Name: ____________________________

St George Girls High School

Trial Higher School Certificate Examination

2014

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen.
- Write your student number on each booklet.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- The mark allocated for each question is listed at the side of the question.
- Marks may be deducted for careless or poorly presented work.

Total Marks – 70

Section I – Pages 2 – 5
10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.
- Answer on the sheet provided.

Section II – Pages 6 – 9
60 marks

- Attempt Questions 11 – 14.
- Allow about 1 hour 45 minutes for this section.
- Begin each question in a new booklet.
- Show all necessary working in Questions 11 – 14.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Section I

10 marks **Marks**
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1. If $p(x) = (x+2)(x+k)$ and if the remainder is 12 when $p(x)$ is divided by $x - 1$, then $k =$

   (A) 2

   (B) 3

   (C) 6

   (D) 11

2. In the diagram drawn below $PB = 12$ cm and $BA = 20$ cm.

   P divides AB externally in the ratio

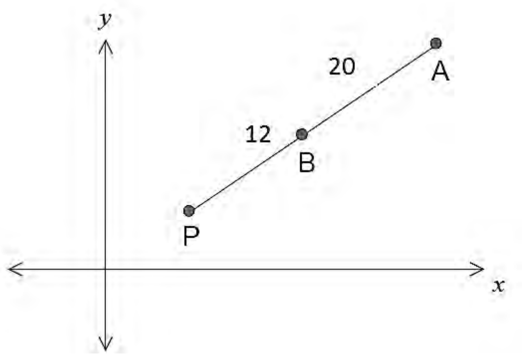


Diagram not to scale

   (A) $3 : 5$

   (B) $3 : 8$

   (C) $5 : 3$

   (D) $8 : 3$

Section I (cont'd)

3. If the function f is defined by $f(x) = x^5 - 1$, then f^{-1}, the inverse function of f, is defined by $f^{-1}(x) =$

   (A) $\frac{1}{\sqrt[5]{x}+1}$

   (B) $\frac{1}{\sqrt[5]{x+1}}$

   (C) $\sqrt[5]{x+1}$

   (D) $\sqrt[5]{x} - 1$

4. The coefficient of x^2 in the expansion of $(2x-3)^5$ is equal to:

   (A) -1080

   (B) -540

   (C) -10

   (D) 1080

5. Which of the following is always true of the perpendicular bisectors of non-parallel chords in the same circle?

   (A) The perpendicular bisectors never intersect

   (B) The perpendicular bisectors are always parallel

   (C) The perpendicular bisectors are always perpendicular to each other

   (D) The perpendicular bisectors always intersect at the centre of the circle

6. What is the domain and range of $y = 3\sin^{-1}(2x)$?

   (A) Domain : $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Range $-\frac{1}{3} \leq y \leq \frac{1}{3}$

   (B) Domain : $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Range $-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$

   (C) Domain : $-2 \leq x \leq 2$. Range $-\frac{1}{3} \leq y \leq \frac{1}{3}$

   (D) Domain : $-2 \leq x \leq 2$. Range $-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$

Section I (cont'd)

