St George Girls High School

Trial Higher School Certificate Examination

2007

Mathematics Extension 1

Total Marks – 84

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Attempt ALL questions.
- Begin each question on a new booklet
- Write your student number on each page
- All necessary working must be shown.
- Diagrams are not to scale.
- Board-approved calculators may be used.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – 12 marks (Start a new booklet) **Marks**

a) Differentiate $\log\left(xe^{x}\right)$ 2

b) Find the equation of the normal on the curve $y=\ln(x+2)$ at the point $(0, \ln 2)$ 3

c) Solve $\dfrac{2x+3}{x-4} \geq 1$

d) Let A be the point $(3, -1)$ and B be the point $(9, 2)$. Find the coordinates of the point P which divides the interval AB externally in the ratio $5:2$ 2

e) Evaluate the limit $\lim\limits_{x \to 0} \dfrac{\sin\frac{x}{5}}{2x}$ 2

<u>Question 2</u> – 12 marks (Start a new booklet) **Marks**

a) Find $\int \frac{19dx}{4+8x^2}$ 2

b) Find the acute angle between the lines $x+2y=5$ and $x-3y=-3$ 2

c) Find $\frac{d}{dx}\left(\cos^{-1}\left(2\cos^2 x-1\right)\right)$ in simplest terms for $0 \le x \le \frac{\pi}{2}$ 2

d) Evaluate $\int_0^2 \frac{2x}{\sqrt{x^2+1}}\,dx$ by using the substitution $u=x^2+1$ 4

e) Differentiate $x^2 \tan 5x$ 2

Question 3 – 12 marks (Start a new booklet) **Marks**

a) Write down the period and amplitude of $y = 2\cos\frac{1}{3}x$ 2

b) Use the change of base formula to evaluate $\log_3 14$ correct to one decimal place. 1

c) (i) Write $\sqrt{3}\sin x - \cos x$ in the form $r\sin(x-\alpha)$ 1

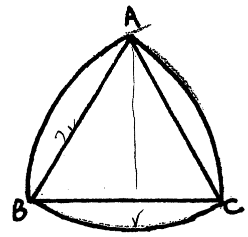
(ii) And hence or otherwise solve $\sqrt{3}\sin x - \cos x = 1 \qquad 0 \le x \le 2\pi$ 2

d) Sketch a graph of $y = \frac{3}{\pi}\cos^{-1}\frac{x}{2}$ indicating its domain and range. 4

e) Find $\frac{d}{dx}\left(2\sin^{-1}x\right)$ 1

Question 4 – 12 marks (Start a new booklet) **Marks**

a) ABC is an equilateral triangle, side $2r$.
The circular arcs AB, BC and CA have centres at C, A and B respectively.



Show that for the figure bounded by the arcs:

(i) The perimeter is equal to that of a circle of radius r 2

(ii) The area is approximately 90% of that of a circle, radius r 3

b) Use the principle of mathematical induction to prove that $3^{2n} - 1$ is divisible by 8 when n is a positive integer. 4

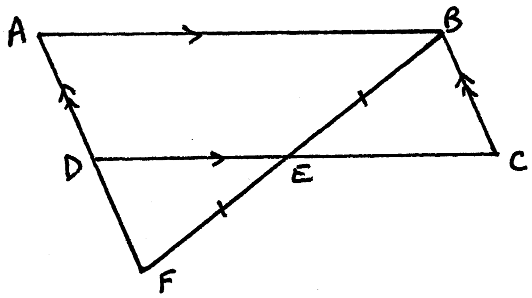
c) Given that $f'(x) = 1 - \dfrac{2}{x}$ and the graph of $y = f(x)$ passes through the point $(e, -2)$ find $f(x)$ 3

Question 5 – 12 marks (Start a new booklet) **Marks**

a) The remainder when $x^3 + ax^2 - 3x + 5$ is divided by $(x+2)$ is 11. Find the value of a. 2

b) Find the general solution of $2\cos x + \sqrt{3} = 0$ 2

c) In the diagram below $ABCD$ is a parallelogram, $BE = EF$ and AD is produced to F. 3



