wall that is 12m from the point of projection.

Assume acceleration due to gravity is $10m/\sec^2$. Assume the equations
of displacement are:

$x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$.

(i)   Find V and θ, to the nearest degree.

(ii)   Find the maximum height reached by the projectile.

(iii)   Find the range in the horizontal plane through the point of projection.

b)   Assume that at time t years the population $P(t)$ of a town is given by   **6**
$P(t) = 50000 + Ae^{kt}$ where A and k are constants.

(i)   Show that $P(t)$ satisfies the equation $P^{1}(t) = k[P(t) - 50000]$

(ii)   Given that $P(0) = 70000$, evaluate A.

(iii)   If $P(8) = 150000$, find k correct to 4 decimal places.

(iv)   Find, to the nearest year, the value of t for which $P(t+1) = 1.2 \times P(t)$.

Standard Integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \; n \neq -1; \; x \neq 0, \text{ if } n < 0.$$

$$\int \frac{1}{x} dx = \log_e x, \; x > 0.$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \; a \neq 0.$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \; a \neq 0.$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \; a \neq 0.$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \; a \neq 0.$$

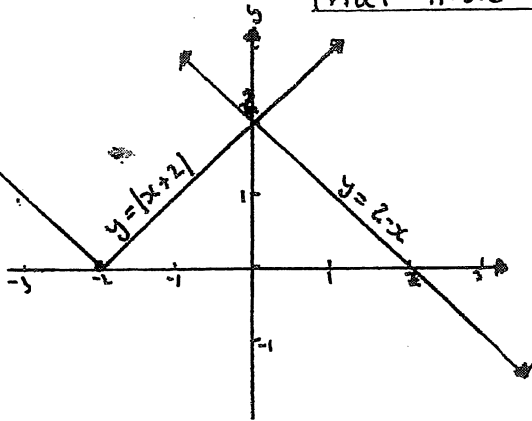$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \; a \neq 0.$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \; a \neq 0.$$

$$\int \frac{1}{\sqrt{(a^2 - x^2)}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \; -a < x < a.$$

$$\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log_e \left\{ x + \sqrt{(x^2 - a^2)} \right\}, \; |x| > |a|.$$

$$\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log_e \left\{ x + \sqrt{(x^2 + a^2)} \right\}.$$
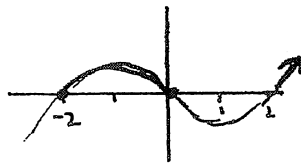
21 (a) (i)

(ii) From graph, $|x+2| < 2-x$

for $\underline{x < 0}$

(b) $x^3 \geq 4x$

$x(x-2)(x+2) \geq 0$

Critical values
0, 2, -2

Soln

$\underline{-2 \leq x < 0}$

and $\underline{x \geq 2}$

(c) $\sin\theta = \dfrac{\sqrt{3}}{2}$

General Soln $\theta = n\pi + (-1)^n \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

$\underline{\theta = n\pi + (-1)^n \cdot \dfrac{\pi}{3}}$

(d) $\displaystyle\int_0^{\sqrt{2}} \dfrac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\left(\dfrac{x}{2}\right)\right]_0^{\sqrt{2}}$

$= \sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) - \sin^{-1}(0)$
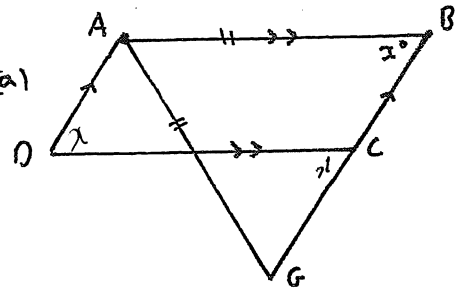
$= \underline{\dfrac{\pi}{4}}$

(e) Let $f(x) = \cos x - x$; $f(0.7) = 0.648\ldots$

then $f'(x) = -\sin x - 1$; $f'(0.7) = -1.644\ldots$

Next approx.