7. If $x = t^3 - t$ and $y = \sqrt{3t+1}$, then $\dfrac{dy}{dx}$ at $t = 1$ is:

(A) $\dfrac{1}{8}$

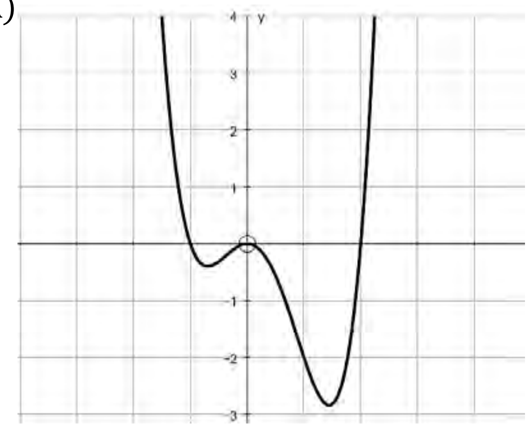
(B) $\dfrac{3}{8}$

(C) $\dfrac{3}{4}$

(D) $\dfrac{8}{3}$

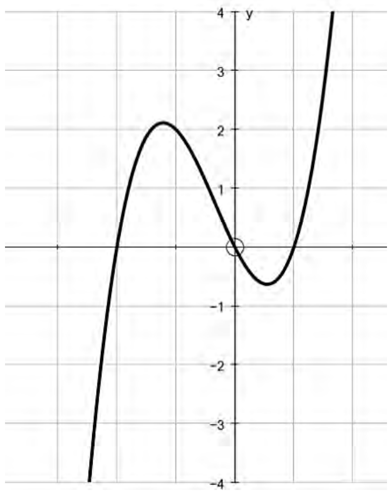
8. Which graph best represents $y = x^4 - x^3 - 2x^2$?

(A)

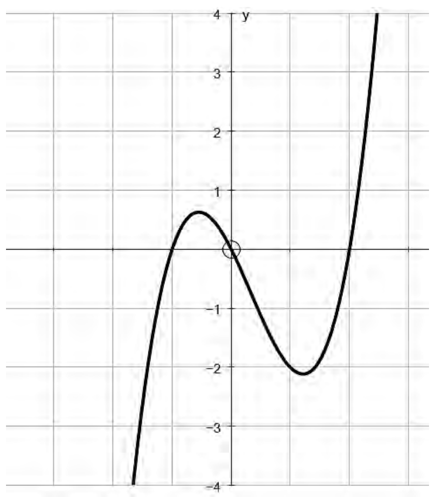


(B)

(C)



(D)



Section I (cont'd)

9. If $y = \sin^{-1}\left(\frac{5}{x}\right)$, $x > 5$, then $\frac{dy}{dx}$ is equal to

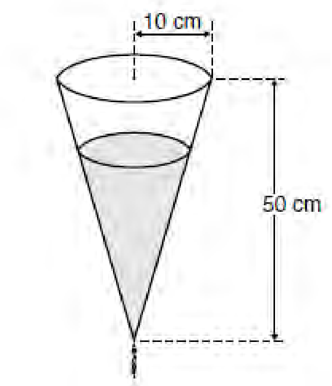
(A) $\frac{-5}{\sqrt{x^2-25}}$

(B) $\frac{x}{\sqrt{x^2-25}}$

(C) $\frac{-5}{x\sqrt{x^2-5}}$

(D) $\frac{-5}{x\sqrt{x^2-25}}$

10.



Water is draining from a cone-shaped funnel at the constant rate of 600 cm^3/ min.
The cone has height 50 cm and base radius 10 cm.
Let h cm be the depth of water in the funnel at time t min.
The rate of **decrease** of h, in cm/min, is given by

(A) 12

(B) $\frac{100\pi}{3}$

(C) $\frac{15000}{\pi h^2}$

(D) $24\pi h^2$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

a) The polynomial $4x^3 - 2x^2 + 3x - 5$ has roots α, β and γ. 1

Find $\alpha\beta + \alpha\gamma + \beta\gamma$.

b) Find the remainder when $P(x) = 3x^2 - 2x + 1$ is divided by $x - 3$. 2

c) The graphs of $y = 8 - x^3$ and $x - 2y + 13 = 0$ intersect at the point $(1, 7)$. 3

Find the size of the acute angle between the tangent to the curve and the line at the point of intersection. (answer to the nearest minute)

d) Find the exact value of $\cos[\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)]$. 1

e) Find $\lim_{x\to 0} \dfrac{\sin 3x}{2x}$. 1

f) Find $\int \cos^2 2x \, dx$. 2

g) Differentiate $\cos^{-1}(6x^2)$. 2

h) Solve $\dfrac{4}{6-x} \leq 1$. 3

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

a) Use the substitution $u = 5 - x^2$ to evaluate 3

$$\int_0^2 \frac{x}{(5-x^2)^3}\,dx\,.$$

b) What is the coefficient of x^3 in the expansion of $(4x - \frac{2}{x})^5$? 3

c) Prove the identity $\frac{\cos x - \cos 2x}{\sin 2x + \sin x} = \operatorname{cosec} x - \cot x$. 3

d) Use mathematical induction to prove that $9^n - 3$ is divisible by 6 for all positive integers n . 3

