Question 1 (4 + 3 + 2 + 3 = 12 Marks)

[4] a) Evaluate the following integrals correct to 2 Decimal places.

(i) $\int_0^2 \frac{x^3}{x^4 + 1}\,dx$

(ii) $\int_{-2}^{4} \left(1 + \frac{x}{2}\right)^5 dx$

[3] b) Solve the following for x : $\frac{4}{5 - x} \geq 1$

[2] c) Find the co - ordinates of the point which divides the interval AB with A(1,4) and B(5,2) externally in the ratio 1 : 3

[3] d) On the same diagram make a neat sketch of the graphs

$y = |x - 2|$

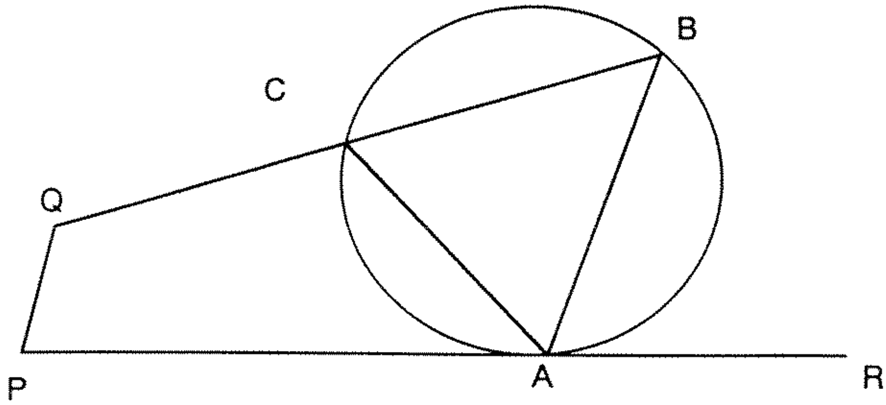
$y = \frac{3}{x}$

For what values of x is $|x - 2| < \frac{3}{x}$.

Question 2 (5 + 5 + 2 = 12 Marks)

[5] a) ABC is a triangle inscribed in a circle. PA is a tangent to the circle. PQ is drawn parallel to AB and meets BC produced to Q. Copy the diagram and prove APQC is a cyclic quadrilateral.

Diagram not to scale



[5] b) (i) Expand $(x-a)^3$

(ii) Given $P(x) = 0$ has a triple root at $x = a$,

that is $P(x) = (x-a)^3 Q(x)$ where $Q(x)$ is a another polynomial.

Show $P'(a) = 0$.

(iii) Hence or otherwise find the value of a and b if $x = 1$ is a triple root of

$$x^5 + x^4 + ax^3 + bx^2 - 5x + 1 = 0.$$

[2] c) Use Newton's method to find a second approximation to a root of

$$x - e^{-x} = 0,$$

given that $x = 0.5$ is the first approximation. Give the answer correct to three decimal places.

Question 3 (4 + 4 + 4 = 12 Marks)

[4] a) If the displacement x centimetres from 0 at time t seconds is given by

$$x = 5 + 4\sin^2 t$$

(i) Show that the motion is Simple Harmonic Motion,
(ii) Find the Centre and Amplitude of the motion,
(iii) Find the value of x for which the speed is maximum and determine this speed.

[4] b) Evaluate, leaving your answer in exact form

$$\int_{\frac{1}{8}}^{\frac{\sqrt{3}}{8}} \frac{dx}{\sqrt{1-16x^2}}$$

[4] c) From a group of seven men and five women a team of six is to be formed

(i) How many possible teams are there,

(ii) What is the probability that the team will have at least four men.

Question 4 (7 + 5 = 12 Marks)

[7] a) Molten plastic at a temperature of $250°C$ is poured into moulds to form car parts. After 20 minutes the plastic has cooled to $150°C$. If the temperature after t minutes is $T°C$, and if the temperature of the surroundings is $30°C$, then the rate of cooling is approximately given by

$$\frac{dT}{dt} = -k(T-30) \quad \text{, where k is a positive constant.}$$

(i) Show that a solution of this equation is $T = 30 + Ae^{-kt}$, where A is a constant.

