SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS

Year 12
3/4 Unit

Time allowed Two hours.
(plus 5 minutes reading time)

Examiners : T.A.Donnellan, B.J.Genner.

Directions to Candidates

- All question may be attempted
- All questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Start each question on a new page, clearly showing your name and class.
- If required, addition paper may be obtained upon request.
- This is a trial paper only and does not necessarily reflect the content or the format of the final Higher School Certificate Paper for this subject.

QUESTION 1. (Start a new writing booklet) **Marks**

(a) Differentiate $\sin^{-1} 2x$ with respect to x. [1]

(b) Find $\tan^{-1}(-1)$. [1]

(c) Find the acute angle between the lines $5x - y - 9 = 0$ and $2x - 3y + 12 = 0$ [1]

(d) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 4x}{7x}$ [2]

(e) If α, β and γ are roots of the equation $x^3 + x^2 - 3 = 0$, write down the value of [4]

i) $\alpha + \beta + \gamma$

ii) $\alpha\beta + \beta\gamma + \alpha\gamma$

iii) $\alpha^2 + \beta^2 + \gamma^2$

(f) Evaluate $\int_0^{\pi/3} \cos^2 x \, dx$ [3]

QUESTION 2. (Start a new writing booklet)

(a) Given $f(x) = \frac{1}{3}\cos^{-1} 2x$; [4]

i) write down the domain.

ii) write down the range, and hence

iii) sketch $y = f(x)$

(b) Divide the interval AB externally in the ratio 2:3, where A is the point (3,1) and B is (−1,4). [2]

(c) Find [4]

i) $\displaystyle\int \frac{dx}{1 + 4x^2}$

ii) $\int x\sqrt{2 - x} \, dx$ using $u = 2 - x$

(d) Given that $\log_4 9 = 1{\cdot}585$ (to 3 decimal places), find $\log_4 144$. [2]

QUESTION 3. (Start a new writing booklet) **Marks**

(a) Find the term independent of x in the expansion of $\left(x - \dfrac{2}{x^2}\right)^9$. [3]

(b) Show that the graph $y = x^3 + 3x^2 + 4x$ cuts the x-axis only once. [2]

(c) Prove $\cos^4 x + \sin^2 x \equiv \cos^2 x + \sin^4 x$ [3]

(d) Use the method of mathematical induction to prove that $4 \times 6^n + 1$ is a multiple of 5 when n is a positive integer. [4]

QUESTION 4. (Start a new writing booklet)

(a) i) Express $\sqrt{3}\sin 3t - \cos 3t$ in the form $R\sin(3t - \alpha)$ where α is acute and. $R > 0$ [3]

ii) Hence or otherwise find in exact form the general solution of the equation

$$\sqrt{3}\sin 3t - \cos 3t = 0$$

(b) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$. The tangent at P and a line through Q parallel to the y-axis meet at point R. The tangent at Q and the line through P parallel to the y-axis meet at S. [9]

i) Draw a neat diagram showing all information given above.

ii) Prove the gradient at P is p and the equation of the tangent is $y = px - ap^2$.

iii) Show that PQRS is a parallelogram.

iv) Show that the area of this parallelogram is $2a^2|p - q|^3$ square units.

QUESTION 5. (Start a new writing booklet) **Marks**

(a) A particle is moving on a straight line in such a way that its displacement x metres from the origin at time t seconds is given by [4]

$$x = 5\sin 2t$$

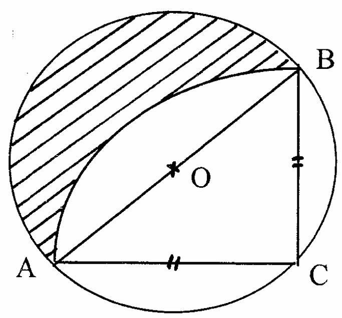
i) Show that $\dfrac{d^2x}{dt^2} = -4x$

ii) Find the maximum speed of the particle.

iii) Find the maximum acceleration of the particle.

iv) What will be the acceleration of the particle when its displacement is 0?