(i) Prove that ΔDEF is congruent to ΔBEC

(ii) Hence prove that $DE = \frac{1}{2}DC$

d) If $y = \dfrac{x}{\operatorname{cosec} x}$ find $\dfrac{dy}{dx}$ 1

e) Given that $\log_b\left(\dfrac{p}{q}\right) = 3$ and $\log_b\left(\dfrac{p}{r}\right) = 1.6$ 2

Find $\log_b\left(\dfrac{p}{r}\right)$

Question 6 – 12 marks (Start a new booklet) **Marks**

a) Use Newton's method to find a second approximation to the positive root of $x - 2\sin x = 0$. Take $x = 1.7$ as the first approximation. 2

b) One of the roots of the equation $x^3 + 6x^2 - x - 30 = 0$ is equal to the sum of the other two roots. Find the value of the three roots. 3

c) A spherical balloon is being inflated and its radius is increasing at a constant rate of 3cm/min. At what rate is its volume increasing when the radius of the balloon is 5cm? 3

d) From a point P due south of a vertical tower, the angle of elevation of the top of the tower is 20° and from a point Q due east of the tower it is 35°. If the distance from P to Q is 40 metres, find the height of the tower. 4

Question 7 – 12 marks (Start a new booklet) **Marks**

a) A particle is moving in simple harmonic motion. Its displacement x metres at any time t seconds is given by $x = 3\cos(2t+5)$ 6

(i) Find the period and amplitude of the motion.

(ii) Find the maximum acceleration of the particle.

(iii) Find the speed of the particle when $x = 2$

b) A projectile is fired with initial velocity V m/s at an angle of projection θ from a point O on horizontal ground. After 2 seconds it just passes over a 10 metre high wall that is 12 metres from the point of projection. 6

Assume acceleration due to gravity is 10m/sec^2. Assume the equations of displacement are $x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$

(i) Find V and θ to the nearest degree.

(ii) Find the maximum height reached by the projectile.

(iii) Find the range in the horizontal plane through the point of projection.

End of Paper

TABLE OF STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

Trial Solutions Mathematics 2007

Question 1

(a) $\frac{d}{dx}\log(xe^x) = \frac{d}{dx}(\ln x + \ln e^x)$

$= \frac{d}{dx}(\ln x + x)$

$= \frac{1}{x} + 1$

(b) $y = \ln(x+2)$

$y' = \frac{1}{x+2}$

$x = 0 \quad y' = \frac{1}{2}$

$\therefore m_T = \frac{1}{2} \quad m_N = -2$

$\therefore m = -2 \quad (0, \ln 2)$

$y - y_1 = m(x - x_1)$

$y - \ln 2 = -2(x - 0)$

$y - \ln 2 = -2x$

$2x + y - \ln 2 = 0$

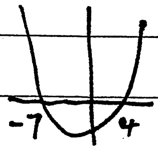
(c) $\frac{2x+3}{x-4} \geq 1 \quad x \neq 4$

$(x-4)(2x+3) \geq (x-4)^2$

$2x^2 - 5x - 12 \geq x^2 - 8x + 16$

$x^2 + 3x - 28 \geq 0$

$(x+7)(x-4) \geq 0$



$x > 4$

$x \leq -7$

(d) A $(3, -1)$ B $(9, 2)$ $-5 : 2$

$P = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

$= \left(\frac{-5.9 + 2.3}{-5+2}, \frac{-5.2 + 2.-1}{-5+2}\right)$

$= (13, 4)$

(e) $\lim_{x \to 0} \frac{\sin \frac{x}{5}}{2x}$

$= \lim_{x \to 0} \frac{1}{2} \cdot \frac{1}{5} \cdot \frac{\sin \frac{x}{5}}{\frac{x}{5}}$

