SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

VERITATE ET VIRTUTE

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 1998

MATHEMATICS

3 UNIT ADDITIONAL
(3/4 UNIT COMMON)

Time allowed:	*3 Hours* *(plus five minutes reading time)*
Examiners:	P.R. Bigelow & P.S. Parker

DIRECTIONS TO CANDIDATES

- *ALL* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Start each question in a new answer booklet. Indicate your name, class and teacher on each new booklet
- Additional answer booklets may be obtained from the supervisor upon request.

<u>NOTE</u>: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate Examination Paper for this subject.

Question 1 (Start a new page) **Marks**

(a) Find the value of a such that $P(x) = x^3 - 2x^2 - ax + 6$ is divisible by $x + 2$ **2**

(b) For a given series $T_{n+1} - T_n = 7$, $T_1 = 3$, find the value of S_{100}, where $S_n = T_1 + T_2 + \cdots + T_n$. **2**

(c) The interval joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is trisected by the points $P(-2, 3)$ and $Q(1, 0)$.
Write down the coordinates of A and B. **3**

(d) Find the acute angle (to the nearest degree) between the lines $x - y = 2$ and $2x + y = 1$. **2**

(e) Solve $|2x - 1| - |x| \leq 0$ **3**

Question 2 (Start a new page) **Marks**

(a) Find:

(i) $\displaystyle\int \frac{dx}{4+x^2}$ **1**

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2\frac{t}{2}\,dt$ **3**

(b) Given $f(x)=\sin^{-1}2x$ **3**

(i) Write down the domain and range of $y=f(x)$

(ii) Sketch the curve.

(iii) Find the exact value of $f'(0\cdot 25)$

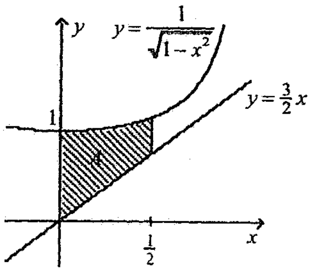
(c) Solve $1+\cos 2x=\sqrt{3}\sin 2x$ where $-\pi<x<\frac{\pi}{4}$ **3**

(d) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x e^{\tan x}\,dx$ using the substitution $u=\tan x$ **2**

Question 3 (Start a new page) **Marks**

(a) (i) Show algebraically that the line $y = \frac{3}{2}x$ does not meet the curve with equation $y = \frac{1}{\sqrt{1-x^2}}$ **3**

(ii) Find the area of the region A, bounded by the curve $y = \frac{1}{\sqrt{1-x^2}}$ and the lines $x = 0$, $x = \frac{1}{2}$ and $y = \frac{3}{2}x$. **3**

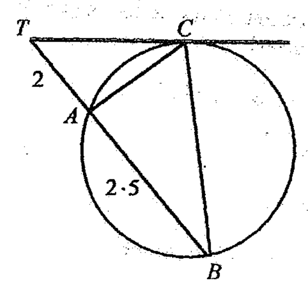


(b) A spherical balloon is expanding so that its volume V mm^3 increases at a constant rate of 72 mm^3 per second. What is the rate of increase of its surface area A mm^2, when the radius is 12 mm. **2**

(c) Given $(3x-2)^{100} = a_{100}x^{100} + a_{99}x^{99} + \cdots + a_1x + a_0$, where a_i $(i = 0, \cdots, 100)$ is a real number. **2**
Evaluate $a_{100} + a_{99} + \cdots + a_1 + a_0$

(d) Differentiate $x^2 \cos^{-1} x$ **2**

Question 4 (Start a new page) **Marks**

(a) 

TC is a tangent. $TA = 2$ units, $AB = 2\cdot 5$ units.

Find the length of TC. **2**

(b) Find the coefficient of y^{10} in the expansion $(1+y)(3y^2-2)^7$ **3**

(c) (i) Show that the equation of the tangent at $T(-2t, t^2)$ on the parabola $y = \frac{1}{4}x^2$ is given by $y + tx - t^2 = 0$ **2**

(ii) If the point $M(x, y)$ is the midpoint of the interval TA where A is the x intercept of the tangent at T. Find the equation of the locus of M as T moves on the parabola. **3**

(d) Evaluate $\lim_{x \to 0} \frac{5x\cos 2x}{\sin x}$ **2**

Question 5 (Start a new page) **Marks**

(a) A particle is moving in a straight line with Simple Harmonic Motion. If the amplitude of the motion is 8 cm and the period of the motion is 6 seconds. 4

(i) Express the displacement, x, of the particle as a function of time, t.