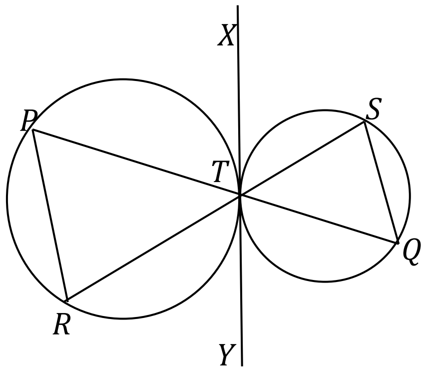
e) For the polynomial $P(x) = x^3 + 5x^2 + 17x - 10$

(i) Show it has a root that lies between 0 and 2 . 1

(ii) Use one application of Newton's method with an initial estimate of 1, to find a better approximation the root. 2

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

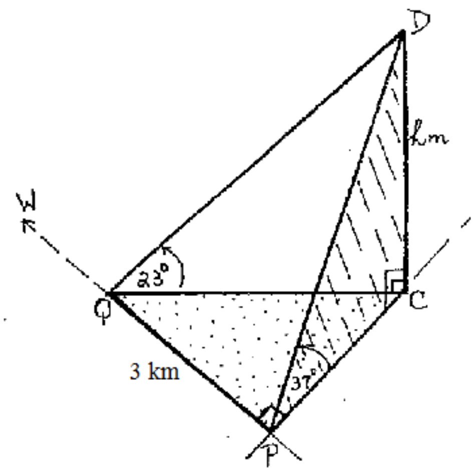
a) 3



Two circles touch externally at T. XY is the common tangent.

PTQ and RTS are straight lines. Prove that PR is parallel to SQ.

b) The angular elevation of a hill at a place P due south of it is $37°$ and at a place Q due west of P the elevation is $23°$ as shown in the diagram below. If the distance from P to Q is 3 km, find the height of the hill to the nearest 10 metres. 4



c) A particle is projected from a point O on a horizontal plane with an initial velocity of 60 metres/second at an angle of $30°$ to the horizontal. Assume acceleration due to gravity is 10 m/s^2 .

(i) Derive the equations (in exact form) for velocity and displacement of the particle in both the horizontal and vertical directions. 4

(ii) Find the range of the particle. 2

(iii) At the same time a second particle is projected in the opposite direction with an initial velocity of 50 metres/second from a point on the same horizontal level as O. Find the angle of projection of the second particle if the particles collide (to the nearest degree). 2

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

a) $P(2p, p^2)$ and $Q(2q, q^2)$ are two points on the parabola $x^2 = 4y$.

(i) Find the coordinates of M, the mid point of PQ. 1

(ii) Show $pq = -4$ if PQ subtends a right angle at the origin. 2

(iii) Using your answers to parts (i) and (ii), find the equation of the locus of M as P and Q move on the parabola if $\angle POQ = 90°$. 2

b) A particle moves in such a way that its displacement x cm from the origin O after time t seconds is given by:

$$x = \sqrt{3}\cos 3t - \sin 3t .$$

(i) Show that the particle moves in simple harmonic motion. 2

(ii) Evaluate the period of the motion. 2

(iii) Find the time when the particle first passes through the origin. 3

c) By equating the coefficient of x^n on both sides of the identity 3

$(1+x)^n(1+x)^n = (1+x)^{2n}$,

Show that $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots \binom{n}{n-1}^2 + \binom{n}{n}^2 = \dfrac{(2n)!}{(n!)^2}$.