$= \frac{1}{10} \lim_{x \to 0} \frac{\sin \frac{x}{5}}{\frac{x}{5}}$

$= \frac{1}{10}$

Question 2

(a) $\int \frac{19\,dx}{4+8x^2} = \frac{19}{8}\int \frac{dx}{\frac{1}{2}+x^2}$

$= \frac{19}{8} \cdot \frac{1}{\frac{1}{\sqrt{2}}} \tan^{-1} \frac{x}{\frac{1}{\sqrt{2}}} + C$

$= \frac{19\sqrt{2}}{8} \tan^{-1} \sqrt{2}x + C$

(b) $\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}$

$x + 2y = 5$ | $x - 3y = -3$
$2y = -x + 5$ | $3y = x + 3$
$y = -\frac{1}{2}x + \frac{5}{2}$ | $y = \frac{1}{3}x + 1$

$\therefore m_1 = -\frac{1}{2} \quad m_2 = \frac{1}{3}$

$\tan\alpha = \frac{-\frac{1}{3} - \frac{1}{3}}{1 + \frac{-1}{2}\cdot\frac{1}{3}}$

$= \frac{-\frac{5}{6}}{\frac{5}{6}}$

$= -1$

$\therefore \tan\alpha = -1$

$\alpha = 135$

$\therefore$ acute angle $= 45^\circ$

(c) $\frac{d}{dx}\left(\cos^{-1}(2\cos^2 x - 1)\right)$

$= \frac{d}{dx}\left(\cos^{-1}(\cos 2x)\right)$

$= \frac{d}{dx}(2x)$

$= 2$

(d) $\int_0^2 \frac{2x}{\sqrt{x^2+1}}\,dx$

let $u = x^2 + 1$ | $x = 2 \quad u = 5$
$du = 2x\,dx$ | $x = 0 \quad u = 1$

$\therefore I = \int_1^5 \frac{du}{\sqrt{u}}$

$= \int_1^5 u^{-\frac{1}{2}}\,du$

$= \left.\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^5$

$= \left.2u^{\frac{1}{2}}\right]_1^5$

$= 2\sqrt{5} - 2\sqrt{1}$

$= 2\sqrt{5} - 2$

(e) $y = x^2 \tan 5x$

$y' = 2x \tan 5x + x^2 \cdot 5\sec^2 5x$

$= 2x \cdot \tan 5x + 5x^2 \cdot \sec^2 5x$

Question 3

(a) $y = 2\cos\frac{1}{3}x$

period $= \dfrac{2\pi}{\frac{1}{3}}$

$= 6\pi$

amp $= 2$

(b) $\log_3 14 = \dfrac{\log 14}{\log 3}$

$\doteqdot 2.4$

(c)(i) $\sqrt{3}\sin x - \cos x$

$r\sin(x-\alpha) = r(\sin x\cos\alpha - \cos x\sin\alpha)$

$= r\cos\alpha\sin x - r\sin\alpha\cos x$

$\therefore \sqrt{3}\sin x - \cos x = r\cos\alpha\sin x - r\sin\alpha\cos x$

$\therefore$ $r\cos\alpha = \sqrt{3}$ $\quad r^2\cos^2\alpha = 3$

$r\sin\alpha = 1$ $\quad r^2\sin^2\alpha = 1$

$r^2 = 4$

$\tan\alpha = \dfrac{1}{\sqrt{3}}$ $\quad r = 2$

$\alpha = \dfrac{\pi}{6}$

$\therefore \sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$

(ii) $\sqrt{3}\sin x - \cos x = 1$

$\therefore 2\sin\left(x - \frac{\pi}{6}\right) = 1$

$\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

$x - \frac{\pi}{6} = \frac{\pi}{3}, \frac{2\pi}{3}$

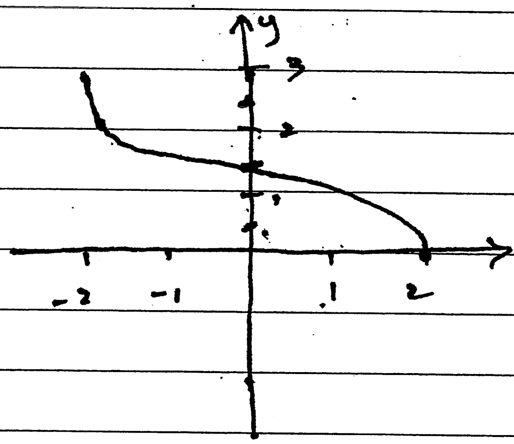
(d) $y = \dfrac{3}{\pi}\cos^{-1}\dfrac{x}{2}$

