2011

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time- 5 minutes
- Working Time – 2 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question in a new booklet.

Total marks (**84**)

- Attempt Questions 1-7
- All questions are of equal value

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QUESTION 1 (12 MARKS) Begin a NEW booklet. **Marks**

a) Calculate $\lim_{x \to 0} \frac{\tan 3x}{2x}$. **1**

b) Solve $\frac{x}{2x-1} \geq 3$. **3**

c) If α, β, γ are the roots of the equation $4x^3 - 6x^2 - 3x + 8 = 0$, find the value of

(i) $\alpha\beta + \alpha\gamma + \beta\gamma$ **1**

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **2**

d) If $\log_5 10 = 2 \cdot 48$ find the exact value of $\log_5 4$. **2**

e) Use the substitution $u = x^2 - 3$ to evaluate $\int_2^6 \frac{x}{\sqrt{x^2 - 3}}\, dx$. **3**

QUESTION 2 (12 MARKS) Begin a NEW booklet.

a) (i) Prove that $\dfrac{1+\cos 2\theta}{\sin 2\theta} = \cot\theta$. **2**

(ii) Hence calculate the exact value of $\cot\dfrac{\pi}{12}$. **2**

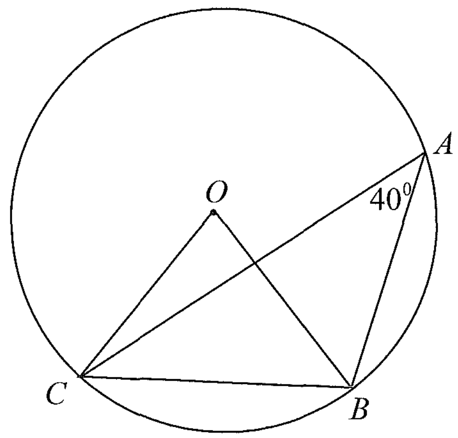
b) A polynomial is given by $P(x) = x^3 + ax^2 + bx + 8$. **3**

Determine the values of a and b if $(x+4)$ is a factor of $P(x)$

and 18 is the remainder when $P(x)$ is divided by $(x+1)$.

c) Find the exact value of $\displaystyle\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{6+3x^2}$. **3**

d) In the diagram below *A*, *B* and *C* are points on the circumference of a circle centre *O*. If $\angle CAB = 40^\circ$, find the size of $\angle OBC$ giving reasons for your answer. **2**



QUESTION 3 (12 MARKS) Begin a NEW booklet. **Marks**

a) Use one application of Newton's method to find a second approximation to the root of the equation $3\sin x - 2x = 0$, by taking $1\cdot 56$ as your first approximation. **3**

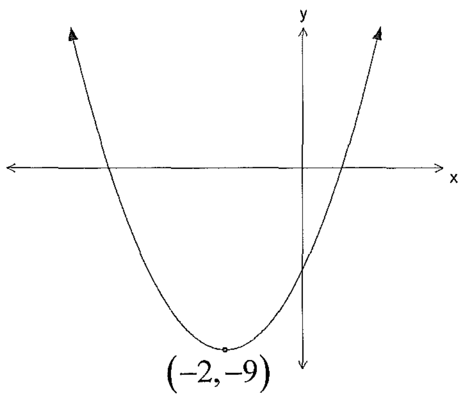
Write your answer correct to 2 decimal places.

b) Find the term independent of x in the expansion of **3**

$$\left(x^2 - \frac{1}{x^3}\right)^{10}.$$

c) Show that the function $f(x) = \dfrac{e^x}{4 - e^x}$ is monotonically increasing over the domain of x. **2**

d) The graph of $g(x) = x^2 + 4x - 5$ is shown in the diagram.



