St George Girls High School

Trial Higher School Certificate Examination

2005

Mathematics
Extension 1

Total Marks – 84

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Attempt ALL questions.
- Begin each question on a new booklet
- Write your student number on each page
- All necessary working must be shown.
- Diagrams are not to scale.
- Board-approved calculators may be used.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 (12 marks) **Marks**

a) Find the coordinates of the point P that divides AB internally in the ratio 2 : 3 where A is $(-3, 5)$ and B is $(-6, -10)$ 2

b) Find the possible values of a if the lines $2x + 3y - 5 = 0$ and $ax + 2y + 3 = 0$ are inclined to each other at 45° 4

c) Solve for x: $\dfrac{2}{x-1} > 3$ 3

d) Find $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$ using the substitution $x = u + 1$ 3

Question 2 (12 marks)

a) (i) Express $\sqrt{3}\sin x + \cos x$ in the form $R\sin(x + \alpha)$ 2

(ii) Hence, sketch the graph of $y = \sqrt{3}\sin x + \cos x$ for $0 \le x \le 2\pi$ 2

b) (i) Show that $f(x) = 2\log_e x + 2x$ has a zero between $x = 0.5$ and $x = 1$ 1

(ii) Starting with $x = 0.5$, use one application of Newton's method to find a better approximation for this zero. Write your answer correct to three significant figures 3

check

c) Find $\int \frac{dx}{\sqrt{9 - 4x^2}}$ 2

d) Find $\int \cos^2 4x \, dx$ 2

Question 3 (12 marks) **Marks**

a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $4ay = x^2$ such that the chord PQ subtends a right angle at the vertex O

(i) Show that $pq = -4$ 2

(ii) Find the locus of the mid-point of PQ 3

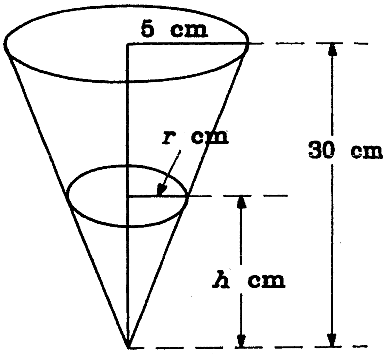
b) Show that $\int_0^3 \left(\frac{x}{x^2+9} + \frac{1}{x^2+9} \right) dx = \log_e \sqrt{2} + \frac{\pi}{12}$ 3

c) If the roots of the equation $x^3 + bx^2 + cx + d = 0$ are in geometric progression show that $c^3 = b^3 d$ 4

Question 4 (12 marks)

a) A container is in the shape of an inverted right circular cone of base radius 5cm and height 30cm. Water is poured into the container at a rate of 2cm³/min

(i) Show that $r = \frac{h}{6}$ 1



(ii) Find the rate at which the level of water is rising when the water is 10cm deep 3

b) (i) State the domain and range of $y = 2\cos^{-1}\left(\frac{x}{3}\right)$ 2

(ii) Hence sketch $y = 2\cos^{-1}\left(\frac{x}{3}\right)$ 1

c) Given $f(x) = \sqrt[3]{x-1}$ for $x > 1$

(i) Show that the function is monotonic increasing for all x in the given domain 2

(ii) State the domain and range of $f^{-1}(x)$ 1

(iii) Find $f^{-1}(x)$ and explain why the inverse is a function 2

Question 5 (12 marks)

Marks

a) By induction show that $7^n - 3^n$ is divisible by 4 for all integers $n \geq 1$ 3

b) The velocity v and position x of a particle moving in a straight line are connected by the relation $v = 3 + 5x$. Show that the acceleration a of the particle is $5v$ 2

9 c) Find the term independent of x in the expansion of $(3-x)^4\left(1+\dfrac{2}{x}\right)^7$ 4

0 d) Evaluate $\cos\left(2\tan^{-1}\dfrac{3}{4}\right)$ without the use of a calculator 3

Question 6 (12 marks)

a) The cooling rate of a body is proportional to the difference between the temperature of the body and that of a surrounding medium ie. $\frac{dT}{dt} = -k(T - T_1)$ where T is the temperature of the cooling body and T_1 is the temperature of the surrounding medium

(i) Show that $T - T_1 = Ae^{-kt}$ satisfies this equation 2

(ii) A cup of coffee cools from 80° to 40° in 10 minutes when placed in a room with temperature 18°. How long will it take for the coffee's temperature to fall to 20°? 4

b) A particle is moving in a straight line such that its acceleration at time t seconds is $\ddot{x} = -4x$, where x is the displacement in metres from the origin. The particle is initially 6m to the right of the origin.