(b)



AB is the diameter of the circle ABC whose centre is O. C is equidistant from A and B and the arc AB is drawn with C as the centre. Show that the shaded area is equal to the area of the triangle ABC [3]

(c) Let T be the temperature of an object at time t and let D be the temperature of the surrounding medium. Newton's Law of Cooling states that the rate of change of T is proportional to (T – D) [5]

$$\text{i.e.} \quad \frac{dT}{dt} = -k(T - D)$$

i) Show that $T = D + Ce^{-kt}$ (where C and k are constants) satisfies Newton's Law of Cooling.

ii) A packet of meat with an initial temperature of 25°C is placed in a freezer whose temperature is kept at a constant -10°C. It takes 12 minutes for the temperature of the meat to drop to 15°C. How much additional time is needed for the temperature of the meat to fall to 0°C? Give you answer in minutes, correct to 1 decimal place.

Question 6. (Start a new writing booklet) **Marks**

(a) 6 white and 2 red marbles are arranged at random in a straight line. [4]
Find the probability that

i) The red marbles are at the ends of the line.

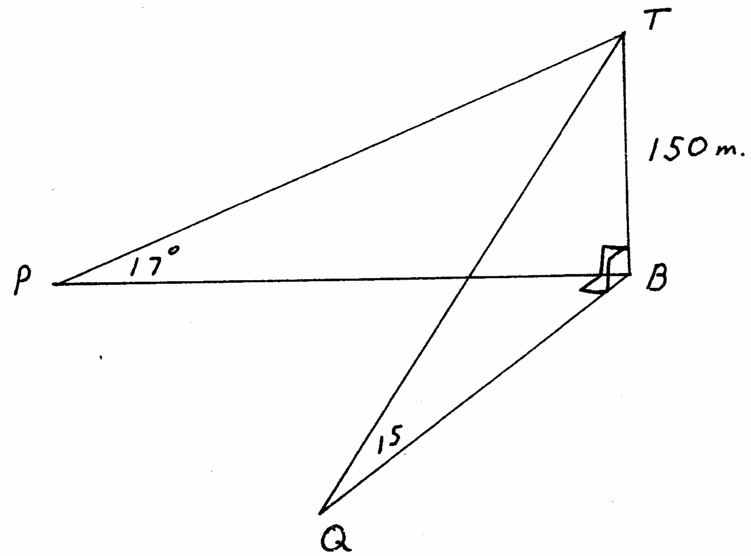
ii) The red marbles are separated by at least 3 white marbles.

(b) Kim wishes to solve $x^4 - 110 = 0$ correct to 2 decimal places and guesses that the solution is close to 3·2. Use Newton's method once to refine Kim's result, and demonstrate that to use it a second time does not improve the result to two decimal places. [4]

(c) A transmitter tower TB is 150 metres tall and is observed from Q (due South of B) with an angle of elevation of 15° and from P (due West of B) with an angle of elevation of 17°. [4]

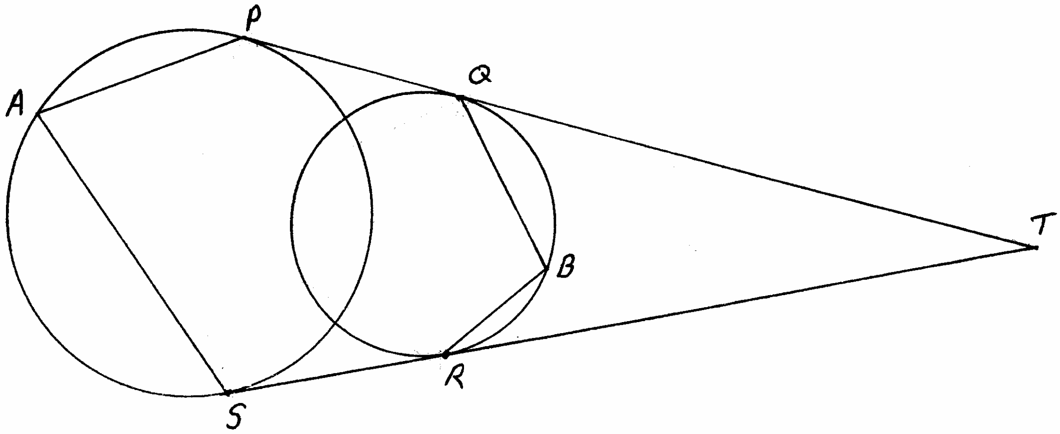
i) Find the distance PQ.