(i) Sketch the graph of the inverse function of $g(x) = x^2 + 4x - 5$, for $x \geq -2$. **1**

(ii) State the domain of the inverse function $g^{-1}(x)$. **1**

(iii) Find an expression for $y = g^{-1}(x)$ in terms of x. **2**

QUESTION 4 (12 MARKS) Begin a NEW booklet. **Marks**

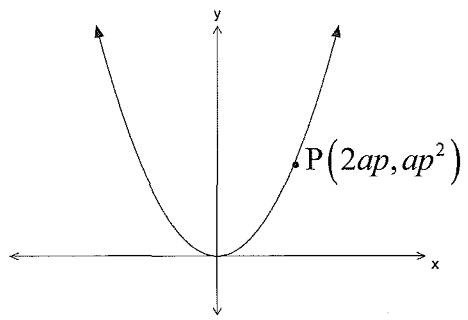
a) Find $\int \sin\theta \cos^2\theta \, d\theta$ by using the substitution $u = \cos\theta$. **2**

b) Evaluate $\sin\left[\tan^{-1}\left(-\sqrt{3}\right)\right]$. **2**

c) A spherical balloon is inflated at a constant rate of $12 \cdot 6 \text{cm}^3/\text{s}$. At what rate is the surface area increasing when the radius of the balloon is 12 cm? **3**

$SA = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$.

d) $P\left(2ap, ap^2\right)$ is a point on the parabola $x^2 = 4ay$ as shown in the diagram drawn below.



The equation of the normal to the curve at P is $x + py = 2ap + ap^3$. DO NOT prove this.

(i) Find the co-ordinates of the point Q where the normal at P meets the y–axis. **1**

(ii) Show that the co-ordinates of the point R, which divides the interval PQ externally in the ratio $1:2$ are given by $\left(4ap, ap^2 - 2a\right)$ **2**

(iii) Find the Cartesian equation of the locus of R. **2**

QUESTION 5 (12 MARKS) Begin a NEW booklet. **Marks**

a) Consider the function $y = 4\sin^{-1}\left(\frac{x}{3}\right)$

(i) State the domain and range of the function. **2**

(ii) Sketch the graph of the function showing all essential features. **1**

(iii) Calculate the gradient of the tangent to the curve at the point where $x = \sqrt{5}$. **2**

b) The area bounded by the curve $y = \sin 2x$,

the x-axis and the line $x = \frac{\pi}{4}$ is rotated about the x-axis. **3**

Calculate the exact volume of the solid of revolution.

c) The rate of growth of bacteria in a culture is given by $\frac{dN}{dt} = k(N - 800)$, where N is the number of bacteria and t is time, in seconds.

(i) Show that $N = 800 + Ae^{kt}$ is a solution of this equation. **1**

(ii) Initially there are 1 000 bacteria and five seconds later there are 1 700 bacteria present in the culture. Calculate the number of bacteria present after ten seconds. **3**

QUESTION 6 (12 MARKS) Begin a NEW booklet. **Marks**

a) (i) Express $\sqrt{3}\sin\theta - \cos\theta$ in the form $\mathrm{R}\sin(\theta-\alpha)$ **2**

where R is positive and α is acute.

(ii) Hence solve $\sqrt{3}\sin\theta - \cos\theta = -1$ for $0 \le \theta \le 2\pi$. **2**

b) Write the binomial expansion of $(3a-2b)^4$ in simplified form. **2**

c) Use the table of standard integrals to show that $\int_6^{10} \frac{dx}{\sqrt{x^2-36}} = \log_e 3$. **2**

d) Use the principle of Mathematical Induction to prove that for all positive integers

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\dots\dots\dots\dots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}.$$ **4**

QUESTION 7 (12 MARKS) Begin a NEW booklet. **Marks**

a) The acceleration of a particle P is given by $\ddot{x} = 4x\left(x^2 - 1\right)$, **3**

where x is the displacement of the particle from the origin, in metres, after t seconds. Initially the particle is at the origin, moving to the right with a velocity of $\sqrt{2}\ \mathrm{m/s}$.

Prove that the velocity of the particle is $v = -\sqrt{2}\left(x^2 - 1\right)$.

b) Consider the expansion of $\left(x + \dfrac{3}{x^2}\right)^8$ with the general term T_{k+1}.

(i) Show that $\dfrac{T_{k+1}}{T_k} = \dfrac{9-k}{k} \times \dfrac{3}{x^3}$ **3**

(ii) Hence calculate the greatest co-efficient in the expansion. **2**

c) A particle is moving in simple harmonic motion about a fixed point, with a velocity measured in metres/second, given by $v^2 = 21 + 4x - x^2$.