(i) Find its displacement in terms of time 3

(ii) Find the position and time when the particle first obtains a velocity of 6m/s 3

Question 7 (12 marks) **Marks**

a) (i) Differentiate $x(1+x)^n$ 1

(ii) Write the binomial expansion for $x(1+x)^n$ 1

(iii) Hence show that $\sum_{r=0}^{n} (r+1)\ {}^{n}C_{r} = (n+2)\, 2^{n-1}$ 3

b) A particle is projected from a point O with an initial velocity of 60m/s at an angle of 30° to the horizontal. At the same instant a second particle is projected in the opposite direction with an initial velocity of 50m/s from a point level with O and 100m from O.

(i) Show that the horizontal and vertical displacement equations of the first particle are given by: 2

$x = 60\cos 30° t$ and $y = 60\sin 30° t - ½\, g t^2$
where g is acceleration due to gravity

(ii) Find the angle of projection of the second particle if they collide 3

(iii) Find the time at which the two particles collide 2

End of Paper

Extension 1 – Trial HSC Solutions 2005

①a) $A(-3,5)$ $B(-6,-10)$ Ratio $2:3$

$x = \frac{3\times -3 + 2\times -6}{2+3}$ $\quad y = \frac{3\times 5 + 2\times -10}{2+3}$

$= \frac{-21}{5}$ $\quad = -1$

$\therefore P$ is $(-4\frac{1}{5}, -1)$

b) $2x+3y-5=0$ $\quad ax+2y+3=0$

$m_1 = -\frac{2}{3}$ $\quad m_2 = -\frac{a}{2}$

$\therefore \tan 45 = 1 = \left| \frac{-\frac{2}{3} + \frac{a}{2}}{1 + -\frac{2}{3} \times -\frac{a}{2}} \right|$