$x_1 = 0.7 - \dfrac{f(0.7)}{f'(0.7)}$

$\underline{x_1 = 0.74 \quad [2\ d.pl.]}$

Q 2. (a)

(i) let $\angle ABC = x°$

then $\angle ADC = x°$ (opp $\angle$'s of //gram)

and $\angle AGB = x°$ (equal $\angle$'s opp equal sides)

$\therefore$ AC subtends equal angles at D and G.

Thus, ACGD is cyclic

(ii) Given ACGD is cyclic, AD subtends equal angles

$\therefore \angle ACD = \angle AGD$.

(b) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$

let $x = 3\sin\theta$

then $dx = 3\cos\theta\,d\theta$

and $x = 0, \theta = 0$

$x = 3, \theta = \dfrac{\pi}{2}$

Substitute

$= \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta} \cdot 3\cos\theta\,d\theta$

$= 9\displaystyle\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \quad$ (use $\cos^2\theta = \dfrac{1}{2}(\cos 2\theta + 1)$

$= \dfrac{9}{2}\displaystyle\int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)\,d\theta$

$= \dfrac{9}{2}\left[\dfrac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}}$

$= \dfrac{9}{2}\left[\left(0 + \dfrac{\pi}{2}\right) - 0\right]$

$= \underline{\dfrac{9\pi}{4}}$

(c)(i) given $\dfrac{d^2x}{dt^2} = 1 - 4x$

then $a = -4\left(x - \dfrac{1}{4}\right)$

ie acceleration as a function of displacement in form $a = -n^2(x-b)$ gives motion as S.H.M.

2(c) cont.

(ii) Since starts from rest then
$$x = b + a\cos nt$$
ie $1.25 = .25 + a\cos 2 \times 0$
$$a = 1$$

__Amplitude__ is 1   and __Centre__
__Period__ : $T = \dfrac{2\pi}{2}$ $b = \dfrac{1}{4}$
$$\qquad\quad = \pi$$

(iii) $\dfrac{dx}{dt} = \dot{x} = -2\sin 2t$

at $t = 0.2$, $\dot{x} = -2\sin 0.4$
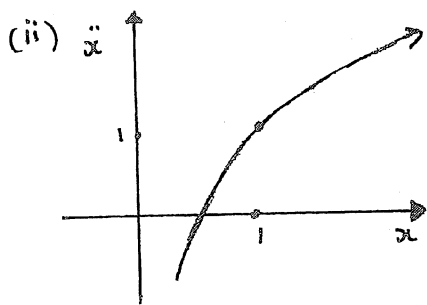$$\qquad\qquad = \underline{0.8\,cm/s}\ \ (1\,d.p.)$$

Q3 (a)(i) No. of arrangements
$$\dfrac{9!}{4! \times 2!} = \underline{7560}$$

(ii) Arrangements with E's and R's
together is $5!$
$$\therefore \text{Prob.}\quad \dfrac{5!}{7560} = \underline{\dfrac{1}{63}}$$

(b) (i) $\dfrac{d}{dx}(x.\log x) = 1.\log x + x.\dfrac{1}{x}$
$$\qquad\qquad\qquad = \log x + 1$$

(ii) $\ddot{x}$

Acceleration Diagram.

__Now__, $\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 1 + \log x$
$$\tfrac{1}{2}v^2 = x.\log x + C \qquad [\text{from (i)}]$$

at $x = 1$, $v = 0$ so $C = 0$
$$\therefore\ v^2 = 2x.\log x$$

(iii) Initially at rest at $x = 1$.
    acceleration always positive, from (i
therefore velocity will always move
to the right
$$\left[\ v^2 = 0,\quad \begin{matrix} 2x = 0 \\ x = 0 \end{matrix}\ \text{or}\ \begin{matrix} \log x = 0 \\ x = 1 \end{matrix}\ \right]$$

So $v > 0$ for $t > 0$
since $x \geq 1$.

