St George Girls High School

Trial Higher School Certificate Examination

2008

Mathematics Extension 1

Total Marks – 84

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Attempt ALL questions.
- Begin each question on a new booklet
- Write your student number on each page
- All necessary working must be shown.
- Diagrams are not to scale.
- Board-approved calculators may be used.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

TABLE OF STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note $\ln x = \log_e x,\quad x > 0$

Question 1 – (12 marks) Marks

a) Find $\frac{d}{dx}(e^x \sin^{-1} x)$ 2

b) Evaluate $\int_0^2 \frac{dx}{(4+x^2)}$ 2

c) Solve for x: $\frac{x+3}{x-2} \geq 2$ 2

d) Find the general solution of $2\sin\theta + 1 = 0$ 2

e) Use the substitution $u = 1 + x$ to evaluate $\int_0^1 \frac{x}{\sqrt{(1+x)^3}}\,dx$ 4

Question 2 – (12 marks) Marks

a) The curves $y = x^2 - 4x + 2$ and $y = e^x + 1$ intersect at the point $(0, 2)$. Find the acute angle between the two curves at this point. 3

b) If $\log_a 2 = 0.75$, find the value of $\log_a 3$ correct to two decimal places. 2

c) Solve for x: $\ln(\ln x) = 0$ 2

d) (i) Sketch the graph of $y = 3\cos^{-1}\left(\frac{x}{2}\right)$ clearly indicating the domain, the range and any intercepts. 2

(ii) The region bounded by this curve, the x-axis and the y-axis, is rotated about the y-axis. Show that the volume of the solid so formed is given by $\pi\int_0^{\frac{3\pi}{2}} 4\cos^2\left(\frac{y}{3}\right)dy$ 1

(iii) Hence find the volume of this solid. 2

Question 3 – (12 marks) **Marks**

a) (i) Find the value of b if $x-2$ is a factor of $P(x) = 2x^3 + x^2 - bx + 6$ **1**

(ii) Hence solve for x: $P(x) = 0$ **2**

b) A particle is moving along the x-axis such that its velocity, v m/s, at displacement x metres, is given by $v = \sqrt{(5x - x^2)}$. Find the acceleration of the particle when $x = 4$ **2**

c) Use mathematical induction to prove that

$(1 \times 1!)+(2 \times 2!)+(3 \times 3!)+...+(n \times n!) = (n+1)!-1$ for all positive integers n. **3**

d) (i) Write down the co-efficient of x^r in the expansion of $(5+2x)^{12}$ in simplest terms. **1**

(ii) Hence find the greatest co-efficient in the expansion of $(5+2x)^{12}$ **3**

Question 4 – (12 marks) **Marks**

a) (i) Find the largest possible domain of positive values for which the function $f(x) = x^2 - 4x + 9$ has an inverse which is a function. **1**

(ii) Find $f^{-1}(x)$ clearly stating the domain and range. **2**

(iii) Find $f^{-1}[f\,(2 - a^2)]$ **1**

b) Jenny borrowed \$400 000 over 30 years at 6.6%pa reducible monthly. If the outstanding balance at the end of n months is $\$B_n$ and the monthly repayment is $\$R$

(i) Show that $B_n = 400\,000(1.0055)^n - \frac{R(1.0055^n - 1)}{0.0055}$ **2**

(ii) Find the value of R required to repay the loan and interest over 30 years. **1**

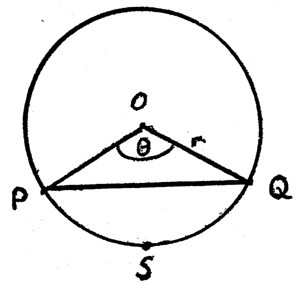
(iii) Before making the first repayment, Jenny decides to increase her monthly repayments to \$2 800. What time period is required to pay out the loan? **3**

c) The temperature of a cooling body, T° C, at time t minutes, is given by $T = 20 + 40\,e^{-0.04t}$. At what rate is the temperature changing when the temperature is 30° C? **2**

Marks

Question 5 – (12 marks)

a) In the diagram below, the points P and Q lie on the circle centre O of radius r. The chord PQ divides the sector $OPSQ$ into two regions of equal area.