(ii) Show that the value of the constant A is 220 .

(iii) Find the value of k correct to 3 significant figures.

(iv) The plastic can be taken out of the moulds when the temperature has dropped to $80°C$. How long after the plastic has been poured will this temperature be reached. Give the answer to the nearest minute.

[5] b) You are walking along a straight level fire track heading south at 6km/hr. The hut you are heading for is 12km along this track and 8km east into the bush. It is getting late and you wish to minimise the time to get to the hut. If you can travel through the bush at only 3km/hr, how far should you continue down this track (to the nearest metre) before you turn into the bush towards the hut ?

Question 5 (8 + 4 = 12 Marks)

[8] a) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

(i) Derive the equation of the tangent to the parabola at P,

(ii) Find the coordinates of the point of intersection T of the tangents to the parabola at P and Q,

(iii) You are given that the tangents at P and Q in Part (ii) intersect at an angle of 60°

Show that $p - q = \sqrt{3}(1 + pq)$

(iv) By evaluating the expression $x^2 - 4ay$ at T, find the locus of T when the tangents at P and Q intersect as given in Part (iii).

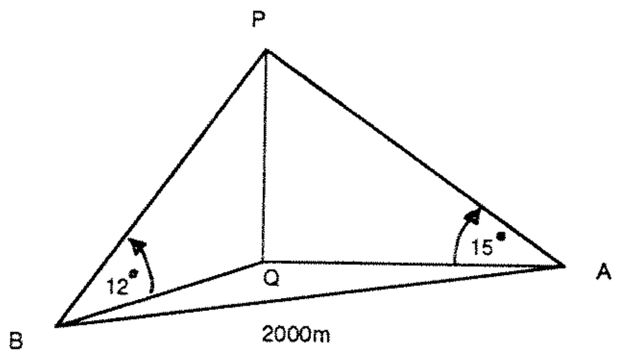
[4] b) Prove by Mathematical Induction that for $n \geq 1$

$$1^2 + 3^2 + \ldots + (2n-1)^2 = \frac{1}{3}(2n-1)n(2n+1)$$

Question 6 (2 + 4 + 6 = 12 Marks)

[2] a) Solve $\sin 2x + \sin x = 0$, $0 \leq x \leq 2\pi$

[4] b)

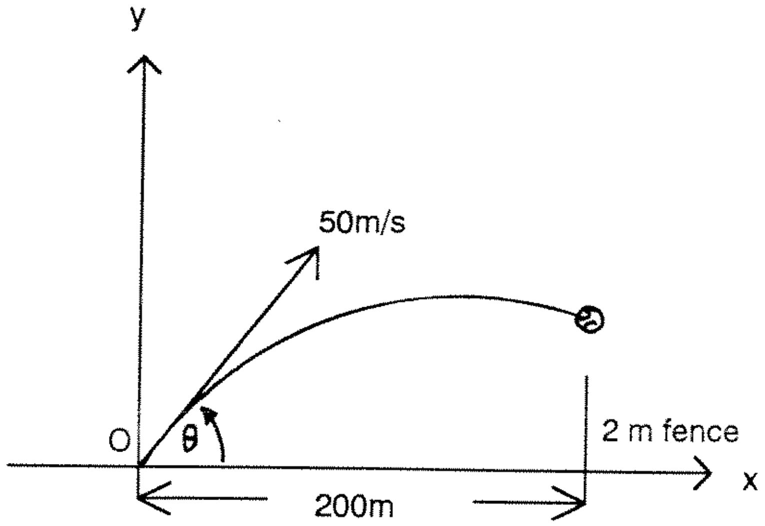


The angle of elevation of a tower PQ of height h metres at a point A due East of it is 15°. From another point B, due South of the tower the angle of elevation is 12°. The point A and B are 2000 metres apart on level ground. Find the height h of the tower to the nearest metre.