ii) Hence or otherwise find ∠PTQ to the nearest minute



Question 7. (Start a new writing booklet) **Marks**

(a) [8]



PQ and SR are tangents to both circles. Show that;

i) PQ = SR.

ii) PQRS is a Trapezium.

iii) P, Q, R and S are concyclic

iv) $\angle PAS + \angle QBR = 180°$

(b) Two guns at the same fortification shoot simultaneously and hit the same target at different times. They have the same muzzle velocity of $150ms^{-1}$ but different angles of elevation. One gun has an angle of elevation of 30°. (Assume $g = 10\ ms^{-2}$) [4]

i) Find the distance of the target from the guns.

ii) Find the angle of elevation of the other gun.

iii) Find the time which elapses between the fall of the two shots to the nearest $\frac{1}{10}$ s.

<u>END OF PAPER</u>

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1999

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Mathematics Extension 1

Sample Solutions

① (a) $\dfrac{d(\sin^{-1}2x)}{dx} = \dfrac{2}{\sqrt{1-4x^2}}$

(b) $\tan^{-1}(-1) = -\tan^{-1}1$

$= -\pi/4$

(c) $5x - y - 9 = 0$ | $2x - 3y + 12 = 0$

$y = 5x - 9$ | $3y = 2x + 12$

$m_1 = 5$ | $m_2 = \frac{2}{3}$

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\dfrac{5 - 2/3}{1 + 5\times\frac{2}{3}}\right|$

$= 1$ $\therefore \theta = 45°$

(d) $\sin x \doteqdot x$, x small

$\therefore \lim\limits_{x\to 0} \dfrac{\sin 4x}{7x} = \lim\limits_{x\to 0} \dfrac{4x}{7x}$

$= \dfrac{4}{7}$

(e) $x^3 + x^2 - 3 = 0$

(i) $\alpha + \beta + \gamma = -1$

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma = 0$

(iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 1 - 2\times 0$

$= 1$

(f) $\displaystyle\int_0^{\pi/3} \cos^2 x\, dx = \frac{1}{2}\int_0^{\pi/3} 2\cos^2 x\, dx$

$\displaystyle = \frac{1}{2}\int_0^{\pi/3} (1 + \cos 2x)\, dx$

$\displaystyle = \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/3}$

$\displaystyle = \frac{1}{2}\left[\pi/3 + \frac{1}{2}\sin\frac{2\pi}{3}\right]$

$\displaystyle = \frac{\pi}{6} + \frac{1}{4}\times\frac{\sqrt{3}}{2}$

$\displaystyle = \pi/6 + \frac{\sqrt{3}}{8}$

② (a) $f(x) = \frac{1}{3}\cos^{-1}2x$

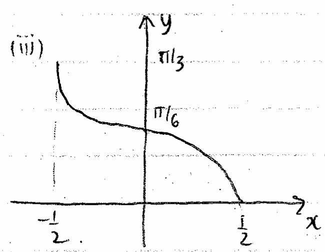
(i) $-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$

(ii) $0 \le 3y \le \pi$

$0 \le y \le \pi/3$

(iii)