$1 = \left| \frac{\frac{-4+3a}{6}}{\frac{6+2a}{6}} \right|$

$\therefore |6+2a| = |3a-4|$

$\therefore 6+2a = 3a-4$ or $6+2a = 4-3a$

$10 = a$ $\quad 5a = -2$

$a = -\frac{2}{5}$

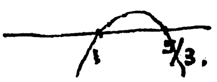
c) $\frac{2}{x-1} > 3$ $\quad x \neq 1$

$2(x-1) > 3(x-1)^2$

$2(x-1) - 3(x-1)^2 > 0$

$(x-1)[2 - 3(x-1)] > 0$

$(x-1)(5-3x) > 0$



$\therefore 1 < x < \frac{5}{3}$

d) $\int \frac{x}{\sqrt{x-1}}\, dx$ $\quad x = u+1$

$dx = du$

$= \int \frac{u+1}{\sqrt{u}}\, du$

$= \int \sqrt{u} + \frac{1}{\sqrt{u}}\, du$

$= \int u^{\frac{1}{2}} + u^{-\frac{1}{2}}$

$= \frac{2u}{3}^{3/2} + 2u^{1/2} + c$

$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + c$

②a) i) $\sqrt{3}\sin x + \cos x \equiv R\sin(x+\alpha)$

$\sqrt{3}\sin x + \cos x \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

$\therefore R\cos\alpha = \sqrt{3}$ $\quad R\sin\alpha = 1.$

$\therefore R^2(\cos^2\alpha + \sin^2\alpha) = 3+1$

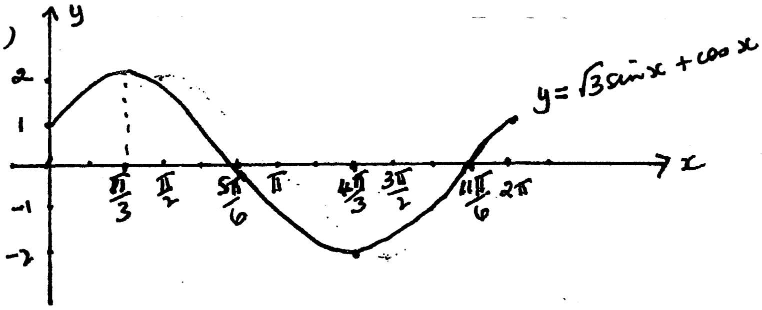
$\therefore R = 2$ $\quad R > 0.$

and $\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{3}}$

$\therefore \tan\alpha = \frac{\pi}{6}.$

$\therefore \sqrt{3}\sin x + \cos x = 2\sin(x + \frac{\pi}{6})$

(ii)



b)i) $f(x) = 2\log_e x + 2x.$

$f(0\cdot 5) \doteqdot -0\cdot 386$

$f(1) = 2.$

$\therefore$ Since sign change a zero lies between $\frac{1}{2}$ and 1.

ii) $f'(x) = \frac{2}{x} + 2.$

If $z_1 = 0\cdot 5$

then $z_2 = 0\cdot 5 - \frac{f(0\cdot 5)}{f'(0\cdot 5)}$

$= 0\cdot 5 - \frac{(2\ln 0\cdot 5 + 1)}{6}$

$\doteqdot 0\cdot 56438\ldots.$

$\doteqdot 0\cdot 564$ (to 3 sig. fig)

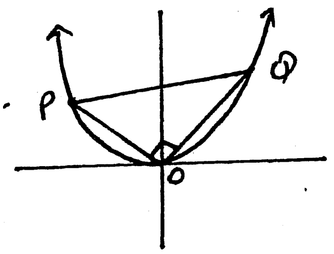
c) $\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$

d) $\int \cos^2 4x\, dx = \frac{1}{2}\int 1 + \cos 8x\, dx$

$= \frac{1}{2}\left(x + \frac{\sin 8x}{8}\right) + c$

$= \frac{x}{2} + \frac{\sin 8x}{16} + c$

1)a) $P(2ap, ap^2)$ $Q(2aq, aq^2)$



i) m of OP $= \frac{ap^2-0}{2ap-0}$ m of OQ $= \frac{aq^2-0}{2aq-0}$

$= \frac{p}{2}$ $= \frac{q}{2}$

Since $P\hat{O}Q = 90^0$

$\frac{p}{2} \times \frac{q}{2} = -1$

$\therefore pq = -4$

ii) midpt PQ $= \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

$x = a(p+q)$

$\therefore p+q = \frac{x}{a}$

$y = \frac{a(p^2+q^2)}{2}$

$\frac{2y}{a} = (p+q)^2 - 2pq$

$= \left(\frac{x}{a}\right)^2 - 2\times -4$

$\frac{2y}{a} = \frac{x^2}{a^2} + 8$

$2ay = x^2 + 8a^2$

$x^2 = 2a(y-4a)$

(b) $\int_0^3 \frac{x}{x^2+9} + \frac{1}{x^2+9} \, dx$

$= \left[\frac{1}{2}\ln(x^2+9) + \frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$

$= \left(\frac{1}{2}\ln 18 + \frac{1}{3}\tan^{-1} 1\right) - \left(\frac{1}{2}\ln 9 + 0\right)$

$= \frac{1}{2}\ln 2 + \frac{1}{3} \times \frac{\pi}{4}$

$= \ln\sqrt{2} + \frac{\pi}{12}$ as req.

c) $x^3 + bx^2 + cx + d = 0$

Let roots be $\frac{\alpha}{r}, \alpha, \alpha r$

$\therefore \frac{\alpha}{r} + \alpha + \alpha r = -b$ —(1)

$\frac{\alpha^2}{r^2} + \alpha^2 + \alpha^2 r = c$ —(2)

$\frac{\alpha}{r} \times \alpha \times \alpha r = -d$ —(3)

$\therefore \alpha^3 = -d$

From (1) : $\alpha\left(\frac{1}{r} + 1 + r\right) = -b$

From (2) : $\alpha^2\left(\frac{1}{r} + 1 + r\right) = c$

$\therefore \frac{1}{\alpha} = -\frac{b}{c}$

$\alpha = \frac{c}{-b}$

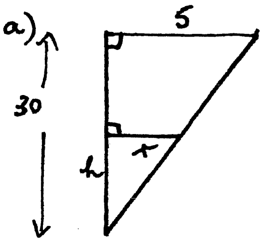
$\therefore \left(\frac{c}{-b}\right)^3 = -d$

$\frac{c^3}{-b^3} = -d$

$\therefore c^3 = b^3 d$

as req.