$-1 \leq \frac{x}{2} \leq 1$

$-2 \leq x \leq 2$

$0 \leq \cos^{-1}\frac{x}{2} \leq \pi$

$0 \leq \frac{3}{\pi}\cos^{-1}\frac{x}{2} \leq 3$



domain $\quad -2 \leq x \leq 2$

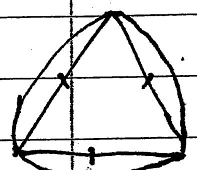
range $\quad 0 \leq y \leq 3$

(e) $\dfrac{d}{dx}(2\sin^{-1}x) = 2 \cdot \dfrac{1}{\sqrt{1-x^2}}$

$= \dfrac{2}{\sqrt{1-x^2}}$

Question 4

(a) i



$60^\circ = \frac{\pi}{3}$

length of one arc

radius $= 2r$

$\theta = \frac{\pi}{3}$

arc length $= r\theta$

$= 2\pi \cdot \frac{\pi}{3}$

3 arc lengths $= 3 \cdot \frac{2\pi r}{3}$

$= 2\pi r$

(ii) Area of one segment

$= \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2} (2r)^2 \left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$

$= \frac{1}{2} \cdot 4r^2 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= r^2 \left(\frac{2\pi}{3} - \sqrt{3}\right)$

Area of $\Delta = \frac{1}{2} ab \sin C$

$= \frac{1}{2} \cdot 2r \cdot 2r \cdot \sin\frac{\pi}{3}$

$= 2r^2 \cdot \frac{\sqrt{3}}{2}$

$= r^2\sqrt{3}$

Total area $= \sqrt{3} r^2 + 3\left[r^2\left(\frac{2\pi}{3} - \sqrt{3}\right)\right]$

$= \sqrt{3} r^2 + 3r^2\left(\frac{2\pi}{3} - \sqrt{3}\right)$

$= r^2\left[3\left(\frac{2\pi}{3} - \sqrt{3}\right) + \sqrt{3}\right]$

$= r^2(2\pi - 2\sqrt{3})$

% of circle $= \frac{r^2(2\pi - 2\sqrt{3})}{\pi r^2} \times 100\%$

$= 89.7\%$

$\approx 90\%$

(b)

~~Show true for n=1~~

$3^2 - 1 = 8 \quad \therefore$ divisible by 8

let statement be true for $n = k$

$\therefore \quad 3^{2k} - 1 = 8M$

Need to show true for $n = k+1$

$3^{2(k+1)} - 1 = 3^{2k+2} - 1$

$= 3^{2k} \cdot 3^2 - 1$

$= 3^{2k} \cdot 9 - 1$

$= 3^{2k} \cdot 9 - 9 + 8$

$= 9(3^{2k} - 1) + 8$

$= 9 \cdot 8M + 8$

$= 8(9M + 1)$

$\therefore$ divisible by 8

Since true for $n=1$ then must be true for $n = 1+1 = 2$ then $n = 2+1 = 3$ $\therefore$ true for all n $n \geq 1$

(c) $f'(x) = 1 - \frac{2}{x}$

$f(x) = \int 1 - \frac{2}{x}\, dx$

$= x - 2\ln x + c$

$y = -2$ when $x = e$

$-2 = e - 2\ln e + c$

$c = e$

$\therefore f(x) = x - 2\ln x + e$

Question 5

(a) $P(x) = x^3 + ax^2 - 3x + 5$

$P(-2) = 11$

$P(-2) = (-2)^3 + a(-2)^2 - 3.-2 + 5$

$= -8 + 4a + 6 + 5$

$= 3 + 4a$

$\therefore 3 + 4a = 11$

$4a = 8$

$\therefore a = 2$

b)