(ii) Calculate the maximum velocity of the particle.

(iii) Calculate the maximum acceleration of the particle.

(iv) Calculate the speed when it is 4 cm from the centre of the motion.

(b) Twelve students sit around a circular table. 4

(i) How many ways can they be arranged?

(ii) If 4 students wish to sit together, how many seating arrangements can be made?

(iii) Let three of the students be A, B and C. Find the probability that A does not sit next to either B or C.

(c) A particle is projected from a point O. After 5 seconds its horizontal and vertical displacements from O are 60 m and $57 \cdot 5$ m respectively. 4
If the particle is still rising, find its initial velocity.
(You may take $g = 10$ m/s^2)

Question 6 (Start a new page) **Marks**

(a) How many times should a die be thrown so that the probability of obtaining at least one multiple of 3 exceeds $0\cdot95$? **2**

(b) At any time t, the rate of cooling of the temperature T of a body when the surrounding temperature is S is given by the equation **5**

$\frac{dT}{dt} = -k(T-S)$, for some constant k

(i) Show that $T = S + Ae^{-kt}$, for some constant A, satisfies this equation.

(ii) A metal rod has a temperature of 1390°C and cools to 1060°C in 10 minutes when the surrounding temperature is 30°C.
Find how much longer it will take the rod to cool to 110°C, giving your answer correct to the nearest minute.

(c) A particle is moving along the x axis with velocity v m s^{-1}, and acceleration $\ddot{x}$ m s^{-2}. **5**

(i) Show that $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$

(ii) If $v^2 = 24 - 6x - 3x^2$ find the acceleration of the particle at the particle's greatest displacement from the origin O.

Question 7

(a) (i) Show that $P(x) = 8x^3 - 12x^2 + 6x + 13$ has only one zero x_1, and that this zero is negative. **4**

(ii) Find the least value of c, where c is a positive integer, such that $-c < x_1 < 0$

(iii) With $-\frac{c}{2}$ as a first approximation, find a better approximation to x_1, using Newton's Method once. Express your answer correct to two decimal places.

(b) Consider $\tan^{-1} y = 2\tan^{-1} x$ **4**

(i) Express y as a function of x.

(ii) Show that the function has no turning points.

(iii) State the domain of the function.

(iv) Sketch the graph of the function.

(c) If $\tan\alpha$ and $\tan\beta$ are the two values of $\tan\theta$ which satisfy the quadratic equation: **4**

$a\tan^2\theta + b\tan\theta + c = 0$

(i) Find $\tan(\alpha+\beta)$

(ii) Show that $\tan^2(\alpha-\beta) = \dfrac{b^2-4ac}{(a+c)^2}$

END OF THE PAPER

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Mathematics Extension 1

Sample Solutions

① a) let $x+2=0$

$x=-2$

So $P(-2) = -16+6+2a = 0$

$a = 5$ ②

(b) $T_{n+1} - T_n = 7$

$T_2 - T_1 = 7$ $\quad T_1 = 3$.

$T_2 - 3 = 7 \Rightarrow T_2 = 10$

$T_3 - T_2 = 7$

$T_3 - 10 = 7 \Rightarrow T_3 = 17$

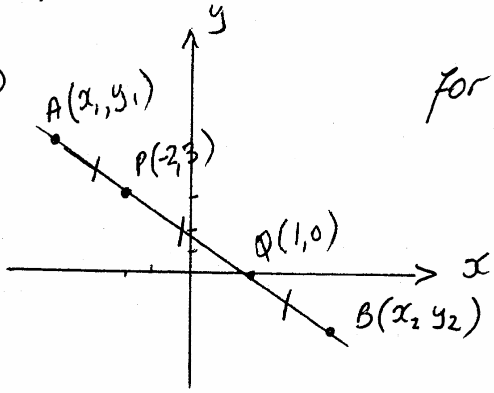
So $\{3+10+17+\cdots\}$.

$a = 3$

$d = 7$ $\qquad S_{100} = \frac{100}{2}\{6+99\times 7\} = 34950$ ②

$n = 100$

(c)



for AQ, midpoint P.