[6] c) (i) Draw a sketch of $y = \sin^{-1} x$. State the Domain and Range.

(ii) A region R is bounded by the curve $y = \sin^{-1} x$, the x axis and the line x=1. Use Simpson's rule with five function values to give an approximation for the area R. Give your answer correct to two decimal places.

Question 7 (4 + 2 + 2 + 4 = 12 Marks)
One method to score a home run in a baseball game is to hit the ball over the boundary fence on the full.



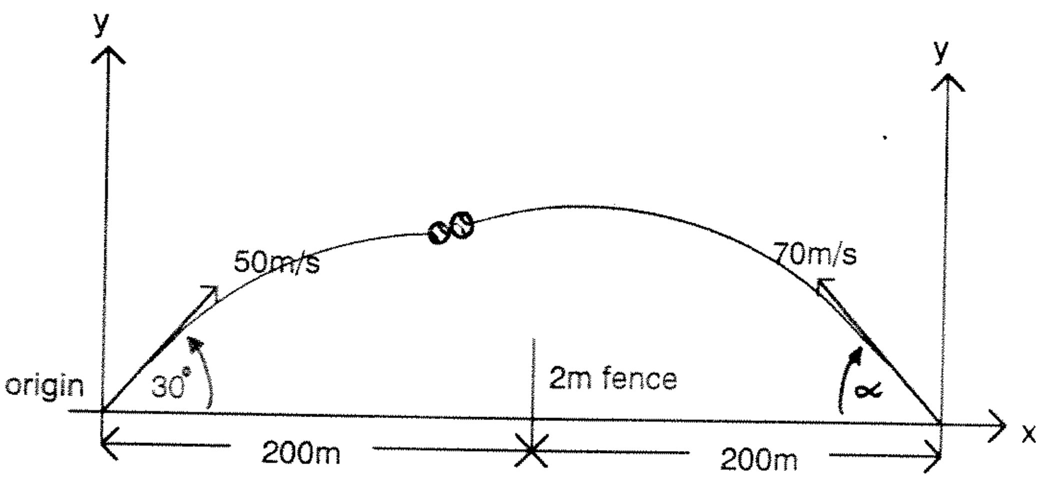
A ball is hit at 50 metres per second. The fence 200 metres away is 2 metres high.

[4] (i) Derive the equations of motion of the ball in the x and y directions. (neglect air resistance and acceleration due to gravity can be taken as 10ms^{-2}) .

[2] (ii) Show the ball would just clear the 2 metre boundary fence when
$80\tan^2\theta - 200\tan\theta + 82 = 0$, where θ is the angle of projection.

[2] (iii) In what range must theta (θ) lie to score a home run by this method.

[4] (iv) In an adjacent field another ball is hit at the same instant at 70ms^{-1} and the balls collide.
Assuming theta $\theta = 30°$, Find the angle alpha (α) the angle of projection of the second ball and the time and position where the balls collide.



THSC 1995 3UNIT Solutions

Question 1 (4 + 3 + 2 + 3 = 12 Marks)

a) (i) $\int_0^2 \frac{x^3}{x^4+1}\,dx = \frac{1}{4}\int_0^2 \frac{4x^3}{x^4+1}\,dx = \left[\frac{1}{4}\ln(x^4+1)\right]_0^2$

2M

$= \frac{1}{4}\ln 17 \doteqdot$

(ii) $\int_{-2}^{4}\left[1+\frac{x}{2}\right]^5 dx$ let $u = 1+\frac{x}{2}$ $\quad \frac{du}{dx} = \frac{1}{2}$ $\quad \therefore 2du = dx$