② (b) $\overset{x_1\,y_1}{A(3,1)}$ $\overset{x_2\,y_2}{B(-1,4)}$ $\overset{m:n}{-2:3}$

$$P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

$$=\left(\frac{2+9}{1}, \frac{-8+3}{1}\right)$$

$$=(11,-5)$$

(c) (i) $\int \frac{dx}{1+4x^2}$

$$=\frac{1}{4}\int\frac{dx}{\frac{1}{4}+x^2}$$

$$=\frac{1}{4}\times 2\tan^{-1}(2x)+C$$

$$=\frac{1}{2}\tan^{-1}(2x)+C$$

(c) (ii) $\int x\sqrt{2-x}\,dx$

$$\left[u=2-x \Rightarrow x=2-u \quad dx=-du\right]$$

$$=\int -(2-u)\sqrt{u}\,du$$

$$=\int(u^{3/2}-2u^{1/2})\,du$$

$$=\frac{2}{5}u^{5/2}-\frac{4}{3}u^{3/2}+C$$

$$=\frac{2}{5}(2-x)^{5/2}-\frac{4}{3}(2-x)^{3/2}+C$$

(d) $\log_4 9 \doteqdot 1.585$

$$\log_4 144 = \log_4(9\times16)$$

$$=\log_4 9+\log_4 16$$

$$\doteqdot 1.585+2\log_4 4$$

$$\doteqdot 1.585+2$$

$$=3.585$$

③ (a) $\left(x-\frac{2}{x^2}\right)^9 = \sum_{k=0}^{9}\binom{9}{k}x^{9-k}(-2x^{-2})^k$

$$=\sum_{k=0}^{9}\binom{9}{k}(-2)^k x^{9-3k}$$

constant term : $9-3k=0$

$\therefore k=3$

constant term $=\binom{9}{3}(-2)^3$

$=-672$

(b) $y=x^3+3x^2+4x$

y is a cubic so cuts at least once.

$y'=3x^2+6x+4$

$$\Delta=36-4\times3\times4$$

$$=-12$$

$$<0$$

$\therefore$ y is strictly increasing

$\therefore$ y has only one x-intercept.

(3) (c) $LHS = \cos^4 x + \sin^2 x$

$= (1-\sin^2 x)^2 + \sin^2 x \quad [\because \cos^2 x = 1-\sin^2 x]$

$= \sin^4 x - 2\sin^2 x + 1 + \sin^2 x$

$= \sin^4 x + 1 - \sin^2 x$

$= \sin^4 x + \cos^2 x$

$= RHS$

(d) $4 \times 6^n + 1 = 5M$, $n > 0$ (where M is an integer)

Test $n=1$: $LHS = 4 \times 6 + 1 = 25$

$= 5 \times 5$

$\therefore$ The statement is true for $n=1$

Assume the statement is true for some integer $n=k$

i.e. $4 \times 6^k + 1 = 5P$ (P an integer) $(*)$

We need to prove the statement true when $n=k+1$

ie. $4 \times 6^{k+1} + 1 = 5Q$ (Q an integer)

$LHS = 4 \times 6^{k+1} + 1$

$= 4 \times 6 . 6^k + 1$

$= 4 \times 6 . 6^k + 1$

$= 6(4 \times 6^k + 1) - 6 + 1$

$= 6(5P) - 5$

$= 5(6P - 1)$

$= 5Q$ [Q is an integer, since P is an integer]

$= RHS$

So when the statement is true for $n=k$, it is true for $n=k+1$

So by the principle of mathematical induction $4 \times 6^n + 1 = 5M$, for $n > 0$

④ (a) (i) $\sqrt{3}\sin 3t - \cos 3t \equiv R\sin(3t-\alpha)$

$= R\sin 3t\cos\alpha - R\sin\alpha\cos 3t$

$R\cos\alpha = \sqrt{3}$ — ① $\Rightarrow$ $\boxed{R = 2}$

$R\sin\alpha = 1$ — ②

$\Downarrow$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\boxed{\alpha = \pi/6}$

(ii) "otherwise" is better approach as

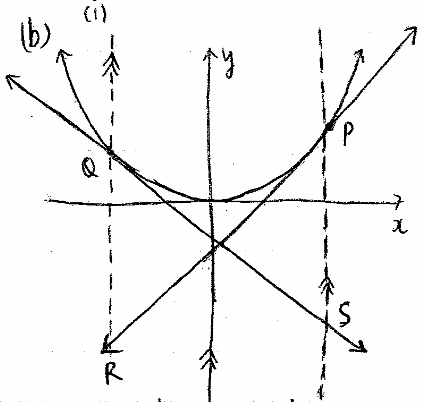
$\sqrt{3}\sin 3t = \cos 3t$

$\tan 3t = \frac{1}{\sqrt{3}}$

$3t = n\pi + \pi/6$ $(n \in \mathbb{Z})$

$\boxed{t = \frac{n\pi}{3} + \pi/18}$

(b) (i)