(i) Between what two points is the particle oscillating? **1**

(ii) What is the centre of the motion? **1**

(iii) Write the amplitude of the motion. **1**

(iv) Calculate the particle's maximum speed. **1**

2011 TRIAL HSC EXT 1 MATHEMATICS - SOLUTIONS

QUESTION 1.

a) $\displaystyle \lim_{x\to 0} \frac{\tan 3x}{2x} = \frac{3}{2} \lim_{x\to 0} \frac{\tan 3x}{3x}$

$= \frac{3}{2} \times 1$

$= \frac{3}{2}$

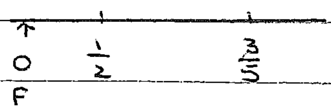
b) $\dfrac{x}{2x-1} \geq 3 \quad , \quad x \neq \frac{1}{2}$

$x(2x-1) \geq 3(2x-1)^2$

$3(2x-1)^2 - x(2x-1) \leq 0$

$(2x-1)(6x-3-x) \leq 0$

$(2x-1)(5x-3) \leq 0$



$\therefore \quad \frac{1}{2} < x \leq \frac{3}{5}$

c) (i) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= -\frac{3}{4}$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ $\qquad \alpha\beta\gamma = \frac{-d}{a} = -2$

$= -\frac{3}{4} \div -2$

$= \frac{3}{8}$

d) $\log_5 10 = 2 \cdot 48$

$\log_5 (2 \times 5) = 2 \cdot 48$

$\log_5 2 + \log_5 5 = 2 \cdot 48$

$\log_5 2 = 1 \cdot 48$

$2 \log_5 2 = 2 \cdot 96$

$\therefore \log_5 4 = 2 \cdot 96$

e) $\displaystyle\int_2^6 \frac{x}{\sqrt{x^2-3}}\,dx$

$u = x^2 - 3$

$du = 2x\,dx$

$x = 2, \quad u = 1$

$x = 6, \quad u = 33$

$= \dfrac{1}{2}\displaystyle\int_1^{33} u^{-1/2}\,du$

$= \dfrac{1}{2} \cdot 2\left[u^{\frac{1}{2}}\right]_1^{33}$

$= \sqrt{33} - 1$

QUESTION 2

a) (i) $\text{LHS} = \dfrac{1+\cos 2\theta}{\sin 2\theta}$

$= \dfrac{1 + 2\cos^2\theta - 1}{2\sin\theta\cos\theta}$

$= \dfrac{\cancel{2}\cos^2\theta}{\cancel{2}\sin\theta\cos\theta}$

$= \dfrac{\cos\theta}{\sin\theta}$

$= \cot\theta$

$= \text{RHS}$

(ii) $\cot\theta = (1+\cos 2\theta) \div \sin 2\theta$

$\cot\dfrac{\pi}{12} = \left(1 + \cos\dfrac{\pi}{6}\right) \div \sin\dfrac{\pi}{6}$

$= \left(1 + \dfrac{\sqrt{3}}{2}\right) \div \dfrac{1}{2}$

$= 2 + \sqrt{3}$

b) $P(-4) = 0$ ie $-64 + 16a - 4b + 8 = 0$

$16a - 4b = 56$

$4a - b = 14$ —— (1)

$P(-1) = 18$ ie $-1 + a - b + 8 = 18$

$a - b = 11$ —— (2)

(1) − (2) $3a = 3$

$\therefore$ $a = 1$, $b = -10$

c) $\displaystyle\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{6+3x^2} = \frac{1}{3}\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{2+x^2}$

$a^2 = 2$

$a = \sqrt{2}$

$$= \frac{1}{3}\cdot\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{x}{\sqrt{2}}\right]_{\sqrt{2}}^{\sqrt{6}}$$

$$= \frac{1}{3\sqrt{2}}\left(\tan^{-1}\sqrt{3} - \tan^{-1} 1\right)$$

$$= \frac{1}{3\sqrt{2}}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{36\sqrt{2}}$$

d) $\angle COB = 80^\circ$ (angle at the centre is twice the angle at the circumference, subtended by the same arc)

$\triangle OCB$ is isosceles as $OC = OB$ (radii)

$\therefore$ $\angle OBC = 50^\circ$ (base angle in isosceles triangle)

QUESTION 3

a) Let $f(x) = 3\sin x - 2x$ $\qquad x_1 = 1\cdot 56$

$f'(x) = 3\cos x - 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1\cdot 56 - \left(\frac{3\sin 1\cdot 56 - 2\times 1\cdot 56}{3\cos 1\cdot 56 - 2}\right)$$

$$\doteqdot 1\cdot 4989\ldots\ldots$$

$\therefore$ a second approx^n to the root is $1\cdot 50$, to 2 dec. pl.