When $x = e^2$, $v^2 = 2e^2.\log e^2$
$$\qquad\qquad\qquad v^2 = 4e^2$$
$$\qquad\qquad\qquad \underline{v = 2e}\quad (v >)$$

(c) • let $n = 1$
    then $\cos(x + \pi) = \cos x\cos\pi - \sin x\sin$
$$\qquad\qquad\qquad = -\cos x$$
$$\qquad\qquad\qquad = (-1)^1\cos x$$
    True for $n = 1$
  • Assume result true for $n = k$
  (k positive integer)
    ie $\cos(x + k\pi) = (-1)^k\cos x$
  • Test result for next n,
      ie $n = k+1$
$$\cos(x + (k+1)\pi) = \cos\left[(x + k\pi) + \pi\right]$$
$$\qquad\qquad = \cos(x+k\pi)\cos\pi - S\dots$$
$$\qquad\qquad = \cos(x+k\pi)\times -1$$
From above $\qquad = (-1)^k\cos x \times -1$
$$\qquad\qquad = (-1)^{k+1}\cos x\quad (\text{as re}$$

  • From above, if result is true for
$n = k$ then it is true for next n, $n =$
    Since true for $n = 1$ it is true
for $n = 2$. By induction true for $n =$
and __all__ n

Q4 (a) (i) $\quad y = \cos^{-1} 2x$

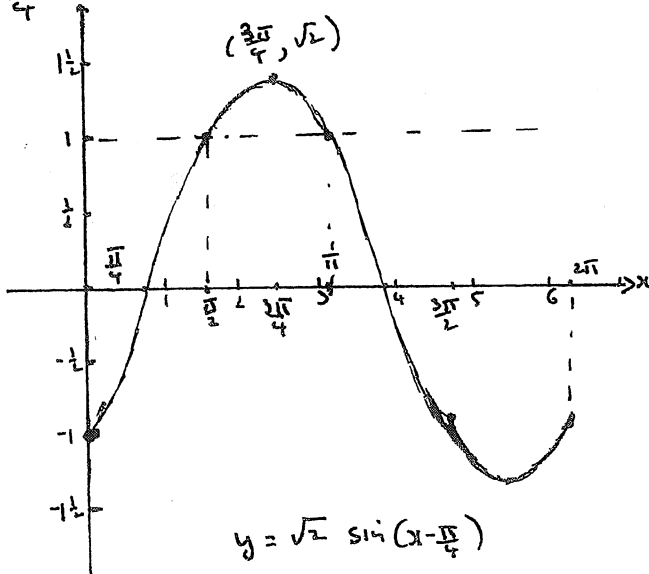$$y' = \frac{-2}{\sqrt{1-4x^2}}$$

(ii) $\quad y = \frac{1}{2} \ln(2x+1)$

$$y' = \frac{1}{2x+1}$$

(b) (i)

$\sin\theta = \frac{1}{\sqrt{2}}$

$\cos\theta = \frac{1}{\sqrt{2}}$

$\tan\theta = 1$

$\therefore \theta = \frac{\pi}{4}$

$\sin x - \cos x = \sqrt{2}\left(\sin x \cos\theta - \cos x \sin\theta\right)$

$$= \sqrt{2}\,\sin(x-\theta)$$

$$= \sqrt{2}\,\sin\left(x - \frac{\pi}{4}\right)$$

(ii)

$\left(\frac{3\pi}{4}, \sqrt{2}\right)$

$y = \sqrt{2}\,\sin\left(x - \frac{\pi}{4}\right)$
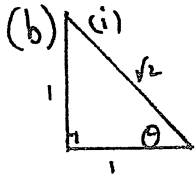
$x=0, y=-1$; $x = \frac{\pi}{4}, y=0$; $x = \frac{\pi}{2}, y=1$

Amplitude $\sqrt{2}$

Period 2π

let $\sqrt{2} = \sqrt{2}\,\sin\left(x - \frac{\pi}{4}\right)$

$1 = \sin\left(x - \frac{\pi}{4}\right)$

$\therefore x - \frac{\pi}{4} = \frac{\pi}{2}$

$x = \frac{3\pi}{4}$

(iii)

(c) (i) $\dfrac{h}{PC} = \tan 60°$ (※) $\dfrac{h}{PD} = \tan 45°$

$\therefore PC = \dfrac{h}{\sqrt{3}}$ --(A) $\quad PD = h$ --(B)