④ a)



i) Since Δ's are similar

$$\frac{r}{5} = \frac{h}{30}$$

$$\therefore r = \frac{5h}{30} = \frac{h}{6}$$

ii) vol of cone $= \frac{1}{3}\pi r^2 h$

$$\therefore V = \frac{1}{3}\pi \times \left(\frac{h}{6}\right)^2 \times h$$

$$= \frac{\pi h^3}{108}$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$2 = \frac{3\pi h^2}{108} \times \frac{dh}{dt}$$

$\therefore$ when $h = 10$: $2 = \frac{300\pi}{108} \times \frac{dh}{dt}$

$$\therefore \frac{dh}{dt} = \frac{216}{300\pi}$$

$$= \frac{18}{25\pi}$$

$\therefore$ water is rising at $\frac{18}{25\pi}$ cm/min

(b) i) $y = 2\cos^{-1}\frac{x}{3}$

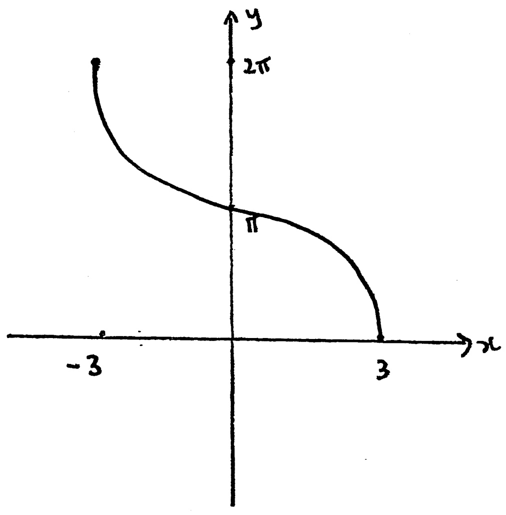
D: $-1 \leq \frac{x}{3} \leq 1$

$-3 \leq x \leq 3$

R: $0 \leq \frac{y}{2} \leq \pi$

$0 \leq y \leq 2\pi$

ii)



c) $f(x) = \sqrt[3]{x-1}$ $\quad x > 1$

i) $f(x) = (x-1)^{1/3}$

$$f'(x) = \frac{1}{3}(x-1)^{-2/3}$$

$$= \frac{1}{3\sqrt[3]{(x-1)^2}}$$

Since $(x-1)^2$ is positive for all x

$\sqrt[3]{(x-1)^2} > 0$

$\therefore \frac{1}{3\sqrt[3]{(x-1)^2}} > 0$ for all x

$\therefore f(x)$ is montonic increasing

(ii) For $f(x)$: D: $x > 1$

R: $y > 0$

$\therefore$ For $f^{-1}(x)$: D: $x > 0$

R: $y > 1$

(iii) $y = (x-1)^{1/3}$

For inverse: $x = (y-1)^{1/3}$

$x^3 = y - 1$

$\therefore y = x^3 + 1$

Since $f(x)$ is montonic increasin and it passes horizontal line tes inverse will also be a function

5\. a) Assertion: that $7^n - 3^n$ is divisible by 4 for $n \geqslant 1$

For $n=1$: $7^1 - 3^1 = 4$ which is divisible by 4

$\therefore$ Assertion is true for $n=1$.

Assume assertion is true for $n=k$

i.e. that $7^k - 3^k$ is divisible by 4

i.e. $7^k - 3^k = 4M$ (where M is a positive integer)

We need to prove that:

$7^{k+1} - 3^{k+1}$ is also divisible by 4.

$$7^{k+1} - 3^{k+1} = 7^k.7 - 3^k.3$$
$$= (8-1).7^k - (4-1).3^k$$
$$= 8.7^k - 7^k - 4.3^k + 3^k$$
$$= 8.7^k - 4.3^k - (7^k - 3^k)$$
$$= 8.7^k - 4.3^k - 4M \text{ using assumption}$$
$$= 4(2.7^k - 3^k - M)$$
$$= 4J \text{ where J is a positive integer)}$$

$\therefore 7^{k+1} - 3^{k+1}$ is divisible by 4.