End of Paper

2014 – EXT 1 TRIAL

MC

1) $P(1) = 12 \qquad 12 = 3(1+k)$

$\therefore k = 3$ B

2) $AP : PB = 32 : 12$

$= 8 : 3$ D

3) $f: y = x^5 - 1$

$f^{-1}: x = y^5 - 1 \qquad y = \sqrt[5]{x+1}$ C

$x + 1 = y^5$

4) $(2x-3)^5 = \sum_{i=0}^{5} {}^5C_i \, (2x)^{5-i}(-3)^i$

$x^2: \quad 5 - i = 2 \quad \therefore i = 3$

${}^5C_3 \, 2^2 (-3)^3 = 10 \times 4 \times -27$ A

$= -1080$

5) D

6) $y = 3\sin^{-1}(2x)$

$\frac{y}{3} = \sin^{-1}(2x)$ B

for $y = \sin^{-1} x$

$R: -\frac{\pi}{2} \le y \le \frac{\pi}{2}$ so $-\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2}$ $\qquad -\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

$D: -1 \le x \le 1 \qquad -1 \le 2x \le 1 \qquad -\frac{1}{2} \le x \le \frac{1}{2}$

7) $x = t^3 - t \qquad y = (3t+1)^{1/2}$ B

$\frac{dx}{dt} = 3t^2 - 1 \qquad \frac{dy}{dt} = \frac{3}{2}(3t+1)^{-1/2}$

$\frac{dy}{dx} = \frac{3}{2}(3t+1)^{-1/2} \times \frac{1}{3t^2-1}$ at $t = 1$ $\frac{dy}{dx} = \frac{3}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{3}{8}$

8) $x \to \infty \quad y \to \infty$
$x \to -\infty \quad y \to \infty$
$y = x^2(x^2 - x - 2)$
$= x^2(x-2)(x+1)$ $\therefore$ A.

9) $y = \sin^{-1}\left(\frac{5}{x}\right)$

Let $\frac{5}{x} = m$ $\qquad y = \sin^{-1} m$

$-\frac{5}{x^2} = \frac{dm}{dx}$ $\qquad \frac{dy}{dm} = \frac{1}{\sqrt{1-m^2}}$

$$\frac{dy}{dx} = \frac{dy}{dm} \times \frac{dm}{dx}$$
$$= \frac{1}{\sqrt{1-\left(\frac{5}{x}\right)^2}} \times -\frac{5}{x^2}$$
$$= \frac{-5}{x\sqrt{x^2-25}}$$

D

10) $\frac{dV}{dt} = 600 \qquad V = \frac{1}{3}\pi r^2 h$

where $5r = h$ $\qquad = \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 h$
$r = \frac{h}{5}$ $\qquad = \frac{\pi h^3}{75}$

$\frac{dV}{dh} = \frac{\pi h^2}{25}$

C

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$
$$= \frac{25}{\pi h^2} \times 600$$
$$= \frac{15000}{\pi h^2}$$

f) $\int \cos^2 2x \, dx = \frac{1}{2}\int (\cos 4x + 1) \, dx$

$= \frac{1}{2}\left[\frac{1}{4} \sin 4x + x\right] + C$

$= \frac{1}{8} \sin 4x + \frac{1}{2} x + C$

g) $y = \cos^{-1}(6x^2)$

let $m = 6x^2$ $\qquad y = \cos^{-1} m$

$\frac{dm}{dx} = 12x$ $\qquad \frac{dy}{dm} = -\frac{1}{\sqrt{1-m^2}}$

$$\frac{dy}{dx} = \frac{dy}{dm} \times \frac{dm}{dx}$$

$$= -\frac{1}{\sqrt{1-(6x^2)^2}} \times 12x$$

$$= \frac{-12x}{\sqrt{1-36x^4}}$$

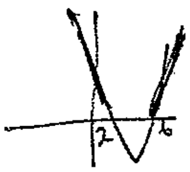
h) $\frac{4}{6-x} \leq 1 \qquad x \neq 6$