(i) Show that $\theta = 2\sin\theta$ 2

(ii) The first approximation to the solution of the equation $\theta - 2\sin\theta = 0$ is $\theta = 1.91$ radians. Use one application of Newton's method to find a better approximation correct to 4 decimal places. 2

b) A ladder, 12 metres long, leans against a vertical wall with its lower end on horizontal ground. The lower end is slipping away from the wall at 3m/s. Find the rate at which the upper end is slipping down the wall when the lower end is 7.2 m from the wall. 5

c) The velocity, v m/s, of a particle moving in a straight line is given by $v = \frac{e^{-2x}}{2}$. Initially the particle was at the origin. Find its displacement after 2 seconds. 3

Marks

Question 6 – (12 marks)

a) A particle moves along a straight line such that its displacement, x metres, at time t seconds, is given by $x = 5\sin 2t + 5\cos 2t$

(i) Show that this motion is simple harmonic by showing that $\ddot{x} = -4x$ 2

(ii) Find the period of the motion. 1

(iii) Show that the velocity function can be written in the form $\dot{x} = R\cos(2t + \alpha)$ where $R > 0$ and $0 < \alpha < \pi$ 2

(iv) Find the first occasion when the velocity is $5\sqrt{2}$ m/s 1

b) The point $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$. The normal at P cuts the y-axis at Q. PQ is then produced to R such that $PQ = QR$.

(i) Show that the equation of the normal at P is $x + ty = at^3 + 2at$ 2

(ii) Find the co-ordinates of Q and R. 2

(iii) Deduce the equation of the locus of R 2

Question 7 – (12 marks) **Marks**

a) The top of a tower is viewed from two points, A and B. A is due East of the tower and B is due South of A. The angles of elevation of the top from A and B are 40° and 20° respectively. If the distance from A to B is 100m, find the height of the tower. **4**

b) A ball is kicked with velocity V m/s at an angle of 45° to the ground towards a person who will catch it 2 metres above ground level. At the instant the ball is kicked, the person is 20m from the kicker and is running away at a speed of 2m/s. The person continues to run away at this speed. Using $g = 10$ m/s²

(i) Derive the six equations of motion for the ball. **2**

(ii) Find the maximum height reached by the ball in terms of V. **2**

(iii) Find V correct to one decimal place. **4**

End of Paper

Ext 1 2008 Trial Solutions. Q5

Q1 (a) $\frac{d}{dx}\left(e^{x}\cdot\sin^{-1}x\right) = e^{x}\cdot\sin^{-1}x + \frac{1}{\sqrt{1-x^2}}\cdot e^{x}$ 1 1

$= e^{x}\left(\sin^{-1}x + \frac{1}{\sqrt{1-x^2}}\right)$

2

(b) $\int_0^2 \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$ 1

$= \frac{1}{2}\left(\tan^{-1}1 - 0\right)$ 1

$= \frac{1}{2}\times\frac{\pi}{4}$

$= \frac{\pi}{8}$

2

(c) $\frac{x+3}{x-2} \geq 2$ $[\times (x-2)^2]$ 1 or $\frac{x+3}{x-2} - 2 \geq 0$

$(x+3)(x-2) \geq 2(x-2)^2$ $\frac{x+3-2x+4}{x-2} \geq 0$

$x^2+x-6 \geq 2x^2-8x+8$ 1

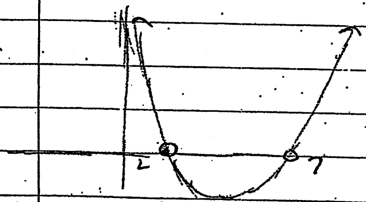
$\therefore x^2-9x+14 \leq 0$ $\frac{7-x}{x-2} \geq 0$

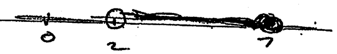
$(x-7)(x-2) \leq 0$

4

Critical values

2, 7





1

Test $x=3$

$4 \geq 0$ ✓

$x \neq 2$ $\therefore 2 < x \leq 7$ $2 < x \leq 7$ 1

(d) $2\sin\theta + 1 = 0$

$\sin\theta = -\frac{1}{2}$ $\quad \theta = (-1)^n \sin^{-1}\left(-\frac{1}{2}\right) + n\pi$