$\therefore$ If statement is true for $n=k$, it is true for $n=k+1$.

$\therefore$ Since statement is true for $n=1$, it is true for $n=2$ and by induction it is true for all $n \geqslant 1$.

b) $v = 3 + 5x$

Since $\frac{d}{dx}(\frac{1}{2}v^2) = \ddot{x}$

$$\frac{d}{dx}\left(\tfrac{1}{2}(3+5x)^2\right) = 2 \times \tfrac{1}{2}(3+5x) \times 5$$
$$= 5(3+5x)$$
$$= 5v$$

$\therefore$ acceleration $= 5v$.

c) $(3-x)^4 \left(1+\frac{2}{x}\right)^7$

$(3-x)^4 = {}^4c_0 3^4 - {}^4c_1 3^3 x + {}^4c_2 3^2 x^2 - {}^4c_3 3x^3 + {}^4c_4 x^4$

$\left(1+\frac{2}{x}\right)^7 = {}^7c_0 + {}^7c_1 \frac{2}{x} + {}^7c_2 \frac{4}{x^2} + {}^7c_3 \frac{8}{x^3} + {}^7c_4 \frac{16}{x^4} \ldots\ldots$

$\therefore$ Term Independent of $x = 3^4 \times {}^7c_0 - {}^4c_1 {}^7c_1 27 \times 2 + {}^4c_2 9 \times {}^7c_2 \times 4 - {}^4c_3 3 \times {}^7c_3 \times 8 + {}^7c_4 16$

$= 81 - 1512 + 4536 - 3360 + 560$

$= 305$

d) next page

⑥ a) i) $$\frac{dT}{dt} = -k(T-T_1)$$

$$T - T_1 = Ae^{-kt}$$

$$\therefore T = T_1 + Ae^{-kt}$$

$$LHS = \frac{dT}{dt}$$

$$= -k \times Ae^{-kt}$$

$$RHS = -k(T-T_1)$$

$$= -k(\cancel{T_1} + Ae^{-kt} - \cancel{T_1})$$

$$= -k . Ae^{-kt}$$

$$= LHS$$

$\therefore T - T_1 = Ae^{-kt}$ satifies eqn.

ii) $T_1 = 18$

$t = 0$: $T = 80$

$$\therefore 80 = 18 + A \times 1$$

$$\therefore A = 62$$

$$\therefore T = 18 + 62e^{-kt}$$

$t = 10$, $T = 40$

$$40 = 18 + 62e^{-10k}$$

$$\frac{22}{62} = e^{-10}$$

$$\therefore k = \frac{\ln \frac{11}{31}}{-10.}$$

$T = 20$:

$$20 = 18 + 62e^{-kt}$$

$$\frac{2}{62} = e^{-kt}$$

$$t = \frac{\ln \frac{1}{31}}{-\left(\frac{\ln \frac{11}{31}}{-10}\right)}$$

$= 33.14$ mins (2dp).

b) i) Since $\ddot{x} = -4x$ particle is moving in SHM about origin

$$\therefore x = a\cos(nt + \alpha)$$

$$= 6\cos(2t + \alpha)$$

$t = 0$, $x = 6$:

$$\therefore 6 = 6\cos\alpha$$

$$\cos\alpha = 1$$

$$\therefore \alpha = 0$$

$$\therefore x = 6\cos 2t$$

ii) $v = -12\sin 2t$

$v = 6$: $6 = -12\sin 2t$

$$\therefore \sin 2t = -\frac{1}{2}$$

$$2t = \frac{7\pi}{6}$$

$$t = \frac{7\pi}{12}$$

$t = \frac{7\pi}{12}$: $x = 6\cos 2 \times \frac{7\pi}{12}$

$$= 6\cos\frac{7\pi}{6}$$

$$= 6 \times -\frac{\sqrt{3}}{2}$$

$$= -3\sqrt{3}$$

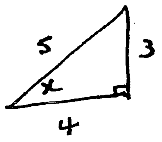
$\therefore$ Particle first reaches 6m/sec after $\frac{7\pi}{12}$ secs., $3\sqrt{3}$ metres to left of origin.