$4(6-x) \leq 1 \cdot (6-x)^2$

$0 \leq (6-x)^2 - 4(6-x)$

$(6-x)(6-x-4) \geq 0$

$(6-x)(2-x) \geq 0$



$x \leq 2$ or $x \geq 6$ BUT $x \neq 6$

$\therefore x \leq 2$ or $x > 6$

Q 11

a) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

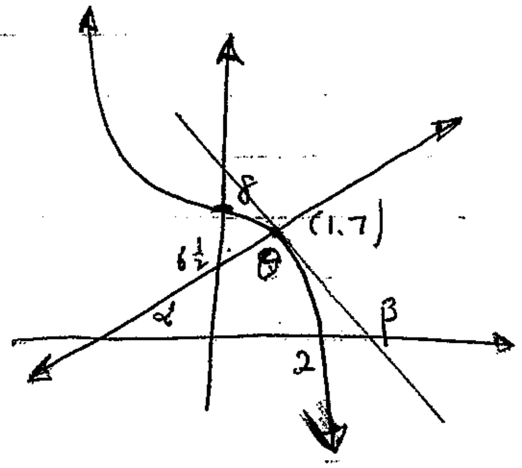
$= \frac{3}{7}$

b) $P(3) = 27 - 6 + 1$

$= 22$

Remainder is 22

c)



$\theta = \beta - \alpha$

$y = 8 - x^3$

$y' = -3x^2$

at $x = 1$ $\quad m = -3 \quad \tan\beta = -3$

$x - 2y + 13 = 0$

$m = \frac{1}{2} \quad \tan\alpha = \frac{1}{2}$

~~OR~~

$\tan\theta = \frac{-3 - \frac{1}{2}}{1 + (-3)(\frac{1}{2})}$

$= \frac{-7/2}{-\frac{1}{2}}$

$= -7$

$\theta = 81°52'$

$\theta = \tan^{-1}(-3) - \tan^{-1}(\frac{1}{2})$

$= 81°52'$

d) $\cos\left(\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right) = \cos\left(-\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}}$

e) $\lim_{x \to 0} \frac{\sin 3x}{2x} = \lim_{x \to 0} \frac{\sin 3x}{3x \times \frac{2}{3}}$

$= \frac{3}{2} \lim_{x \to 0} \frac{\sin 3x}{3x} = \frac{3}{2}$

Q12

a) $u = 5 - x^2$ $\quad x = 2 \quad u = 1$

$du = -2x\,dx$ $\quad x = 0 \quad u = 5$

$$\int_0^2 \frac{x}{(5-x^2)^3}\,dx = -\frac{1}{2}\int_0^2 \frac{-2x}{(5-x^2)^3}\,dx$$

$$= -\frac{1}{2}\int_5^1 \frac{du}{u^3}$$

$$= \frac{1}{2}\int_1^5 u^{-3}\,du$$

$$= \frac{1}{2}\left[-\frac{1}{2}u^{-2}\right]_1^5$$

$$= -\frac{1}{4}\left[5^{-2} - 1^{-2}\right]$$

$$= -\frac{1}{4}\left(\frac{1}{25} - 1\right)$$

$$= -\frac{1}{4} \times -\frac{24}{25}$$

$$= \frac{6}{25}$$

b) $\left(4x - \frac{2}{x}\right)^5 = \sum_{i=0}^{5} {}^5C_i\,(4x)^{5-i}\left(-\frac{2}{x}\right)^i$

x^3 : $\quad x^{5-i}\cdot(x^{-1})^i = x^3$

$5 - i - i = 3$

$i = 1$

Co-efficient $\quad {}^5C_1\,4^4\cdot(-2)^1 = 5 \times 256 \times -2$

$= -2560$

c) $$\frac{\cos x - \cos 2x}{\sin 2x + \sin x} = \frac{\cos x - [2\cos^2 x - 1]}{2\sin x\cos x + \sin x}$$

$$= \frac{-(2\cos^2 x - \cos x - 1)}{\sin x\,(2\cos x + 1)}$$

$$= \frac{-\cancel{(2\cos x + 1)}(\cos x - 1)}{\sin x\,\cancel{(2\cos x + 1)}}$$

$$= \frac{1 - \cos x}{\sin x}$$

$$= \operatorname{cosec} x - \cot x$$

(as required)

d) Step 1: for $n = 1$ $\quad 9^1 - 3 = 9 - 3$
$= 6$

$\therefore$ true for $n = 1$

Step 2: Assume true for $n = k$

$9^k - 3 = 6m$ (for some integer m)

Now for $n = k+1$

$$9^{k+1} - 3 = 9.9^k - 3$$
$$= 9(6m + 3) - 3$$
$$= 54m + 27 - 3$$
$$= 6(9m + 4)$$