$\therefore \theta = \frac{7\pi}{6} + 2n\pi$ (n an integer) $\quad \theta = (-1)^n \times \left(\frac{-\pi}{6}\right) + n\pi$

or $\theta = \left(\frac{-\pi}{6}\right) + 2n\pi$

(e) let $u = 1 + x$

then $x = u - 1$

$1 = \frac{du}{dx}$

When $x = 0$, $u = 1$

$x = 1$, $u = 2$

$$\therefore \int_0^1 \frac{x\,dx}{\sqrt{(1+x)^3}} = \int_1^2 \frac{(u-1)}{u^{\frac{3}{2}}} dx.\frac{du}{dx}$$

$$= \int_1^2 \frac{(u-1)}{u^{\frac{3}{2}}} du$$

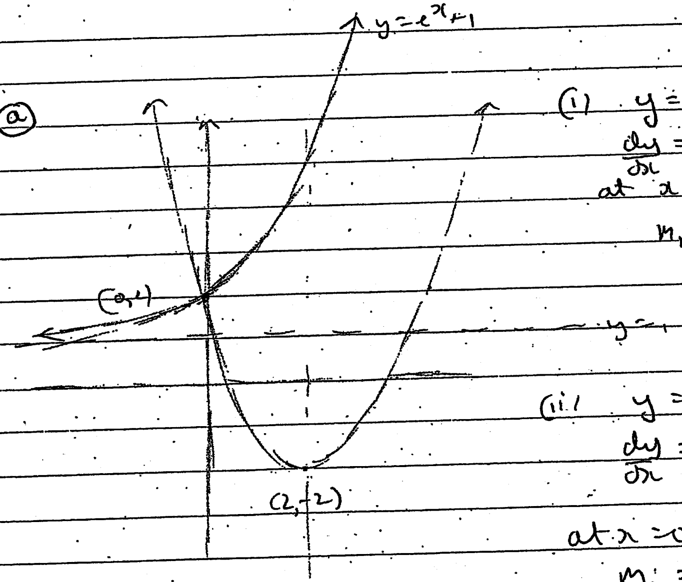
$$= \int_1^2 u^{-\frac{1}{2}} - u^{-\frac{3}{2}} du$$

$$= \left[2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right]_1^2$$

$$= \left(2\times\sqrt{2} + \frac{2}{\sqrt{2}}\right) - (2+2)$$

$$= 3\sqrt{2} - 4$$

Q2 (a)



(i) $y = x^2 - 4x + 2$

$\frac{dy}{dx} = 2x - 4$

at $x = 0$

$m_1 = -4$

(ii) $y = e^x + 1$

$\frac{dy}{dx} = e^x$

at $x = 0$

$m_2 = 1$

then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-4-1}{1 + -4\times1}\right|$

$= \frac{5}{3}$

$\therefore$ acute angle $\theta = 59° 2'$

(b) Given $\log_a 2 = 0.75$

then $\frac{\log_e 2}{\log_e a} = \frac{3}{4} \Rightarrow \log_e a = \frac{4}{3}\log_e 2$ --- (1)

also $\log_a 3 = \frac{\log_e 3}{\log_e a} \Rightarrow \frac{\log_e 3}{\frac{4}{3}\log_e 2}$ from (1)

$= 1.19$ (2 dec. places)