$\leftarrow |2ap - 2aq|$

(ii) $P(2ap, ap^2)$

$x^2 = 4ay$

$\therefore 2x = 4a\frac{dy}{dx}$

P: $2(2ap) = 4a\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = p$

$y - ap^2 = p(x - 2ap)$

$y = px - 2ap^2 + ap^2$

$\boxed{y = px - ap^2}$

(iii) R: $x = 2aq$

$y = px - ap^2$

$\therefore y = 2apq - ap^2$

$\therefore QR = |aq^2 - 2apq + ap^2| = |a|(p-q)^2$

$\therefore x = 2ap$

$y = qx - aq^2 \Rightarrow y = 2apq - aq^2$

$PS = |ap^2 - 2apq + aq^2| = |a|(p-q)^2$

(4) (b) (iii)

RQ ∥ SP and RQ = SP ⇒ PQRS is a parallelogram.

(iv) $\text{Area} = |ap^2 - 2apq + aq^2| \times |2a(p-q)|$

$= 2a^2|p^2 - 2pq + q^2| \times |p-q|$

$= 2a^2|(p-q)^2| \times |p-q|$

$= 2a^2|p-q|^3$

(5) $x = 5\sin 2t$

(a) (i) $\dot{x} = 10\cos 2t$

$\ddot{x} = -20\sin 2t$

$= -4(5\sin 2t)$

$= -4x$

(ii) $\dot{x} = 10\cos 2t$

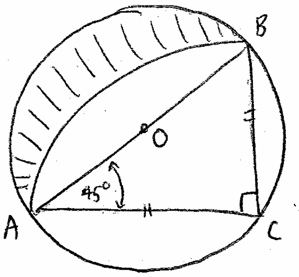
max $|\dot{x}| = 10$ m/s

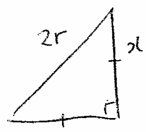
(iii) $\ddot{x} = -20\sin 2t$

max acc $= 20$ m/s^2

(iv) $x = 0 \Rightarrow a = 0$ m/s^2

(b)





$\therefore 2x^2 = 4r^2$

$x = \sqrt{2}r$

$\Delta ABC = \frac{1}{2}(\sqrt{2}r)^2$

$= r^2$

segment AOB $= \frac{1}{4}\pi(\sqrt{2}r)^2 - r^2$

$= \frac{\pi r^2}{2} - r^2$

$= r^2(\pi/2 - 1)$

$= r^2\left(\frac{\pi - 2}{2}\right)$

semi-circle $= \frac{1}{2}\pi r^2$

$\therefore$ shaded area $= \frac{1}{2}\pi r^2 - \left(r^2\left(\frac{\pi}{2} - 1\right)\right)$

$= r^2$

Q.E.D.

(5) (c) (i) $T = D + Ce^{-kt}$

LHS $\dfrac{dT}{dt} = -k \times Ce^{-kt}$

$= -k(T - D)$

$= RHS$

Q.E.D.