2M

If $x=-2$ $u=0$
$\quad x=4$ $u=3$

$\therefore 2\int_0^3 u^5\,du = 2\left[\frac{u^6}{6}\right]_0^3 = \frac{729}{3}$

b) Solve $\frac{4}{5-x} \geq 1$ $\times$ by $(5-x)^2$

$\therefore \frac{4(5-x)^2}{5-x} \geq (5-x)^2$

3M

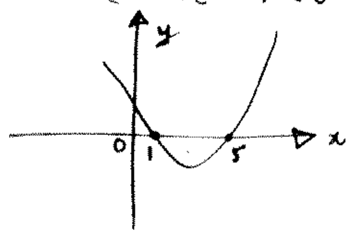
$\therefore 4(5-x) \geq 25-10x+x^2$

$20-4x \geq 25-10x+x^2$

$0 \geq 5-6x+x^2$

$\therefore x^2-6x+5 \leq 0$

$(x-5)(x-1) \leq 0$



$\therefore$ Soln $\{x : 1 \leq x \leq 5\}$

c) External ratio $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$
let $m:n$
be $m:-n$

$A(1, 4)$ (x_1, y_1)
$B(5, 2)$ (x_2, y_2)

$\left(\frac{1.5+(-3).1}{1-3}, \frac{1.2+(-3).4}{1-3}\right)$

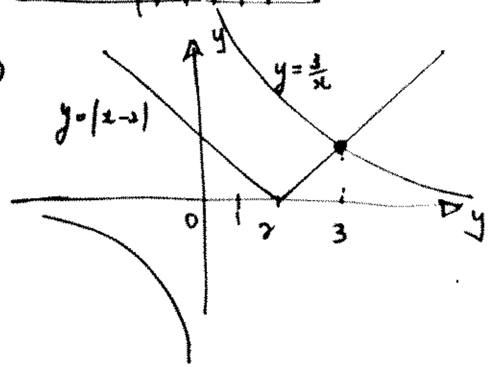
$m : n$
$1 : 3$

2M

$P(-1, 5)$

d)

3M



$x-2 = \frac{3}{x}$

$x^2-2x-3 = 0$

$(x-3)(x+1) = 0$

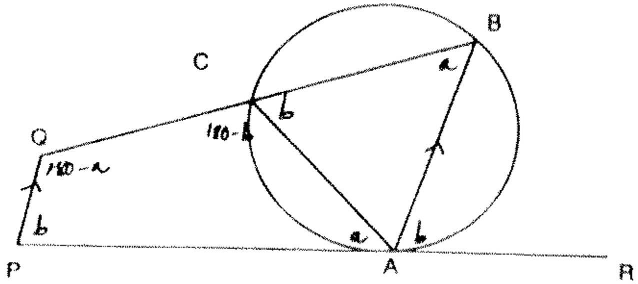
$\therefore x = 3, \cancel{-1}$

Soln $\{x : x < 3\}$

Question 2 (5 + 5 + 2 = 12 Marks)

a)

Diagram not to scale



Let $\angle PAC = a$, $\angle BAR = b$

5M

Now **Aim**: To Prove APQC is a cyclic quadrilateral

Proof:
1. $\angle PAC = \angle ABC$ angle in alternate segment
2. $\angle BAR = \angle ACB$ "
3. $\angle ACQ = 180 - \angle BAR$ as QCB is a straight line
4. $\angle PQC = 180 - \angle ABC$ as cointerior angles are supplementary

$\therefore$ APQC is a cyclic quadrilateral as opposite angles are supplementary

b) (i) $(x-a)^3 = (x-a)(x^2-2ax+a^2) = x^3-2ax^2+a^2x-ax^2+2a^2x-a^3$

1M

$= x^3-3ax^2+3a^2x-a^3$

(ii) $P(x) = (x-a)^3 Q(x)$ $\quad P'(x) = 3(x-a)^2 Q(x) + (x-a)^3 Q'(x)$

2M

$\therefore P'(x) = (x-a)^2[3Q(x) + (x-a)Q'(x)]$

$\therefore P'(a) = 0$

(iii) We know $P(1) = 0$ and $P'(1) = 0$

2M

$P(1) = 0 \rightarrow 1+1+a+b-5+1 = 0$ $\quad a+b = 2 \rightarrow a = 2-b$

$P'(x) = 5x^4+4x^3+3ax^2+2bx-5$

$P'(1) = 0 \rightarrow 5+4+3a+2b-5 = 0$ $\quad 3a+2b = -4$

$\therefore$ sub $a = 2-b$ into $3a+2b = -4$ gives $-b = -10$ $\therefore b = 10$