As m is an integer $9m+4$ is integral

Step 3 $\therefore$ true for $n = k+1$ when true for $n = k$ and as true for $n = 1$ then true for $n = 2$ and all integers n.

e) (i) $P(x) = x^3 + 5x^2 + 17x - 10$

$P(0) = -10$ $\qquad$ $P(2) = 8 + 20 + 34 - 10 = 52$

$\therefore$ root between $x = 0$ and $x = 2$

So use $x = 1$ as an estimate

(ii) $x_1 = 1 - \frac{P(1)}{P'(1)}$

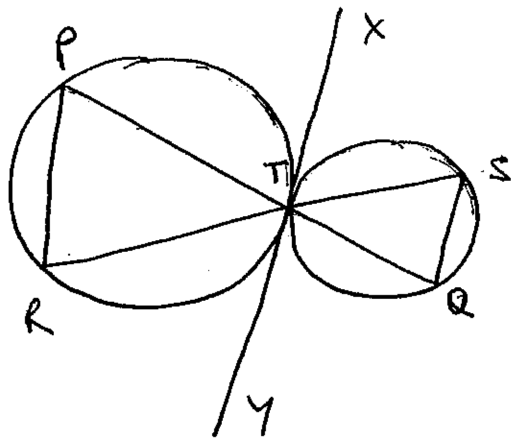
$= 1 - \frac{13}{30}$ $\qquad$ $P'(x) = 3x^2 + 10x + 17$

$P'(1) = 30$

$= \frac{17}{30}$

Q 13

a)



$\angle XTP = \angle YTQ$ (vertically opposite)

$\angle XTP = \angle PRT$ (angle in the alternate segment)

$\angle YTQ = \angle QST$ (angle in alternate segment)

So $\angle PRT = \angle QST$

$\therefore PR \parallel SQ$ (alternate angles equal)

b) In $\triangle PCD$ $\quad \frac{h}{PC} = \tan 37°$ $\qquad PC = \frac{h}{\tan 37°}$

$\triangle QCD$ $\quad \frac{h}{QC} = \tan 23°$ $\qquad QC = \frac{h}{\tan 23°}$

$QC^2 = QP^2 + PC^2$

$\frac{h^2}{\tan^2 23°} = 3^2 + \frac{h^2}{\tan^2 37°}$

Q14

a) $P(2p, p^2)$ $Q(2q, q^2)$ $a = 1$
$x^2 = 4y$

(i) $M\left(\frac{2p+2q}{2}, \frac{p^2+q^2}{2}\right)$ where M is midpoint PQ

$M\left((p+q), \frac{p^2+q^2}{2}\right)$

(ii) $m_{OP} = \frac{p^2}{2p}$ $m_{OQ} = \frac{q^2}{2q}$ m is gradient

$= \frac{p}{2}$ $= \frac{q}{2}$

When $\angle POQ = 90°$ $\frac{p}{2} \times \frac{q}{2} = -1$
$pq = -4$

(iii) $x = p+q$ $x^2 = p^2 + 2pq + q^2$
$= p^2 + q^2 - 8$

$y = \frac{p^2+q^2}{2}$ $\therefore x^2 + 8 = p^2 + q^2$

$y = \frac{x^2+8}{2}$

$y = \frac{1}{2}x^2 + 4$ $x^2 = 2(y-4)$

$\therefore$ locus is a parabola $V(0, 4)$
focal length $\frac{1}{2}$

b) $x = \sqrt{3}\cos 3t - \sin 3t$

(i) $\dot{x} = -3\sqrt{3}\sin 3t - 3\cos 3t$

$\ddot{x} = -9\sqrt{3}\cos 3t + 9\sin 3t$
$= -9(\sqrt{3}\cos 3t - \sin 3t)$

$$h^2\left[\frac{1}{\tan^2 23^\circ}-\frac{1}{\tan^2 37^\circ}\right]=9$$

$$h^2\left[\frac{\tan^2 37^\circ-\tan^2 23^\circ}{\tan^2 23\tan^2 37^\circ}\right]=9$$