(c) $\ln(\ln x) = 0$

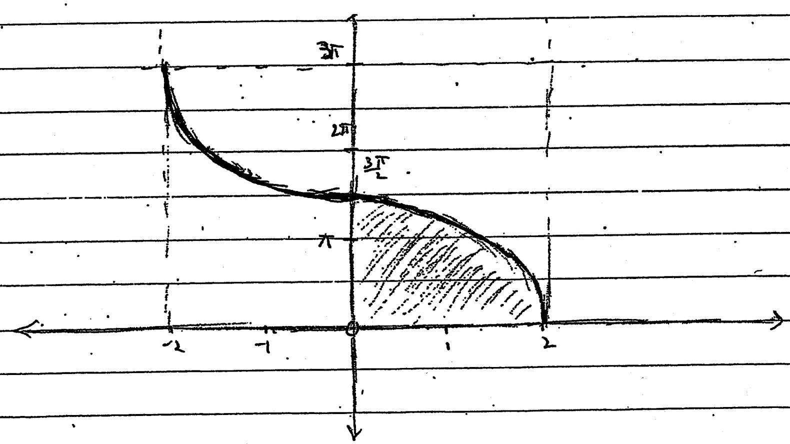
then $\ln x = 1$

$x = e^1$

(d) Domain: $-1 \le \frac{x}{2} \le 1$

(i) $-2 \le x \le 2$

Range: $0 \le 3\cos^{-1}(f(x)) \le 3\pi$



(ii) $V = \pi \int_0^{\frac{3\pi}{2}} (f[y])^2 \, dy$ $\qquad y = 3\cos^{-1}\left(\frac{x}{2}\right)$

$\frac{y}{3} = \cos^{-1}\left(\frac{x}{2}\right)$

$= \pi \int_0^{\frac{3\pi}{2}} \left[2\cos\left(\frac{y}{3}\right)\right]^2 dy$ $\qquad \cos\left(\frac{y}{3}\right) = \frac{x}{2}$

$= \pi \int_0^{\frac{3\pi}{2}} 4\cos^2\left(\frac{y}{3}\right) dy$ $\qquad \therefore 2\cos\left(\frac{y}{3}\right) = x$

(iii) $V = 2\pi \int_0^{\frac{3\pi}{2}} \left[\cos\left(\frac{2y}{3}\right) + 1\right] dy$ $\qquad 2\cos^2\left(\frac{y}{3}\right) = \cos\left(\frac{2y}{3}\right) +$

$= 2\pi \left[\frac{3}{2}\sin\left(\frac{2y}{3}\right) + y\right]_0^{\frac{3\pi}{2}}$

$= 2\pi\left[\left(\frac{3}{2} \times \underbrace{\sin\pi}_{=0} + \frac{3\pi}{2}\right) - 0\right]$

$= 3\pi^2$ cubic units.

Q3 (a) (i) $P(2) = 0$

$\therefore 2 \times 2^3 + 2^2 - 2b + 6 = 0$

$26 - 2b = 0$

So $b = 13$

(ii)
$$\begin{array}{r} 2x^2 + 5x - 3 \\ x-2 \overline{)\, 2x^3 + x^2 - 13x + 6} \\ 2x^3 - 4x^2 \\ \hline 5x^2 - 13x + 6 \\ 5x^2 - 10x \\ \hline -3x + 6 \\ -3x + 6 \end{array}$$

$\therefore P(x) = (x-2)(2x^2+5x-3)$

$= (x-2)(2x-1)(x+3)$

Let $P(x) = 0$

then $x = 2, \frac{1}{2}$ and -3

(b) $v = (5x - x^2)^{\frac{1}{2}}$; $a = \frac{d}{dx}\left[\frac{1}{2}v^2\right]$

$a = \frac{d}{dx}\left[\frac{1}{2}(5x - x^2)\right]$

$a = \frac{1}{2}(5 - 2x)$

When $x = 4$

$a = \frac{1}{2}(5 - 8)$

$= -\frac{3}{2}$ m/sec²

(c) (i) Let $n = 1$

then L.H.S. and R.H.S.

$(1 \times 1!) = 1$ $\qquad (1+1)! - 1 = 2! - 1$

$= 2 - 1$

$= 1$

True for $n = 1$

(ii) Assume result is true for $n = k$, k a positive integer.

Then $(1 \times 1!) + (2 \times 2!) + (3 \times 3!) + \dots + (k \times k!) = (k+1)! -$

(iii) Next value of n, $n = k+1$

has L.H.S.

$(1 \times 1!) + (2 \times 2!) + (3 \times 3!) + \dots + (k \times k!) + [(k+1) \times (k+1)!]$

$= (k+1)! - 1 + [(k+1) \times (k+1)!]$ [From (ii)]

$= (k+1)!\,[(k+1) + 1] - 1$

$= (k+1)! \times (k+2) - 1$

$= (k+2)! - 1$

$= [(k+1)+1]! - 1$

as required for R.H.S

(iv) From above, if result is true for $n = k$ then it is true for next n, $n = k+1$. Since true for $n=1$ then it is true for $n=2$ and by induction true for all n.