$\therefore a = 2-10 = -8$ **Soln** $a = -8$ $b = 10$

c) Newton's Method $x - e^{-x} = 0$ let $f(x) = x - e^{-x}$

$f'(x) = 1 + e^{-x}$

2M

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ with $x_n = 0.5$

$f(0.5) \doteqdot -0.10653$

$f'(0.5) \doteqdot 1.60653$

$\therefore x_{n+1} = 0.5 - \frac{f(0.5)}{f'(0.5)} \doteqdot 0.566$ (to 3DP)

Question 3 (4 + 4 + 4 = 12 Marks)

a) (i) $x = 5 + 4\sin^2 t \quad \frac{dx}{dt} = 8\sin t\cos t$

$\frac{d^2x}{dt^2} = 8[\sin t.-\sin t + \cos t.\cos t] = 8[1 - 2\sin^2 t]$

$= 8 - 16\sin^2 t \quad$ as $x - 5 = 4\sin^2 t$

2M $= 8 - 4(x-5) = 28 - 4x$

$\therefore \frac{d^2x}{dt^2} = -4(x-7) \quad$ if $y = x - 7 \quad \therefore \frac{d^2y}{dt^2} = -4y$

$\therefore$ SHM about $x = 7$. $\qquad$ SHM about $y = 0$

(ii) Centre $x = 7$, For amplitude $v = 0 \quad \therefore 8\sin t\cos t = 0$

$\therefore \sin t = 0$ or $\cos t = 0$

1M $t = 0, \pi, 2\pi, .. \quad t = \frac{\pi}{2}, \frac{3\pi}{2}, .. \qquad t = 0,\ x = 5$; $\ t = \frac{\pi}{2}\ x = 9$

5 — 7 — 9 $\qquad \therefore a = 2$

(iii) the speed is maximum at the centre ie $x = 7$

1M firstly when $t = \frac{\pi}{4} \quad \therefore v = 8\cos\frac{\pi}{4}\sin\frac{\pi}{4} = 4$ m/s

b) Evaluate, exact $\int_{\frac{1}{8}}^{\frac{\sqrt{3}}{8}} \frac{dx}{\sqrt{1-16x^2}}$ $\qquad$ let $u = 4x \quad \frac{du}{4} = dx$

$x = \frac{1}{8}\ u = \frac{1}{2}$

$x = \frac{\sqrt{3}}{8}\ u = \frac{\sqrt{3}}{2}$

4M $\therefore \frac{1}{4}\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\left[\sin^{-1} u\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{1}{4}\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right]$

$= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{1}{4}\cdot\frac{\pi}{6} = \boxed{\frac{\pi}{24}}$

c) Group 7M, 5W group of 6

1M (i) $^{12}C_6 = \frac{12!}{6!6!} = 924$

(ii) At least 4 men

3M
4M 2W $= {}^7C_4 \cdot {}^5C_2 = \frac{7!}{4!3!} \cdot \frac{5!}{3!2!} = 35$

5M 1W $= {}^7C_5 \cdot {}^5C_1 = \frac{7!}{5!2!} \cdot 5 = 21$

6M 0W $= {}^7C_6 \cdot {}^5C_0 = 7 \cdot 1 = 7$

$\qquad\qquad 63$

$\therefore$ Prob at least 4 men $\frac{63}{...} = \frac{3}{...}$

Question 4 (7 + 5 = 12 Marks)

a) $\frac{dT}{dt} = -k(T - 30)$ — (1)

(i) show $T = 30 + Ae^{-kt}$ is a soln.