$2\cos x = -\sqrt{3}$

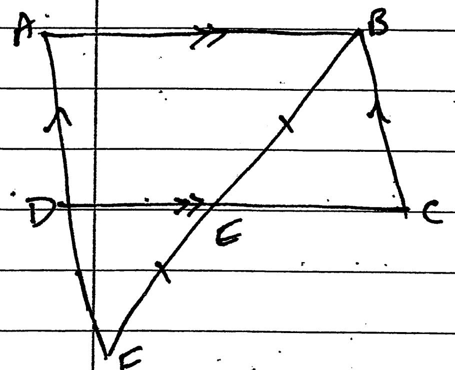
$\cos x = -\frac{\sqrt{3}}{2}$

$\therefore x = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

general solution

$x = 2n\pi \pm \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

c)



) $D\hat{E}F = B\hat{E}C$ (vertically op. angles)

$B\hat{C}E = E\hat{D}F$ (alternate angles on parallel lines)

$FE = BE$ (given)

$\therefore \triangle DEF \equiv \triangle CEB$ (AAS)

(ii) $DE = EC$ (corresponding sides in congruent $\triangle$s)

$\therefore E$ is midpoint of DC

$\therefore DE = \frac{1}{2} DC$

(d) $y = \frac{x}{\operatorname{cosec} x}$

$y = x \sin x$

$y' = \sin x + x \cos x$

(e) $\log\left(\frac{p}{q}\right) = 3 \quad \log\left(\frac{q}{r}\right) = 1.6$

$\frac{p}{r} = \frac{p}{q} \cdot \frac{q}{r}$

$\therefore \log\left(\frac{p}{r}\right) = \log\left(\frac{p}{q} \cdot \frac{q}{r}\right)$

$= \log\frac{p}{q} + \log\frac{q}{r}$

$= 3 + 1.6$

$= 4.6$

Question 6

(a) $x - 2\sin x = 0$

$f(x) = x - 2\sin x$

$f'(x) = 1 - 2\cos x$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$f(1.7) = -0.2833$

$f'(1.7) = 1.2577$

$$= 1.7 - \frac{-0.2833}{1.2577}$$

$$= 1.9253$$

(b) $x^3 + 6x^2 - x - 30 = 0$

$\alpha, \beta, \alpha+\beta$

$\alpha + \beta + \gamma = \frac{-b}{a}$

$\alpha + \beta + \alpha + \beta = -6$

$2\alpha + 2\beta = -6$

$\alpha + \beta = -3$

$\alpha = -3 - \beta$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$\alpha\beta + \alpha(\alpha+\beta) + \beta(\alpha+\beta) = -1$

$\alpha\beta + (\alpha+\beta)^2 = -1$

$\alpha\beta + 9 = -1$

$\alpha\beta = -10$

$\alpha\beta\gamma = -\frac{d}{a}$

$\alpha\beta(\alpha+\beta) = 30$

$\alpha\beta = -10$

$(-3-\beta)\beta = -10$

$\beta^2 + 3\beta - 10 = 0$

$(\beta+5)(\beta-2) = 0$

$\therefore \beta = -5, 2$

$\therefore$ the three roots.

$\therefore \alpha, \beta, \gamma$

$-5, 2, -3$

(c) $\frac{dV}{dt} = ?$ $r = 5$ $\frac{dr}{dt} = 3$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

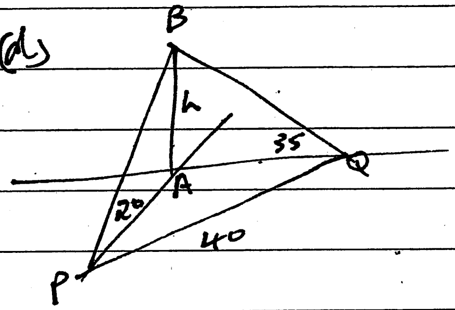
$= 4\pi r^2 \cdot \frac{dr}{dt}$

$r = 5$

$\frac{dV}{dt} = 4 \cdot \pi \cdot 5^2 \cdot 3$

$= 300\pi \text{ cm}^3/\text{min}$

(d)