(2) (i) General term $\binom{12}{r} 5^{12-r} (2x)^r$

$\therefore$ co-efficient on x^r is $\binom{12}{r} 2^r . 5^{12-r}$

(ii) Require $\dfrac{t_{k+1}}{t_k} > 1$

then $$\frac{\binom{12}{k+1} 2^{k+1} . 5^{12-(k+1)}}{\binom{12}{k} 2^k . 5^{12-k}} > 1$$

$$\frac{\frac{12!}{(k+1)!\,[12-(k+1)]!}}{\frac{12!}{k!\,[12-k]!}} \times \frac{2}{5} > 1$$

$$\frac{k!\,(12-k)!}{(k+1)!\,[12-(k+1)]!} > \frac{5}{2}$$

$$\frac{12-k}{k+1} > \frac{5}{2}$$

$$24-2k > 5k+5$$

$$19 > 7k$$

gives $k < 2\frac{5}{7}$

so $t_{k+1} > t_k$ for $k = 0, 1$ and 2 $\Big\}$ $t_0 < t_1 < t_2$

and $t_{k+1} < t_k$ for $k = 3, 4, \ldots$ $\Big\}$ $t_3 > t_4 \ldots$

$t_0 < t_1 < t_2 < t_3 > t_4 > \ldots$

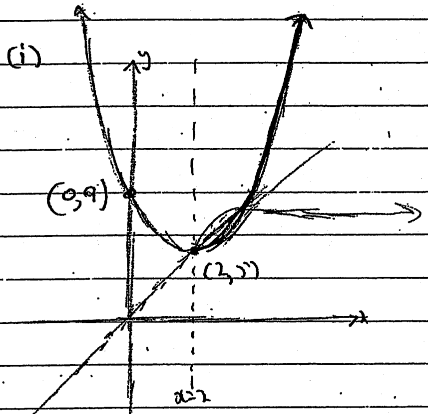
Greatest coeff: $t_3 = \binom{12}{3} 5^9 2^3$

$$= \frac{12!}{3!\,9!} \times 5^9 \times 2^3$$

$$= \frac{12\times11\times10}{3\times2\times1} \times 5^9 \times 2^3$$

$$= 11 . 5^{10} . 2^5$$

Q4 (a) (i)



axis $x = \dfrac{-(-4)}{2\times1}$

$x = 2$

largest possible domain $x \geq 2$

(ii) $y = x^2 - 4x + 4 + 5$

$y = (x-2)^2 + 5$ $\quad \begin{cases} x \geq 2 \\ y \geq 5 \end{cases}$

so $y - 5 = (x-2)^2$

$\sqrt{y-5} = x - 2$

$x = 2 + \sqrt{y-5}$

Inverse $f^{-1}(x) = 2 + \sqrt{x-5}$ $\quad \begin{cases} x \geq 5 & \text{Domain} \\ y \geq 2 & \text{Range} \end{cases}$

(iii) $f^{-1}[f(2-a^2)] = 2 + a^2$ Since $(2+a^2) \geq 2$ for all a

(b) (i) $B_1 = 400\,000 + 400\,000 \times \frac{.066}{12} - R$

$= 400\,000\,(1 + .0055) - R$

$B_1 = 400\,000\,(1.0055) - R$

Then $B_2 = B_1 + B_1 \times \frac{.066}{12} - R$

$= B_1\,[1.0055] - R$

$B_2 = 400000\,(1.0055)^2 - R \times 1.0055 - R$

and $B_3 = B_2 + B_2 \times \frac{.066}{12} - R$

$= B_2\,(1.0055) - R$

$B_3 = 400000\,(1.0055)^3 \underbrace{- R \times 1.0055^2 - R \times 1.0055 - R}_{G.P.}$

$= 400000\,(1.0055)^3 - R\left[\frac{1(1.0055^3 - 1)}{1.0055 - 1}\right]$

$B_3 = 400000\,(1.0055)^3 - R\left[\frac{1.0055^3 - 1}{.0055}\right]$

Sequence gives.