one way LHS $\frac{dT}{dt} = -kAe^{-kt}$

2M RHS $-k(30 + Ae^{-kt} - 30) = -kAe^{-kt}$

$\therefore$ as LHS = RHS $T = 30 + Ae^{-kt}$ is a soln to (1)

1M (ii) $t = 0 \quad T = 250°C \quad \therefore 250 = 30 + Ae^0 \quad \therefore A = 220$

(iii) $t = 20 \quad T = 150°C \quad \therefore 150 = 30 + 220e^{-20k}$

2M $\therefore \frac{120}{220} = e^{-20k} \quad \therefore \ln\frac{6}{11} = -20k \quad \therefore k = \frac{-1}{20}\ln\frac{6}{11}$

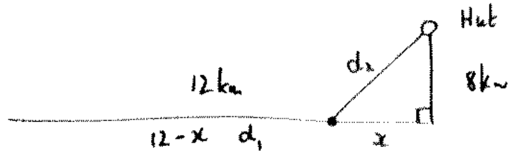
$\doteqdot 0.0303067$

$= 0.030$ (3 sig figs)

(iv) $t = ? \quad T = 80°C$

2M $\therefore 80 = 30 + 220e^{-kt} \quad \therefore \frac{50}{220} = e^{-kt}$ $\qquad (48.886884)$

$\therefore \ln\frac{5}{22} = -kt \quad \therefore t = -\frac{1}{k}\ln\frac{5}{22} = 49$ minutes to nearest minute.

b) 

$v = \frac{d}{t}$

$t = \frac{d}{v}$ $\qquad$ ← N

5M Minimise the time

$t = \frac{d_1}{6} + \frac{d_2}{3} = \frac{12-x}{6} + \frac{\sqrt{8^2+x^2}}{3}$

$\frac{dt}{dx} = -\frac{1}{6} + \frac{1}{2}(64+x^2)^{-\frac{1}{2}}.\frac{2x}{3} = 0$

$\therefore \frac{1}{6} = \frac{x}{3\sqrt{64+x^2}} \quad \therefore \frac{1}{2} = \frac{x}{\sqrt{64+x^2}} \quad \therefore \sqrt{64+x^2} = 2x$

$\therefore 64 + x^2 = 4x^2 \quad \therefore 3x^2 = 64 \quad \therefore x = \pm\sqrt{\frac{64}{3}} \doteqdot 4.61880$

$\therefore$ Continue for 7381 metres (to nearest metre)

a) $P(2ap, ap^2)$ $Q(2aq, aq^2)$ on $x^2 = 4ay$

(i) $y = \frac{x^2}{4a}$ $\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$ at P $\frac{dy}{dx} = p$, at Q $\frac{dy}{dx} = q$

2M ∴ eq^n of Tangent at P

$y - ap^2 = p(x - 2ap)$ ∴ $y = px - ap^2$ — ①

(ii) Eq^n of tangent at Q $y = qx - aq^2$ — ②

Solve ① + ② ∴ $px - ap^2 = qx - aq^2$ ∴ $x(p-q) = a(p+q)(p-q)$

∴ $x = a(p+q)$ ∴ $y = p(a(p+q)) - ap^2 = apq$

2M ∴ $T(a(p+q), apq)$ is point of intersection of tangents at P + Q

(iii) $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$ ∴ $\sqrt{3} = \frac{p-q}{1+pq}$ $m_1 = p$, $m_2 = q$

1M ∴ $p - q = \sqrt{3}(1+pq)$

(iv) $x^2 - 4ay = [a(p+q)]^2 - 4a(apq) = ap^2 - 2a^2pq + a^2q^2$

$= a^2(p-q)^2 = a^2((1+pq)\sqrt{3})^2$

3M $= 3a^2[1 + 2pq + p^2q^2]$ $apq = y$ ∴ $pq = \frac{y}{a}$

∴ $x^2 - 4ay = 3a^2[1 + \frac{2y}{a} + \frac{y^2}{a^2}] = 3a^2 + 3a.2y + 3a^2y^2$