b) $T_{k+1} = {}^nC_k \; a^{n-k} \cdot b^k$

$$= {}^{10}C_k \,(x^2)^{10-k} \cdot (-1)^k \cdot (x^{-3})^k$$

$20 - 2k - 3k = 0$

$k = 4$

$T_{k+1} = {}^{10}C_4 \,(-1)^4$

$\therefore$ the independent term is 210

c) If $f(x)$ is monotonically increasing for all x, then $f'(x) > 0$ for all values of x

$f(x) = \dfrac{e^x}{4 - e^x}$ $\qquad u = e^x \qquad v = 4 - e^x$

$\qquad u' = e^x \qquad v' = -e^x$

$$f'(x) = \frac{(4 - e^x)e^x + e^x \cdot e^x}{(4 - e^x)^2}$$

$$= \frac{4e^x}{(4 - e^x)^2}$$

CONT.

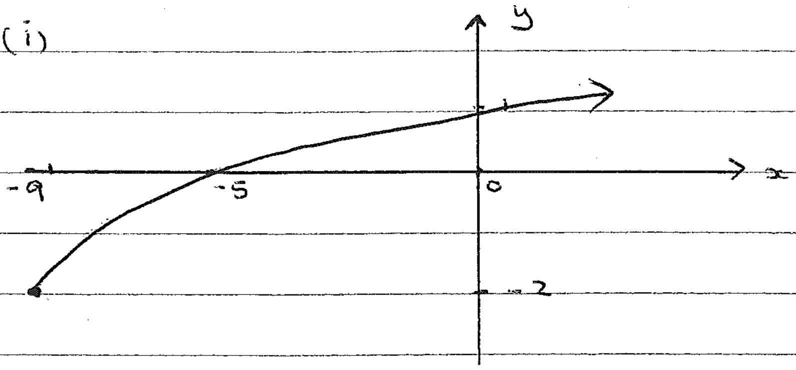
Since $e^x > 0$ for all x

and $(4-e^x)^2 > 0$ for all x

then $f'(x) > 0$

and the function is increasing over the domain of x

d) (i)



(ii) $x \geq -9$

(iii) $x = y^2 + 4y - 5$

$$x + 5 = y^2 + 4y$$

$$x + 5 + 4 = y^2 + 4y + 4$$

$$x + 9 = (y+2)^2$$

$$y + 2 = \pm\sqrt{(x+9)}$$

$\therefore$ $y = -2 + \sqrt{x+9}$ is the inverse fn

QUESTION 4

a) $\int \sin\theta \cos^2\theta \, d\theta$ $\qquad u = \cos\theta$

$\qquad du = -\sin\theta \, d\theta$

$= -\int u^2 \, du$

$= -\frac{1}{3} u^3 + c$

$= -\frac{1}{3} \cos^3\theta + c$

b) $\sin\left[\tan^{-1}(-\sqrt{3})\right] = \sin\left(-\frac{\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2}$

c) $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $\qquad V = \frac{4}{3}\pi r^3$

$12 \cdot 6 = 4\pi \times 12^2 \times \frac{dr}{dt}$ $\qquad \frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{12 \cdot 6}{4\pi \times 12^2}$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ $\qquad A = 4\pi r^2$

$\qquad \frac{dA}{dr} = 8\pi r$

$= 8\pi \times 12 \times \frac{12 \cdot 6}{4\pi \times 12^2}$

$\therefore$ surface area is increasing at the rate of $2 \cdot 1 \ cm^2/s$

d) over page

d) (i) when $x = 0$, $py = 2ap + ap^3$

$y = 2a + ap^2$

$\therefore$ Q is the point $(0, 2a + ap^2)$

(ii) $P(\overset{x_1}{2ap}, \overset{y_1}{ap^2})$ $Q(\overset{x_2}{0}, \overset{y_2}{2a + ap^2})$ $\overset{m}{-1} : \overset{n}{2}$

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right) = \left(\frac{0 + 4ap}{1}, \frac{-2a - ap^2 + 2ap^2}{1}\right)$$