$B_n = 400000\,(1.0055)^n - R\left[\frac{1.0055^n - 1}{.0055}\right]$

(ii) Require $B_{360} = 0$

so $400000\,(1.0055)^{360} - R\left[\frac{1.0055^{360} - 1}{.0055}\right]$

$R = 400000\,(1.0055)^{360} \times \frac{.0055}{1.0055^{360} - 1}$

$= \$2554.64$

Repay \$2554.64 per month

(iii) let $R = 2800 \Rightarrow 400000\,(1.0055)^n - 2800\left(\frac{1.0055^n - 1}{.0055}\right) = 0$

$400000\,(1.0055)^n - \frac{2800}{.0055}(1.0055)^n + \frac{2800}{.0055} = 0$

$(1.0055)^n\left[400000 - \frac{2800}{.0055}\right] = -\frac{2800}{.0055}$

$(1.0055)^n = 4.66\ldots$

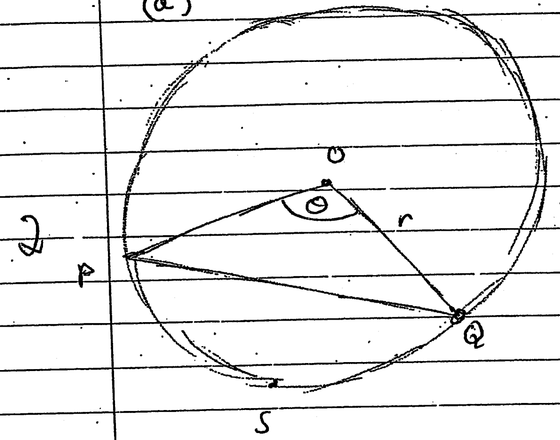
$n = \frac{\log_e(4.66\ldots)}{\log_e(1.0055)}$

$= 280.85\ldots$

Repaid in 281 months [just over 23 years]

Q5

(a)



(i) Area triangle POQ

$= \frac{1}{2}r^2\sin\theta$ $\frac{1}{2}$

1.

and area of segment PSQ

$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$ $\frac{1}{2}$

2 Since areas are equal $\frac{1}{2}$

$\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2\sin\theta$

$\frac{1}{2}r^2\theta = r^2\sin\theta$

$\theta = 2\sin\theta$

(ii) $f(\theta) = \theta - 2\sin\theta$

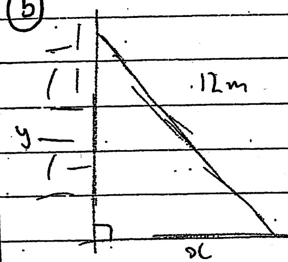
$f'(\theta) = 1 - 2\cos\theta$ 1

2 Given $x_1 = 1.91$ then $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

so $x_2 = 1.91 - \frac{f(1.91)}{f'(1.91)}$ 1

$x_2 = 1.8956$ [4 dec. pl]

(b)



Given $\frac{dx}{dt} = 3m/s$ 1

and $x^2 + y^2 = 12^2$ 1

5 $y = \sqrt{12^2 - x^2}$

$\frac{dy}{dx} = \frac{1}{2}(12^2 - x^2)^{-\frac{1}{2}} \times -2x$ 1

$$\frac{dy}{dx} = \frac{-x}{\sqrt{12^2-x^2}}$$

Reqr: $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$$= \frac{-x}{\sqrt{12^2-x^2}} \times 3$$

at $x = 7.2$ then $\frac{dy}{dt} = \frac{-7.2 \times 3}{\sqrt{12^2-7.2^2}}$

$= -2.25$ m/se

Slipping down at rate of 2.25 m/sec.

(c) $v = \frac{1}{2}e^{-2x}$, at $t=0$, $x=0$.