$\frac{x_1+1}{2} = -2 \Rightarrow x_1 = -5$

$\frac{y_1+0}{2} = 3 \Rightarrow y_1 = 6$

$A(-5, 6)$.

for PB, midpoint Q

$\frac{-2+x_2}{2} = 1 \Rightarrow x_2 = 4$ ③

$\frac{3+y_2}{2} = 0 \Rightarrow y_2 = -3$

$B(4, -3)$.

(d) $x-y=2$ $\quad m_1 = 1$

$2x+y=1$ $\quad m_2 = -2$.

$\tan\theta = \left|\frac{1+2}{1-2}\right|$ $\qquad \theta \doteqdot 72^\circ$ ②

(e) $|2x-1| \le |x|$.

$2x-1 \le x$ or $2x-1 \ge -x$

$x \le 1$ or $x \ge \frac{1}{3}$.

③

a) 1 (i) $\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

3 (ii) $\int_0^{\pi/2} \cos^2\frac{t}{2}\,dt = \int_0^{\pi/2} \frac{1}{2}(\cos t + 1)\,dt$

$= \frac{1}{2}\Big[\sin t + t\Big]_0^{\pi/2}$

$= \frac{1}{2}\Big[\big(\sin\frac{\pi}{2} + \frac{\pi}{2}\big) - (0 + 0)\Big]$

$= \frac{\pi}{4} + \frac{1}{2}$

$\frac{1}{2}(\cos t + 1)$

---

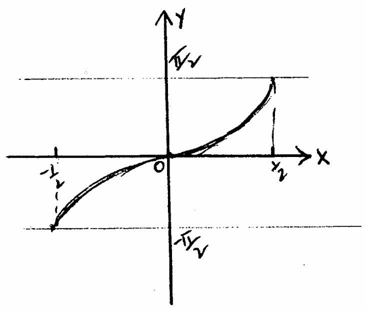
b) 1 (i) $f(x) = \sin^{-1} 2x$

domain: $-1 \le 2x \le 1$

ie $-\frac{1}{2} \le x \le \frac{1}{2}$

range: $-\frac{\pi}{2} \le \sin^{-1} 2x \le \frac{\pi}{2}$

1 (ii)



1 (iii) $f'(x) = \frac{1}{\sqrt{1-4x^2}} \cdot 2$ $\quad \int \frac{1}{\sqrt{\frac{1}{4} - x^2}}$

$f'\left(\frac{1}{4}\right) = \frac{2}{\frac{\sqrt{3}}{2}} = \frac{4}{\sqrt{3}}$

or $\frac{4\sqrt{3}}{3}$

(c) $-\pi < x < \frac{\pi}{4}$

3 $1 + \cos 2x = \sqrt{3}\sin 2x$

$\cos 2x - \sqrt{3}\sin 2x = -1$ — (A)

$R\cos(2x + \alpha) = -1$

$\cos\alpha = 1$ and $\sin\alpha = \sqrt{3}$

$\tan\alpha = \sqrt{3} \Rightarrow \boxed{\alpha = \frac{\pi}{3}}$

$R = \sqrt{1^2 + (\sqrt{3})^2} = 2$

∴ (A) becomes

$2\cos\left(2x + \frac{\pi}{3}\right) = -1$ for $-\frac{5\pi}{3} < 2x + \frac{\pi}{3} < \frac{5\pi}{6}$

$\cos\left(2x + \frac{\pi}{3}\right) = -\frac{1}{2}$

$2x + \frac{\pi}{3} = \frac{2\pi}{3}, -\frac{2\pi}{3}, -\frac{4\pi}{3}$

$\Rightarrow x = \frac{\pi}{6}, -\frac{\pi}{2}, -\frac{5\pi}{6}$

(d) 2 $u = \tan x$, $du = \sec^2 x\,dx$

When $x = 0$, $u = 0$

When $x = \pi/4$, $u = \tan\pi/4 = 1$.

$\therefore I = \int_0^1 e^u\,du$

$= \Big[e^u\Big]_0^1$

$= e^1 - e^0$

$= e - 1$

(a)(i) Curves have a simultaneous solution if they meet.

$\therefore y = \dfrac{1}{\sqrt{1-\frac{4y^2}{9}}}$ or $\dfrac{9x^2}{4} = \dfrac{1}{1-x^2}$

$9y^2 - 4y^4 - 9 = 0$ $\qquad$ $9x^2 - 9x^4 - 4 = 0$

$\Delta = 81 - 144$ $\qquad$ $\Delta = 81 - 144$

< 0 $\qquad$ < 0

$\therefore$ No real solutions for y^2 (or x^2), so no real solution for y (or x)

$\therefore$ No point of intersection.