$\therefore$ R is the point $(4ap, ap^2 - 2a)$

(iii) $x = 4ap$ — ①

$y = ap^2 - 2a$ — ②

from ① $p = \dfrac{x}{4a}$

sub in ② $y = a \cdot \dfrac{x^2}{16a^2} - 2a$

$y + 2a = \dfrac{x^2}{16a}$

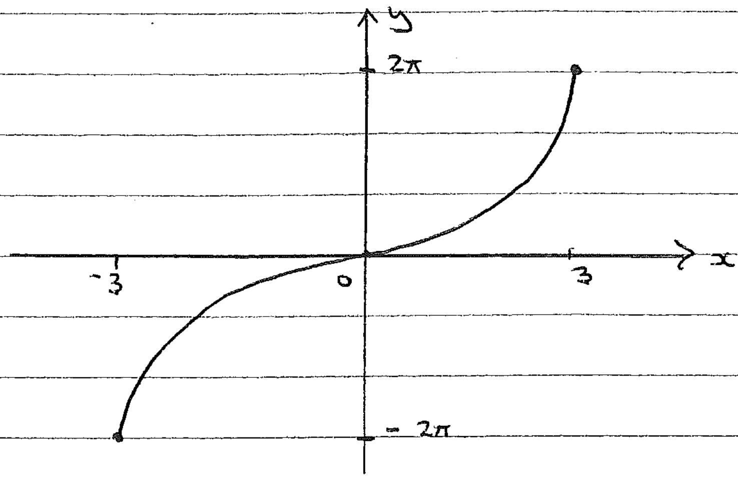
$\therefore$ $x^2 = 16a(y + 2a)$ is the eqn of the locus of R

QUESTION 5

a) (i) domain is $-3 \leq x \leq 3$

range is $-2\pi \leq y \leq 2\pi$

(ii)



(iii) $y = 4\sin^{-1}\left(\frac{x}{3}\right)$

$$\frac{dy}{dx} = 4 \cdot \frac{1}{\sqrt{9-x^2}}$$

when $x = \sqrt{5}$, $\frac{dy}{dx} = 4 \cdot \frac{1}{\sqrt{9-5}}$

$\therefore$ gradient of the tangent is 2

b) $V = \pi \int_0^{\pi/4} \sin^2 2x \, dx$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\sin^2 2x = \frac{1}{2}(1 - \cos 4x)$

$$= \frac{\pi}{2}\int_0^{\pi/4} (1 - \cos 4x)\, dx$$

$$= \frac{\pi}{2}\left[x - \frac{1}{4}\sin 4x \right]_0^{\pi/4}$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{4}\sin\pi - (0 - 0) \right]$$

$\therefore$ the volume is $\frac{\pi^2}{8}$ cubic units

c) (i) $N = 800 + Ae^{kt}$ ie $Ae^{kt} = N - 800$

$$\frac{dN}{dt} = k \cdot Ae^{kt}$$

ie/ $$\frac{dN}{dt} = k(N - 800)$$

(ii) $N = 800 + Ae^{kt}$

$t = 0,\ N = 1000$ so $1000 = 800 + A$

$A = 200$

$N = 800 + 200e^{kt}$

$t = 5,\ N = 1700$ $1700 = 800 + 200e^{5k}$

$$\frac{900}{200} = e^{5k}$$

$5k = \log_e 4 \cdot 5$

$k = \frac{1}{5} \log_e 4 \cdot 5$

when $t = 10$ $N = 800 + 200e^{2\log_e 4 \cdot 5}$

$= 800 + 200(4 \cdot 5)^2$

$= 4850$

$\therefore$ there are 4850 bacteria after 10 s.

QUESTION 6

a) (i) $\sqrt{3}\sin\theta - \cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$R\cos\alpha = \sqrt{3}$ —— ①

$R\sin\alpha = 1$ —— ②

② ÷ ① $\quad \tan\alpha = 1/\sqrt{3}$

$\alpha = \pi/6$

①^2 + ②^2 $\quad R^2 = 4$

$R = 2$

$\therefore \sqrt{3}\sin\theta - \cos\theta = 2\sin\left(\theta - \frac{\pi}{6}\right)$

(ii) $2\sin\left(\theta - \frac{\pi}{6}\right) = -1$ for $-\frac{\pi}{6} \leq \theta - \frac{\pi}{6} \leq 2\pi - \frac{\pi}{6}$