(ii) Cosine rule, noting $\angle NE = 45°$

$CD^2 = h^2 + \left(\dfrac{h}{\sqrt{3}}\right)^2 - 2 \cdot h \times \dfrac{h}{\sqrt{3}} \cos 45°$

$= h^2\left[1 + \dfrac{1}{3} - \dfrac{2}{\sqrt{3}} \times \dfrac{1}{\sqrt{2}}\right]$

$= h^2\left[\dfrac{4}{3} - \dfrac{2}{\sqrt{6}}\right]$

$= 2h^2\left[\dfrac{4}{6} - \dfrac{\sqrt{6}}{6}\right]$

$CD^2 = \dfrac{1}{3}h^2\left(4 - \sqrt{6}\right)$ $\quad$ as required.

Q5 (a) let $X = \sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right)$ and $Y = \sin^{-1}\left(\dfrac{1}{\sqrt{10}}\right)$

then $\sin X = \dfrac{1}{\sqrt{5}}$ and $\sin Y = \dfrac{1}{\sqrt{10}}$

 $\cos X = \dfrac{2}{\sqrt{5}}$ $\quad\quad$ $\cos Y = \dfrac{3}{\sqrt{10}}$

$\underline{Now}$, $\sin(X+Y)$

$= \sin X \cos Y + \cos X \sin Y$

$= \dfrac{1}{\sqrt{5}} \times \dfrac{3}{\sqrt{10}} + \dfrac{2}{\sqrt{5}} \times \dfrac{1}{\sqrt{10}}$

$= \dfrac{5}{5\sqrt{2}}$

$= \dfrac{1}{\sqrt{2}}$

is $\sin(X+Y) = \dfrac{1}{\sqrt{2}}$

$X+Y = \sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$

$= \dfrac{\pi}{4}$

$\therefore \sin^{-1}X + \sin^{-1}Y = \dfrac{\pi}{4}$

(b) $^5C_3 \times {}^7C_4 = 350$ ways.

(c) (i) $f(-x) = \dfrac{6}{3+(-x)^2}$

$= f(x)$

$\underline{\text{Even function}}$ since $\underline{f(-x) = f(x)}$

(c) Cont.

(ii) $f'(x) = \dfrac{-12x}{(3+x^2)^2}$

St.pt when $f'(x) = 0$
at $x = 0$

Stat. at $(0,2)$

Local max. at $(0,2)$

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
$f'(x)$	+ve	—	-ve
	$\frac{6}{}$	0	$-\frac{6}{}$

$\nearrow$ Local $\searrow$
max.

(iii) As $x \to \pm\infty$, $f(x) \to 0$

$(0,2)$
$(-1,1\frac{1}{2})$ $(1,1\frac{1}{2})$
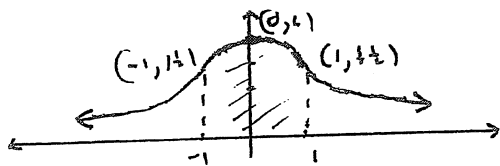
(iv) Area $= 2\displaystyle\int_0^1 \dfrac{6}{3+x^2}\, dx$

$= \left[\dfrac{12}{\sqrt{3}} \tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right)\right]_0^1$

$= \dfrac{12}{\sqrt{3}}\left(\dfrac{\pi}{6} - 0\right)$

$= \dfrac{2\pi}{\sqrt{3}}$ sq units

Q6. (a)

$h = 2r$
$\therefore \dfrac{h}{2} = r$

Given $\dfrac{dV}{dt} = 0.6$ m³/sec.

Now $V = \dfrac{1}{3}\pi r^2 h$

$= \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h$

$= \dfrac{1}{12}\pi h^3$ - - - (A)

So $\dfrac{dV}{dh} = \dfrac{1}{4}\pi h^2$, Require $\dfrac{dh}{dt}$

and $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$

$0.6 = \dfrac{\pi h^2}{4} \cdot \dfrac{dh}{dt}$

$\therefore \dfrac{dh}{dt} = 0.031$ m/sec.