$\frac{dx}{dt} = \frac{\frac{1}{2}e^{-2x}}{1} \Rightarrow \frac{dt}{dx} = 2e^{2x}$

$t = e^{2x} + C$

at $t=0$, $x=0$

$0 = 1 + c$

$c = -1$

3

$\therefore t = e^{2x} - 1$

$t + 1 = e^{2x}$

$\log_e(t+1) = 2x \Rightarrow x = \frac{1}{2}\log_e(t+1)$

at $t=2$; $x = \frac{1}{2}\log_e 3$

Q6 (a) (i) $x = 5\sin 2t + 5\cos 2t$

$v = \dot{x} = 10\cos 2t - 10\sin 2t$

$a = \ddot{x} = -20\sin 2t - 20\cos 2t$

$= -4[5\sin 2t + 5\cos 2t]$

$\ddot{x} = -4x$

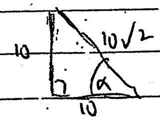
satisfies acceleration proportional to displacement, S.H.M

$\ddot{x} = -n^2 x$

(ii) $T = \frac{2\pi}{n}$ $\therefore T = \pi$ period.

(iii) $\dot{x} = 10\cos 2t - 10\sin 2t$

$\dot{x} = 10\sqrt{2}\left[\frac{1}{\sqrt{2}}\cos 2t - \frac{1}{\sqrt{2}}\sin 2t\right]$



$= 10\sqrt{2}\left[\cos 2t \cos\frac{\pi}{4} - \sin 2t \sin\frac{\pi}{4}\right]$

$\dot{x} = 10\sqrt{2}\cos\left(2t + \frac{\pi}{4}\right)$

$\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

$0 < \alpha < \pi$

(iv) let $\dot{x} = 5\sqrt{2} = 10\sqrt{2}\cos\left(2t + \frac{\pi}{4}\right)$

$\cos\left(2t + \frac{\pi}{4}\right) = \frac{1}{2}$

$t > 0$

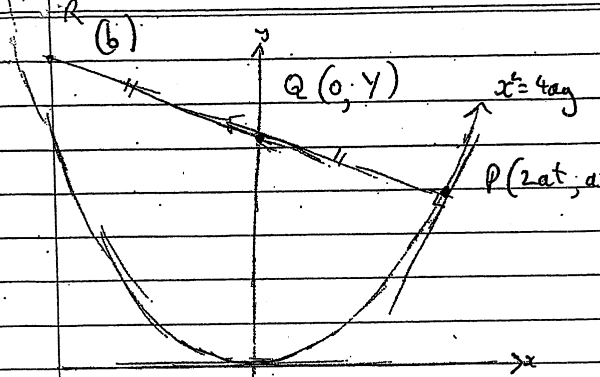
Then $2t + \frac{\pi}{4} = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$

$2t = \frac{\pi}{12}$

$t = \frac{\pi}{24}$

$\dot{x} = 5\sqrt{2}$ m/s first when $t = \frac{\pi}{24}$ se.

(b)



(i) $y = \dfrac{x^2}{4a}$

$\dfrac{dy}{dx} = \dfrac{x}{2a}$

at $x = 2at$

$m = t$

Equation of normal with $m = -\dfrac{1}{t}$

$y - at^2 = -\dfrac{1}{t}(x - 2at)$

$ty - at^3 = -x + 2at$

$x + ty = at^3 + 2at$.

(ii) let $x = 0$

then $0 + ty = at^3 + 2at$

$y = at^2 + 2a$

$\therefore Q\,[0, a(t^2+2)]$

R (X, Y) then $0 = \dfrac{X + 2at}{2}$, $at^2 + 2a = \dfrac{Y + a}{2}$

[midpoint of PR]

$X = -2at$ $\quad$ $2at^2 + 4a = Y + a$

$Y = 4a + at$

$\therefore R\,[-2at,\ 4a + at^2]$

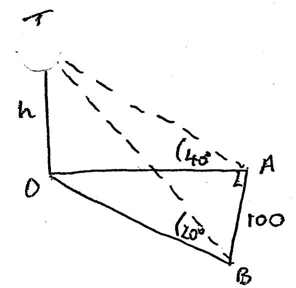
(iii) $X = -2at \quad \rightarrow \quad t = \dfrac{-X}{2a}$

$Y = a\,[4 + t^2]$

Sub $Y = a\left[4 + \dfrac{X^2}{4a^2}\right]$

$Y = 4a + \dfrac{X^2}{4a} \implies$ Equation $X^2 = 4a(Y - 4a)$

Q7 (a)