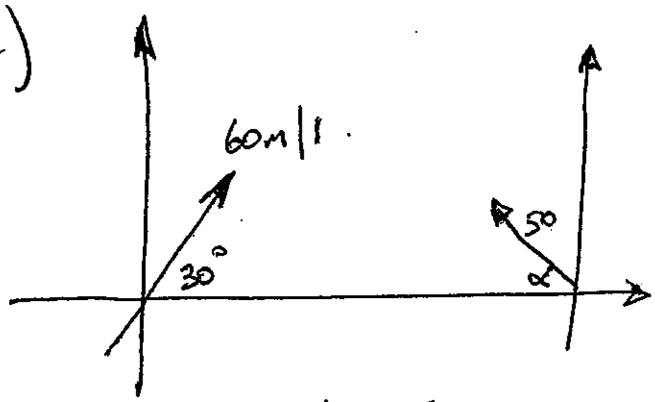
$$h^2 = 9\left[\frac{\tan^2 23^\circ\tan^2 37^\circ}{\tan^2 37^\circ-\tan^2 23^\circ}\right]$$

$$= 2.3753017$$

$$h = 1.5412014$$

$\therefore$ Hill is 1540 m high (to nearest 10 m)

c)

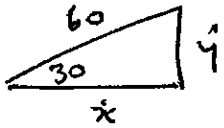


(i) $\ddot{x}=0$ $\qquad \ddot{y}=-10$

$\dot{x}=c$ $\qquad \dot{y}=-10t+c$

$\boxed{\dot{x}=30\sqrt{3}}$ $\qquad \boxed{\dot{y}=-10t+30}$

at $t=0$



$\frac{\dot{x}}{60}=\cos 30^\circ$ $\qquad \frac{\dot{y}}{60}=\sin 30^\circ$

$\dot{x}=60\times\frac{\sqrt{3}}{2}$ $\qquad \dot{y}=60\times\frac{1}{2}$

$=30\sqrt{3}$ $\qquad =30$

$x=30\sqrt{3}t+c$ $\qquad y=-5t^2+30t+c$

$t=0$ $x=0$ $\quad y=0$

$\boxed{x=30\sqrt{3}t}$

$\boxed{y=-5t^2+30t}$

(ii) $y=0$ $\qquad 5t^2-30t=0$

$5t(t-6)=0$

$\therefore t=0$ or 6 $\qquad t=6$ $\quad x=180\sqrt{3}$

$\therefore$ RANGE is $180\sqrt{3}$ m

(iii) $\dot{y}=-10t+c$

$\dot{y}=-10t+50\sin\alpha$

$y=-5t^2+50t\sin\alpha$ $\qquad$ So $30t=50t\sin\alpha$

$\sin\alpha=3/5$

$\alpha=37^\circ$ (nearest degree)

$$\ddot{x} = -9x$$

$\therefore$ Motion is SHM with $n = 3$ centre of motion $x = 0$.

(ii) $T = \frac{2\pi}{n}$

$= \frac{2\pi}{3}$

(iii) $\sqrt{3}\cos 3t - \sin 3t = 0$

$\sqrt{3}\cos 3t = \sin 3t$

$\sqrt{3} = \tan 3t$

$3t = \frac{\pi}{3}, \ldots$

$t = \frac{\pi}{9}$

First passes through origin after $\frac{\pi}{9}$ sec.

c) $(1+x)^n (1+x)^n = (1+x)^{2n}$

LHS $= (1+x)^n(1+x)^n$

$= ({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n)({}^nC_0 + {}^nC_1 x + \ldots {}^nC_n x^n)$

co-eff of x^n: ${}^nC_0\,{}^nC_n + {}^nC_1\,{}^nC_{n-1} + {}^nC_2\,{}^nC_{n-2} + \ldots + {}^nC_{n-1}\,{}^nC_1 + {}^nC_n\,{}^nC_0$

Now as ${}^nC_{n-r} = {}^nC_r$

$= ({}^nC_0)^2 + ({}^nC_1)^2 + \ldots + ({}^nC_{n-1})^2 + ({}^nC_n)^2$

coeff of x^n in RHS ${}^{2n}C_n = \frac{(2n)!}{n!\,n!}$

$\therefore \binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \frac{(2n)!}{n!\,n!}$