$\tan 20 = \frac{h}{PA}$

$\tan 35 = \frac{h}{QA}$

$PA = \frac{h}{\tan 20}$

$QA = \frac{h}{\tan 35}$

$PA^2 + QA^2 = 40^2$

$\left(\frac{h}{\tan 20}\right)^2 + \left(\frac{h}{\tan 35}\right)^2 = 1600$

$h^2\left(\frac{1}{\tan^2 20} + \frac{1}{\tan^2 35}\right) = 1600$

$$h = \sqrt{\frac{1600}{\frac{1}{\tan^2 20} + \frac{1}{\tan^2 35}}}$$

$= \sqrt{166.87}$

$= 12.9$ m.

Question 7

(a) $x = 3\cos(2t+5)$

(i) period $= \frac{2\pi}{n}$ amplitude $= 3$

$= \frac{2\pi}{2}$

$= \pi$

(ii)

$x = 3\cos(2t+5)$

$\dot{x} = -6\sin(2t+5)$

$\ddot{x} = -12\cos(2t+5)$

max $\cos(2t+5) = -1$

$\ddot{x} = 12\,m/s^2$

(iii) $x = 2$

$3\cos(2t+5) = 2$

$\cos(2t+5) = \frac{2}{3}$

$2t + 5 = 5.44$

$2t = .44$

$t = 0.22$

$\dot{x} = -6\sin(2 \times 0.22 + 5)$

$= -6\sin 5.44$

$= 4.47\,m/s$

b) $x = V\cos\theta.t$ $y = V\sin\theta.t - 5t^2$

i) $t=2$ $y=10$ $t=2$ $x=12$

$12 = 2V\cos\theta$

$10 = 2V\sin\theta - 20$ $\therefore \tan\theta = \frac{30}{12}$

$30 = 2V\sin\theta$

$\frac{2V\sin\theta}{2V\cos\theta} = \frac{30}{12}$ $\theta = 68°12'$

$V\cos\theta = 6$ $V^2\cos^2\theta = 36$

$V\sin\theta = 15$ $V^2\sin^2\theta = 225$

$V^2(\sin^2\theta + \cos^2\theta) = 261$

$V^2 = 261$

$V = \sqrt{261}$

$= 16.6\,m/s$

(ii)

$\ddot{y} = -10$ $y = \int -10t + V\sin\theta$

$\dot{y} = \int -10\,dt$ $y = -5t^2 + V\sin\theta t$

$\dot{y} = -10t + c$ $y=0$ $t=0$ $c=0$

$\dot{y} = V\sin\theta$ $t=0$ $\therefore y = -5t^2 + V\sin\theta$

$\therefore \dot{y} = -10t + V\sin 68°12'$

$\dot{y} = 0$ for max height

$-10t + V\sin\theta = 0$

$t = \frac{V\sin\theta}{10}$

$y = V\sin\theta.t - 5t^2$

$= V\sin\theta . \frac{V\sin\theta}{10} - 5\left(\frac{V\sin\theta}{10}\right)^2$

$= \frac{V^2\sin^2\theta}{10} - 5\frac{V^2\sin^2\theta}{100}$

$= \frac{V^2\sin^2\theta}{20}$

but $V = 16.16$ $\theta = 68°12'$

$\therefore y = \frac{(16.16)^2\sin^2 68°12'}{20}$

$= 11.25\,m$

(iii) time of flight $= 2t$

$= \frac{V\sin\theta}{5}$

range $x = V\cos\theta.t$

$= V\cos\theta . \frac{V\sin\theta}{5}$

$= \frac{V^2\cos\theta.\sin\theta}{5}$

$= 17.99 \approx 18\,m$