(ii) $\text{Area} = \displaystyle\int_0^{1/2} \left\{ \frac{1}{\sqrt{1-x^2}} - \frac{3x}{2} \right\} dx$

$= \left[\sin^{-1} x - \dfrac{3x^2}{2} \right]_0^{1/2}$

$= \dfrac{\pi}{6} - \dfrac{3}{16}$

(b) $\dfrac{dV}{dt} = 72 \text{ mm}^3/\text{s}$ $\qquad$ $V = \frac{4}{3}\pi r^3$ $\qquad$ $A = 4\pi r^2$

$\dfrac{dV}{dr} = 4\pi r^2$ $\qquad$ $\dfrac{dA}{dr} = 8\pi r$

$\dfrac{dA}{dt} = \dfrac{dA}{dr} \cdot \dfrac{dr}{dV} \cdot \dfrac{dV}{dt}$

$= \dfrac{8\pi r}{4\pi r^2} \cdot 72$

When $r = 12$, $\dfrac{dA}{dt} = 12 \text{ mm}^2/\text{s}$.

(c) $(3x-2)^{100} = a_{100}x^{100} + a_{99}x^{99} + a_{98}x^{98} + \cdots + a_1 x + a_0$

When $x = 1$

$LHS = (3-2)^{100}$

$= 1$

$RHS = a_{100} + a_{99} + a_{98} + \cdots + a_1 + a_0$

$\therefore a_{100} + a_{99} + \cdots a_0 = 1$.

(d) $D_x\left(x^2 \cos^{-1} x\right) = 2x\cos^{-1} x - \dfrac{x^2}{\sqrt{1-x^2}}$

Q 4

(a) $TC^2 = TB \cdot TA$

$= 4.5 \times 2$

$= 9$

$\therefore \quad TC = 3.$ ——— 2

(b) To find coeff. of y^{10}, we need:

(i) coefficient of y^9 times 1

(ii) coefficient of y^{10} in $(3y^2-2)^7$ times 1.

Now, $(3y^2-2)^7$

$T_{r+1} = \binom{7}{r}(3y^2)^{7-r}(-2)^r$

$= \binom{7}{r} 3^{7-r} \cdot (-2)^r \cdot y^{14-2r}.$

(a) $14-2r = 9 \Rightarrow r = \frac{5}{2}$ (not possible)

(b) $\therefore \; 14-2r = 10 \Rightarrow 2r = 4$ i.e $r = 2.$

$\therefore$ coefficient of y^{10}. ——— 3

$\binom{7}{2} 3^5 (-2)^2 = 20412.$

(c). (i) $y = x^2/4$, $\frac{dy}{dx} = x/2$, $\left.\frac{dy}{dx}\right|_{x=-2t} = -t$

$\therefore$ equation of tg t, $y - t^2 = -t(x+2t).$

$\Rightarrow \quad tx + y + t^2 = 0.$ ——— 2

When $y = 0 \quad x = -t. \Rightarrow$ A $(-t, 0)$.

T $(-2t, t^2)$

$M = \left(\frac{-3t}{2}, \frac{t^2}{2}\right).$ $x = \frac{-3t}{2} \quad \therefore t = -\frac{2x}{3}.$

$y = \frac{1}{2}\left(\frac{4x^2}{9}\right) \Rightarrow$ locus $x^2 = \frac{9y}{2}$ ——— 3

(d). $\lim_{x\to 0} \frac{5x(1-2\sin^2 x)}{\sin x} = 5 \lim_{x\to 0} \frac{1}{\left(\frac{\sin x}{x}\right)} - 10 \lim_{x\to 0} x \sin x$