∴ locus $x^2 - 3a^2y^2 - 10ay - 3a^2 = 0$.

b) Induction $1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{1}{3}(2n-1)n(2n+1)$

STEP 1 Show $n=1$ LHS $1^2 = 1$
RHS $\frac{1}{3}.1.1.3 = 1$ ∴ True for $n = 1$

4M STEP 2 Assume true for $n = k$, prove true for $n = k+1$

∴ LHS $1^2 + 3^2 + \cdots + (2k-1)^2 + (2k+1)^2$ using assumption

$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$

$= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)] = \frac{1}{3}(2k+1)[2k^2 + 5k + 3]$

$= \frac{1}{3}(2k+1)(2k+3)(k+1) = RHS$

STEP 3 Since it is true for $n=1$ and we have established the link between $n=k$ and $n=k+1$ we can induce $n=1+1=2$ and so on for all positive integers n

Question 6 (+ 4 + 6 = 12 Marks)

a) Solve $\sin 2x + \sin x = 0$ ∴ $2\sin x\cos x + \sin x = 0$

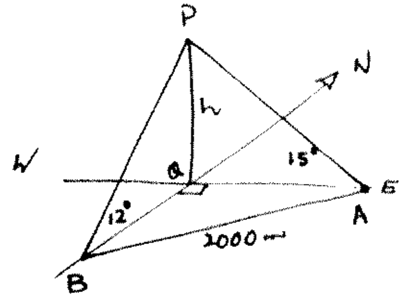
2M ∴ $\sin x(2\cos x + 1) = 0$ ∴ $\sin x = 0$ or $2\cos x + 1 = 0$

for $0 \le x \le 2\pi$ $\sin x = 0$, $x = 0, \pi, 2\pi$.
$\cos x = -\frac{1}{2}$ $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

S | A
T | C

∴ Solns $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$.

b)



4M

In $\triangle ABQ$ $AB = 2000m$

$BQ = h\cot 12°$

$AQ = h\cot 15°$

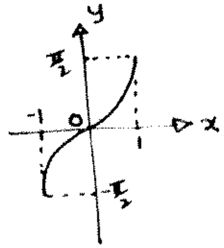
Using Pythagoras' Theorem we have

$2000^2 = h^2\cot^2 12° + h^2\cot^2 15$

∴ $h^2 = \frac{2000^2}{(\cot^2 12 + \cot^2 15)}$ $h > 0$

∴ $h \doteqdot 333$ m (to nearest metre)

c) (i) $y = \sin^{-1}x$

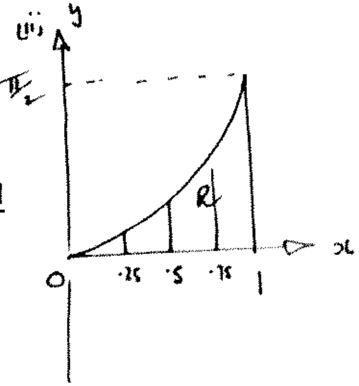


2M

Domain $\{x : -1 \le x \le 1\}$

Range $\{y : -\frac{\pi}{2} \le y \le \frac{\pi}{2}\}$

(ii)



4M

Simpsons Rule 5 function values

$\int_0^1 \sin^{-1}x\,dx \doteqdot \frac{h}{3}[y_0 + y_n + 4(\text{odds}) + 2(\text{evens})]$

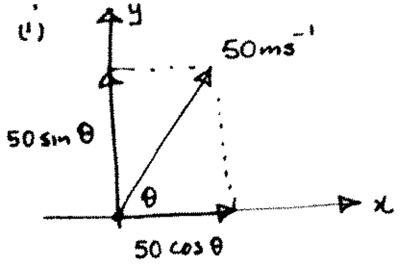
$\doteqdot \frac{1}{12}[\sin^{-1}0 + \sin^{-1}1 + 4[\sin^{-1}(.25) + \sin^{-1}(.75)] + 2[\sin^{-1}(\frac{1}{2})]]$