5d

$$\cos\left(2\tan^{-1}\frac{3}{4}\right) = \cos 2x$$

where $x = \tan^{-1}\frac{3}{4}$

$$\therefore \tan x = \frac{3}{4}$$



$$\therefore \cos 2x = 2\cos^2 x - 1$$

$$= 2\left(\frac{4}{5}\right)^2 - 1$$

$$= \frac{32}{25} - 1$$

$$= \frac{7}{25}$$

⑦ a) i) $\frac{d}{dx}\left(x(1+x)^n\right) = (1+x)^n \times 1 + x \times n(1+x)^{n-1}$

$= (1+x)^n + nx(1+x)^{n-1}.$

ii) $x(1+x)^n = x\,{}^nC_0 + x\,{}^nC_1 x + x\,{}^nC_2 x^2 + \dots + x\,{}^nC_n x^n$

$= {}^nC_0 x + {}^nC_1 x^2 + {}^nC_2 x^3 + \dots + {}^nC_n x^{n+1}$

iii) $\sum_{r=0}^{n} (r+1)\,{}^nC_r = {}^nC_0 + 2\,{}^nC_2 + 3\,{}^nC_3 + \dots + (n+1)\,{}^nC_r.$

from (ii): $\frac{d}{dx} x(1+x)^n = {}^nC_0 + 2\,{}^nC_1 x + 3\,{}^nC_2 x^2 + \dots + (n+1)\,{}^nC_n x^n$

∴ from(i): $(1+x)^n + nx(1+x)^{n-1} = {}^nC_0 + 2\,{}^nC_1 x + 3\,{}^nC_2 x^2 + \dots + (n+1)\,{}^nC_n x^n$

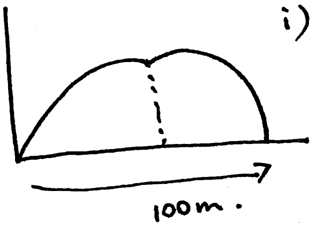
Let $x=1$:

$2^n + n(2)^{n-1} = {}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + \dots + (n+1)\,{}^nC_n$

$\therefore 2^{n-1}(2+n) = {}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + \dots + (n+1)\,{}^nC_n.$

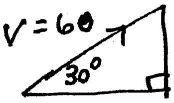
$\therefore \sum_{r=0}^{n} (r+1)^n\,{}^nC_r = 2^{n-1}(n+2)$

b)



i)

First particle.



$\ddot{x} = 0$

$\dot{x} = C = 60\cos 30$

$x = \int 60\cos 30\,dt$

$= 60\cos 30t + k$

$t=0 \quad x=0 \quad \therefore k=0$

$\therefore x = 60\cos 30t$

$\ddot{y} = -g$

$\dot{y} = -gt + M$

$t=0 \quad \dot{y} = 60\sin 30$

$\therefore \dot{y} = -gt + 6$[illegible]$n3.$

$y = \int \dot{y}\,dt$

$= -\frac{gt^2}{2} + 60\sin 30t$

$t=0 \quad y=0 \quad \therefore N=0$

$\therefore y = -\frac{gt^2}{2} + 60\sin 30t$

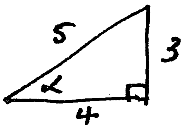
ii) When particles collide height above ground is same

$\therefore 60\sin 30t - \frac{1}{2}gt^2 = 50\sin$ [illegible] $t - \frac{1}{2}gt^2$

$30t = 50\sin\alpha t$

$\therefore \sin\alpha = \frac{3}{5}$

$\therefore \alpha = 36°52'$



iii) x values add to 100 m.

$\therefore 60\cos 30t + 50\,\cancel{\cos}\,36°52'\,t = 100$

$30\sqrt{3}\,t + 50 \times \frac{4}{5}t = 100$

$\frac{100}{} = 1.087$ secs (to 3dp)