$= 5$ ——— 2

ction 5

(a) (i) $x = a\cos(nt+\alpha)$

$a = 8$

$\frac{2\pi}{n} = 6$

$n = \frac{\pi}{3}$ 1

$x = 8\cos(\frac{\pi}{3}t+\alpha)$ cm

(ii) $v^2 = n^2(a^2-x^2)$

Max velocity when $x=0$

$\therefore v^2 = (\frac{\pi}{3})^2 . 64$ 1

$\therefore v = \pm\frac{8\pi}{3}$ cm/s

(iii) $\ddot{x} = -n^2x$

Max acceleration when $x = \pm 8$

$\ddot{x} = -(\frac{\pi}{3})^2 . \pm 8$ 1

$= \mp\frac{8\pi^2}{9}$ cm/s^2

(iv) $v^2 = n^2(a^2-x^2)$

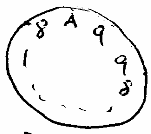
$v^2 = (\frac{\pi}{3})^2(8^2-4^2)$

$v^2 = \frac{\pi^2}{9} . 48$ 1

$\therefore v = \pm\frac{4\pi\sqrt{3}}{3}$ cm/s

(b) (i) $11!$ 1

(ii) $8! \times 4!$ 1

(iii)  $72\times 9!$

2

$\therefore$ Probability is $\frac{72\times 9!}{11!}$

$= \frac{72}{110}$

$= \frac{36}{55}$

(c) $\ddot{x} = 0$ $\ddot{y} = -10$

$\dot{x} = V\cos\theta$ $\dot{y} = -10t + V\sin\theta$

$x = V\cos\theta t$ 1 $y = -5t^2 + V\sin\theta t$ 1

$V\cos\theta = \frac{x}{t}$ $V\sin\theta = \frac{y+5t^2}{t}$.

When $t=5$: $V\cos\theta = 12$ $V\sin\theta = \frac{57.5+125}{5}$

$= 36.5$. 1

$\therefore V^2 = 12^2 + 36.5^2$

$= 1476.25$

$\therefore V = 38.42199....$ 1

≈ 38.4 m/s.

(unfortunately this is on the downward flight not upward)

39916800

967680

26127360

6 (a) $P\left(\begin{matrix}\text{multiple of 3 in}\\ \text{a single throw}\end{matrix}\right) = \frac{2}{6} = \frac{1}{3}$

$P(\text{not a multiple of 3}) = \frac{2}{3}$

$P\left(\begin{matrix}\text{no multiples of 3}\\ \text{in } n \text{ tosses}\end{matrix}\right) = \left(\frac{2}{3}\right)^n$

We need $\left(\frac{2}{3}\right)^n < 0\cdot05$

$$n > \frac{\log(0\cdot05)}{\log\left(\frac{2}{3}\right)}$$ 2

$n > 7\cdot38$

$\therefore$ 8 tosses are needed.

(i) $\frac{dT}{dt} = -K(T-S)$ $\qquad \frac{dT}{dt} = -KAe^{-Kt}$

$T = S + Ae^{-Kt}$ $\qquad \frac{dT}{dt} = -K(T-S)$

(ii) $T = S + Ae^{-Kt}$

$S = 30$

When $t = 0$, $T = 1390$

$1390 = 30 + A$

$A = 1360$

$T = 30 + 1360\, e^{-Kt}$ $\quad$ 1

When $t = 10$ $\quad 1060 = 30 + 1360\, e^{-10K}$

$T = 1060$

$\frac{1030}{1360} = e^{-10K}$

$K = -\frac{1}{10} \ln\left(\frac{1030}{1360}\right)$ $\quad$ 2

$K = 0.02779\ldots$

When $T = 110$ $\qquad t = ?$

$110 = 30 + 1360\, e^{-Kt}$

$\frac{80}{1360} = e^{-Kt}$

$t = -\frac{1}{K} \ln\left(\frac{80}{1360}\right)$

$t = 101.9$ $\qquad$ 4

92 mins later.

(c)(i) RTP $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$$RHS = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$= v \times \frac{dv}{dx} \quad \text{since } v \text{ is a function of } x.$$
$$= \frac{dx}{dt} \times \frac{dv}{dx}$$

2

$$= \frac{dv}{dt} = \ddot{x} = LHS$$

(ii)
$$v^2 = 24 - 6x - 3x^2$$
$$\frac{1}{2}v^2 = 12 - 3x - \frac{3}{2}x^2$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -3 - 3x$$