$\doteqdot \frac{1}{12}[0 + \frac{\pi}{2} + 4.40\ldots + 1.04\ldots]$

$\doteqdot \frac{7.0209632}{12} \doteqdot 0.59$ (2DPs)

Question 7 $(4+2+2+4 = 12 \text{ Marks})$

(i)



$\ddot{x} = 0$ $\quad\quad$ $\ddot{y} = -10$

$\dot{x} = C_1$ $\quad\quad$ $\dot{y} = -10t + C_2$

$t=0$ $\quad C_1 = 50\cos\theta,\ C_2 = 50\sin\theta$

$\therefore \dot{x} = 50\cos\theta$, $\dot{y} = -10t + 50\sin\theta$

$x = 50t\cos\theta + C_3$ $\quad\quad$ $y = -5t^2 + 50t\sin\theta + C_4$

4M

$t=0 \quad x=0,\ y=0$

$\therefore x = 50t\cos\theta$ $\quad\quad$ $y = -5t^2 + 50t\sin\theta$

(ii) Boundary fence $x = 200$ $y = 2$ $\quad \therefore 200 = 50t\cos\theta$

$\therefore t = \dfrac{4}{\cos\theta}$

Now $\quad 2 = -5\left(\dfrac{4}{\cos\theta}\right)^2 + 50\left(\dfrac{4}{\cos\theta}\right)\sin\theta$

$2 = \dfrac{-80}{\cos^2\theta} + 200\tan\theta$ $\quad\quad$ using $\dfrac{1}{\cos^2\theta} = \sec^2\theta$

and $1 + \tan^2\theta = \sec^2\theta$

2M $\quad \therefore \quad 2 = -80\sec^2\theta + 200\tan\theta$

$\therefore 2 = -80(1+\tan^2\theta) + 200\tan\theta$

$2 = -80 - 80\tan^2\theta + 200\tan\theta$

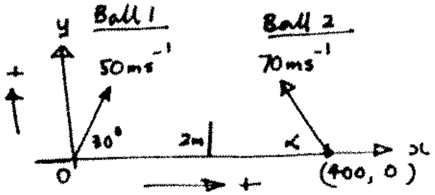
ie $\quad 80\tan^2\theta - 200\tan\theta + 82 = 0$ as required.

(iii) Now Solve for θ $\quad \tan\theta = \dfrac{200 \pm \sqrt{200^2 - 4\cdot 80\cdot 82}}{160}$

2M $\quad \tan\theta \doteqdot \dfrac{200 \pm 117\cdot 303}{160}$ $\quad \therefore \quad 27^\circ 20' < \theta < 63^\circ 14'$ $\quad \theta$ to nearest minute.

range of θ

(iv)



Ball 1 $\quad x_1 = 50t\cos 30 = 25\sqrt{3}\,t$

$y_1 = -5t^2 + 25t$

Ball 2 $\quad \ddot{x} = 0$ $\quad\quad$ $\ddot{y} = -10$

$\dot{x} = -70\cos\alpha$ $\quad\quad$ $\dot{y} = -10t + \sin\alpha$

4M

$x_2 = -70t\cos\alpha + 400$

$y_2 = -5t^2 + 70t\sin\alpha$

for collision $\quad y_1 = y_2$

$\therefore -5t^2 + 25t = -5t^2 + 70t\sin\alpha$

$\therefore \sin\alpha = \dfrac{25}{70}$ $\quad \therefore \alpha \doteqdot 20^\circ 55'$ (angle of projection of second ball)

for t $\quad x_1 = x_2$ $\quad \therefore \quad 25\sqrt{3}\,t = -70t\cos\alpha + 400$ $\quad \therefore t \doteqdot 3\cdot 68$ secs

$t(25\sqrt{3} + 70\cos\alpha) = 400$

Position $x \doteqdot 159\cdot 36$ m $\quad y \doteqdot 24\cdot 28$ m