(b)(i) Gradient $\bar{PQ}$, $m = \dfrac{a(p-q)(p+q)}{2a(p-q)}$

$m = \dfrac{p+q}{2}$

Eqn. $y - ap^2 = \dfrac{p+q}{2}(x - 2ap)$

$\therefore y = \dfrac{p+q}{2}x - apq$

Substitute $(2a, 0)$

$0 = a(p+q) - apq$

$\therefore pq = p+q$ - - - - (A)

(ii) Midpoint $M\left(a(p+q), \dfrac{a}{2}(p^2+q^2\right.$

then $X = a(p+q)$ - - - $(p+q) = \dfrac{X}{a}$ - - -

$Y = \dfrac{a}{2}(p^2+q^2)$ - - $Y = \dfrac{a}{2}\left[(p+q)^2 - 2\right.$

Substitute I into II, use

$Y = \dfrac{a}{2}\left[\left(\dfrac{X}{a}\right)^2 - 2\times\dfrac{X}{a}\right]$

$2Y = a\left[\dfrac{X^2}{a^2} - \dfrac{2X}{a}\right]$

$2Y = \dfrac{1}{a}\left[X^2 - 2aX\right]$

Locus is a parabola $\begin{cases} 2aY = \\ \end{cases}$
is $(x-a)^2 = 2a\left(Y + \dfrac{a}{2}\right)$

(c) $(x+4)(x-3) = x^2 - 7x + 12$.
Remainder must be of degree
than 2. It must be lin

is $(ax+b)$

$\bullet$ $P(x) = Q(x)(x-3)(x-4) + (ax$

$x=3$, $P(3) = 3 = 3a + b$ - -
$x=4$ $P(4) = 4 = 4a + b$ - -

$\therefore$ II $-$ I, $a = 1$

So $b = 0$

Remainder is the linear
polynomial $1x + 0 = x$

Q7

(a)(i) Given

$x = Vt \cos\theta$

$y = -5t^2 + Vt\sin\theta$ and at $t=0$

$x = 12$

$y = 10$

then

$12 = 2V\cos\theta$ ··· (i)

$10 = -20 + 2V\sin\theta$ ·· (ii)

from (i) $V = \dfrac{6}{\cos\theta}$ sub $10 = -20 + 12\tan\theta$

$\tan\theta = \dfrac{5}{2}$

$\theta = 68°$ [nearest degree]

$\therefore \underline{V = 16.2 \text{ m/s}}$ [1 d.p.l]

(ii) Max. height when $y' = 0$

ie $\quad -10t + V\sin\theta = 0$

$t = \dfrac{V\sin\theta}{10}$

$= 1.5$ sec.

$\therefore \quad y = -5 \times (1.5)^2 + 16.2 \times 1.5 \times \sin 68$

$y = 11.3$ m   (1 d.p.l)

Max. height at $\underline{11.3 m}$

(iii) Range, let $t = 2 \times 1.5$

(symmetry of projection)

$t = 3$

then $\quad x = 16.2 \times 3 \times \cos 68$

$x = 18.2$ m

Range is $\underline{18.2}$ m

(b) (i) $P'(t) = k \cdot Ae^{kt}$

since $A \cdot e^{kt} = P(t) - 5000$

then $P'(t) = K[P(t) - 5000]$

(ii) $\quad 70000 = 50000 + Ae^0$

$\therefore \underline{A = 20000}$

(iii) $150000 = 50000 + 20000 \, e^{8k}$

$5 = e^{8k}$

$\therefore k = \dfrac{1}{8} \ln 5$

$\doteqdot \underline{0.2012}$

(iv) let $P(t+1) = 1.2 \times P(t)$

then $50000 + 20000 \, e^{k(t+1)} = 60000 + 24000 \, e^{kt}$

$\therefore e^{kt}[20000 \, e^k - 24000] = 10000$

$t = \dfrac{1}{k} \cdot \ln\left[\dfrac{10000}{20000 \, e^k - 240}\right]$

$= 15.33 \cdots$

It would be after $\underline{15 \text{ years}}$.