$\sin\left(\theta - \frac{\pi}{6}\right) = -\frac{1}{2}$

S	A
T ✓	C ✓

$\theta - \frac{\pi}{6} = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$\therefore \theta = 0, \frac{4\pi}{3}, 2\pi$

b) $(3a - 2b)^4$

$= (3a)^4 + 4(3a)^3(-2b) + 6(3a)^2(-2b)^2 + 4(3a)(-2b)^3 + (-2b)^4$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$= 81a^4 - 216a^3b + 216a^2b^2 - 96ab^3 + 16b^4$

c) $\int_6^{10} \frac{dx}{\sqrt{x^2 - 36}} = \left[\ln\left(x + \sqrt{x^2 - 36}\right)\right]_6^{10}$

$= \ln\left(10 + \sqrt{100 - 36}\right) - \ln 6$

$= \ln 18 - \ln 6$

$= \ln\left(18/6\right)$

$= \ln 3$

d) Step 1 Prove true when $n = 1$

$LHS = \frac{1}{1 \times 3}$ $\quad$ $RHS = \frac{1}{2+1}$

$= \frac{1}{3}$ $\quad$ $= \frac{1}{3}$

$\therefore$ result is true when $n = 1$

Step 2 Assume true when $n = k$

ie/ $S_k = \frac{k}{2k+1}$

Prove true when $n = k+1$

ie/ $S_{k+1} = \frac{k+1}{2k+3}$

Proof: $S_{k+1} = S_k + T_{k+1}$

$$= \frac{k}{(2k+1)} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$$

$$\therefore S_{k+1} = \frac{k+1}{2k+3}$$

Step 3 Since the result is true when $n = 1$, it is also true when $n = 1+1$ ie $n = 2$

Since result is true when $n = 2$, it is also true when $n = 2+1$ ie $n = 3$ etc

$\therefore$ result is true for all positive integers.

QUESTION 7

a) $$\ddot{x} = 4x(x^2-1)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x^3 - 4x$$

$$\frac{1}{2}v^2 = x^4 - 2x^2 + c$$

$x = 0,\ v = \sqrt{2}$

$$\frac{1}{2}\cdot 2 = \cancel{4} - 2\cdot 2 + c \qquad \Rightarrow c = 1$$

$$\frac{1}{2}v^2 = x^4 - 2x^2 + 1$$

$$\frac{1}{2}v^2 = (x^2-1)^2$$

$$v^2 = 2(x^2-1)^2$$

$$v = \pm\sqrt{2}(x^2-1)$$

when $x=0,\ v=\sqrt{2}$

$$\therefore v = -\sqrt{2}(x^2-1)$$

b) (i) $$\frac{T_{k+1}}{T_k} = \binom{8}{k} x^{8-k}\,.\,3^k\,.\,x^{-2k} \div \left[\binom{8}{k-1} x^{9-k}\,.\,3^{k-1}\,.\,x^{-2k+2}\right]$$

$$= \frac{\cancel{8!}}{k!\,(8-k)!} \times \frac{(k-1)!\,(9-k)!}{\cancel{8!}} \times 3 \times x^{-3}$$

$$= \frac{9-k}{k}\cdot\frac{3}{x^3}$$

(ii) $$\frac{T_{k+1}}{T_k} > 1$$

$$\frac{3(9-k)}{k} > 1$$

$$27 - 3k > k$$

$$4k < 27$$

$$k < 6\tfrac{3}{4}$$

$$\therefore k = 6$$

$$\text{co-efficient} = \binom{8}{6}.3^6$$

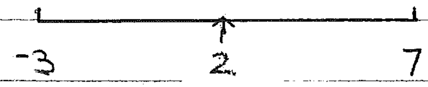
$\therefore$ greatest co-efficient is 20 412

c) (i) $v = 0$ at the end points of the motion

$$x^2 - 4x - 21 = 0$$
$$(x-7)(x+3) = 0$$
$$x = 7, -3$$

$\therefore$ particle is oscillating between $x = -3$ and $x = 7$

(ii) Centre is at $x = 2$



(iii) Amplitude is 5 m

(iv) Max speed occurs at the centre of the motion

when $x = 2$, $v^2 = 21 + 8 - 4$
$v^2 = 25$

$\therefore$ max speed is 5 m/s