$\tan 40° = \dfrac{h}{OA}$

$OA = \dfrac{h}{\tan 40°}$

$\tan 20° = \dfrac{h}{OB}$

$OB = \dfrac{h}{\tan 20°}$

$OB^2 = OA^2 + AB^2$

$AB^2 = OB^2 - OA^2$

$100^2 = \dfrac{h^2}{\tan^2 20°} - \dfrac{h^2}{\tan^2 40°}$

$= h^2\left(\dfrac{1}{\tan^2 20°} - \dfrac{1}{\tan^2 40°}\right)$

$= h^2\left(\dfrac{\tan^2 40° - \tan^2 20°}{\tan^2 20° \tan^2 40°}\right)$

$h^2 = \dfrac{100^2 \tan^2 20° \tan^2 40°}{\tan^2 40 - \tan^2 20°}$

$h = 40.395\text{ m}$ correct to 3 dec. pl.

(b) (i)

Vertical

$\ddot{y} = -10$

$\dot{y} = -10t + C_1$

when $t = 0$ $\dot{y} = \dfrac{V}{\sqrt{2}}$

$\dot{y} = -10t + \dfrac{V}{\sqrt{2}}$

$y = \dfrac{-10t^2}{2} + \dfrac{Vt}{\sqrt{2}} + C_2$

when $t = 0$ $y = 0$

$y = \dfrac{-10t^2}{2} + \dfrac{Vt}{\sqrt{2}}$

$= -5t^2 + \dfrac{Vt}{\sqrt{2}}$

Horizontal

$\ddot{x} = 0$

$\dot{x} = C_3$

when $t = 0$ $\dot{x} = \dfrac{V}{\sqrt{2}}$

$\dot{x} = \dfrac{V}{\sqrt{2}}$

$x = \dfrac{Vt}{\sqrt{2}} + C_4$

when $t = 0$ $x = 0$

$x = \dfrac{Vt}{\sqrt{2}}$

(ii) Maximum height when $\dot{y} = 0$

$$-10t + \frac{V}{\sqrt{2}} = 0$$

$$t = \frac{V}{10\sqrt{2}}$$

Find y when $t = \frac{V}{10\sqrt{2}}$ for maximum height

$$y = \frac{V}{\sqrt{2}} \cdot \frac{V}{10\sqrt{2}} - 5\left(\frac{V}{10\sqrt{2}}\right)^2$$

$$= \frac{V^2}{20} - \frac{5V^2}{200}$$

$$= \frac{V^2}{40}$$

(iii) Require ball and catcher at the same x value when $y = 2$

Let time T elapse for catch to be taken

horizontal distance for ball is $x = \frac{VT}{\sqrt{2}}$

distance from kicker for catcher is $x = 20 + 2T$

$$\therefore \frac{VT}{\sqrt{2}} = 20 + 2T$$

$$\frac{VT}{\sqrt{2}} - 2T = 20$$

$$T\left(\frac{V}{\sqrt{2}} - 2\right) = 20$$

$$T = \frac{20\sqrt{2}}{V - 2\sqrt{2}}$$

At this time $y = 2$

$$y = \frac{V}{\sqrt{2}}t - 5t^2$$

$$2 = \frac{V}{\sqrt{2}}T - 5T^2$$

$$2 = \frac{V}{\sqrt{2}}\left(\frac{20\sqrt{2}}{V - 2\sqrt{2}}\right) - 5\left(\frac{20\sqrt{2}}{V - 2\sqrt{2}}\right)^2$$

$$2(V - 2\sqrt{2})^2 = 20V(V - 2\sqrt{2}) - 5(20\sqrt{2})^2$$

$$2(V^2 - 4\sqrt{2}V + 8) = 20V^2 - 40\sqrt{2}V - 4000$$

$$V^2 - 4\sqrt{2}V + 8 = 10V^2 - 20\sqrt{2}V - 2000$$

$$0 = 9V^2 - 16\sqrt{2}V - 2008$$

$$V = \frac{16\sqrt{2} \pm \sqrt{(16\sqrt{2})^2 + 4 \times 9 \times 2008}}{18}$$

$V > 0$

$V = 16.2$ m/s correct to 1 dec. pl.