1

Greatest displacement when $v = 0$

$$24 - 6x - 3x^2 = 0$$
$$x^2 + 2x - 8 = 0$$

1

$$(x+4)(x-2) = 0$$
$$x = -4, \ 2$$

Acceleration at maximum displacement

is $\ddot{x} = -3 - 3(-4) = 9 \text{ m/s}^2$

or $\ddot{x} = -3 - 3(2) = -9 \text{ m/s}^2$

1

7 (a) (i) $P(x) = 8x^3 - 12x^2 + 6x + 13$

$P'(x) = 24x^2 - 24x + 6$

$= 6(4x^2 - 4x + 1)$

$= 6(2x-1)^2$

Stat pt where $P'(x) = 0$

$(2x-1)^2 = 0$

$x = \frac{1}{2}$

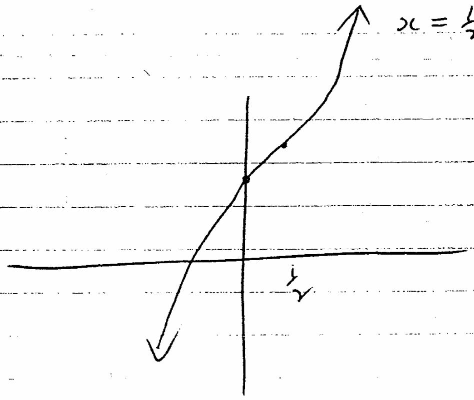
$P'(0{\cdot}4) > 0$ and $P'(0{\cdot}6) > 0$

$\therefore$ Horizontal pt of inflexion where 2

$x = \frac{1}{2}$, $y = 1 - 3 + 3 + 13$

$= 14$

When $x = 0$, $y = 13$



From sketch it is clear that there is only one root which is negative

(ii) $P(-1) = -8 - 12 - 6 + 13 < 0$

$\therefore$ Req'd value of c is $c = -1$

(iii) $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_1 = -\frac{1}{2} - \dfrac{f(-\frac{1}{2})}{f'(-\frac{1}{2})}$

$= -0{\cdot}75$

(b) (i) $\tan^{-1} y = 2\tan^{-1} x$

Take tan of both sides

$$y = \tan(2\tan^{-1} x)$$

$$y = \tan 2\alpha \qquad \text{where } \alpha = \tan^{-1} x$$

$$y = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$y = \frac{2x}{1-x^2}$$

(ii) $$\frac{dy}{dx} = \frac{(1-x^2)2 - 2x(-2x)}{(1-x^2)^2}$$

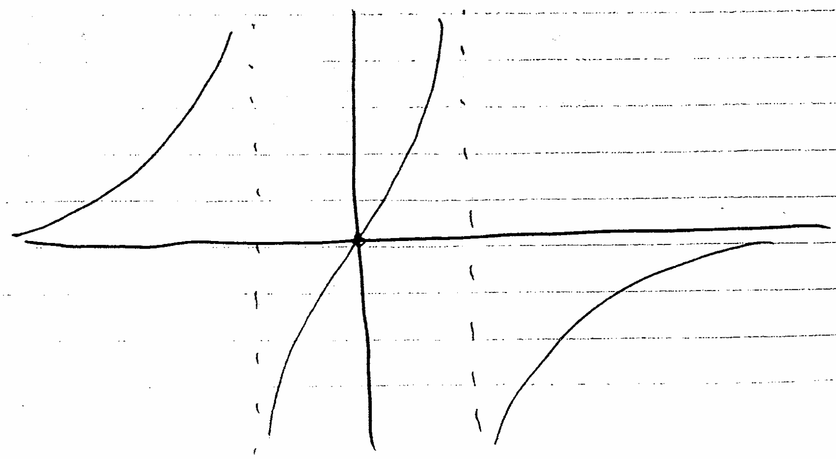
$$= \frac{2-2x^2+4x^2}{(1-x^2)^2}$$

$$= \frac{2x^2+2}{(1-x^2)^2} > 0$$

$\therefore$ No turning pts

(iii) D : all real x except $x=1$ or $x=-1$

(iv)



(c) $\quad a\tan^2\theta + b\tan\theta + c = 0$

(i) $\quad \tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ 1

$$= \frac{\frac{-b}{a}}{1-\frac{c}{a}} = \frac{-b}{a-c}$$ 1   2

(ii) $\quad \tan^2(\alpha-\beta) = \left(\dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right)^2$

$$= \frac{\tan^2\alpha + \tan^2\beta - 2\tan\alpha\tan\beta}{(1+\tan\alpha\tan\beta)^2}$$ 1

$$= \frac{(\tan\alpha+\tan\beta)^2 - 4\tan\alpha\tan\beta}{(1+\tan\alpha\tan\beta)^2}$$

$$= \frac{\left(\frac{-b}{a}\right)^2 - 4\frac{c}{a}}{\left(1+\frac{c}{a}\right)^2}$$

$$= \frac{b^2-4ac}{(a+c)^2}$$ 2