(ii) $D = -10$ $t = 0,\ T = 25$

$t = 12,\ T = 15$

$T = -10 + Ce^{-kt}$

$\therefore T = -10 + 35e^{-kt} = 35e^{-kt} - 10$

$\therefore 15 = 35e^{-12k} - 10$

$35e^{-12k} = 25$

$e^{-12k} = 5/7$

$\therefore -12k = \ln(5/7)$

$k = \frac{1}{12}\ln(7/5)$

$T = 0$

$35e^{-kt} - 10 = 0$

$e^{-kt} = \frac{10}{35} = 2/7$

$-kt = \ln(2/7)$

$t = \frac{1}{k}\ln(7/2) \doteqdot 44.7$

$\therefore$ An additional 32.7 minutes

(6) (a) 6W , 2R

Total arrangements $= \dfrac{8!}{6!\,2!} = \binom{8}{2} = 28$

(i) Ⓡ x x x x x x Ⓡ

$P(\text{Reds at end}) = \dfrac{1}{28}$

(ii) (RR) x x x x x x $\quad \dfrac{7!}{6!} = 7$ ways

(RxR) x x x x x $\quad \dfrac{6 \cdot 6!}{6!} = 6$ ways

(RxxR) x x x x $\quad \dfrac{6 \cdot 5 \cdot 5!}{6!} = 5$ ways

Total = 18 ways

$P(\text{At least 3 separated}) = 1 - \dfrac{18}{28} = \dfrac{10}{28} = \dfrac{5}{14}$

(b) $f(x) = x^4 - 110$, $f'(x) = 4x^3$, $x_0 = 3.2$

$f(3.2) \doteqdot -5.1424$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 3.2 - \dfrac{(3.2^4 - 110)}{4(3.2)^3}$

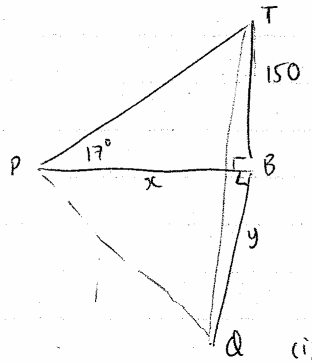
$\doteqdot 3.\underline{239}233398 \quad \doteqdot 3.24$

$f(x_1) \doteqdot 0.953474$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \doteqdot 3.\underline{23}8532068 \quad \doteqdot 3.24$

$f(x_2) \doteqdot 0.0000309$ $\qquad$ No. change to 2 decimal places.

(c)



$\tan 17° = \frac{150}{x}$

$x = \frac{150}{\tan 17°} = 150 \cot 17° = 150 \tan 73°$

$\therefore y = 150 \tan 75°$

(i) $PQ^2 = x^2 + y^2$

$= 150^2 (\tan^2 73° + \tan^2 75°)$

$PQ = 150\sqrt{\tan^2 73° + \tan^2 75}$

(ii) $PT = \frac{150}{\sin 17°}$, $TQ = \frac{150}{\sin 15°}$

$\cos \angle PTQ = \frac{PT^2 + TQ^2 - PQ^2}{2 . PT . TQ}$

$\therefore \angle TPQ = 85° 40'$

7 (a) (i) $PQ = PT - QT = ST - RT = SR$ (tangents from a point)

(ii) ΔTQR & ΔTPS are isosceles

$\because \angle T$ is common $\Rightarrow \angle QRT = \angle PST$

(base angles of isosceles Δ's with common vertex are equal)

$\therefore PS \parallel QR$

$\therefore PSRQ$ is a trapezium

(iii) $\angle QRT = \angle PSR$ (parallel)

$= \angle SPR$ (isos. Δ)

$\therefore$ exterior angle = opposite interior angle

$\therefore PSRQ$ is a cyclic quad (converse of exterior angle theorem of cyclic quads)

(iv) let $\angle PAS = x$ $\Rightarrow \angle PSR = x$ (alternate seg. theorem)

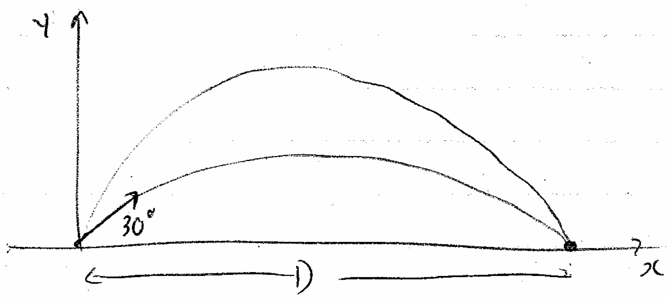
$\Rightarrow \angle PQR = 180 - x$ (opp. angles of cyclic quad)

$\Rightarrow \angle QBR = 180 - x$ (alternate seg. theorem

$\therefore \angle PAS + \angle QBR = 180°$

7(b) Assume the target is on the ground (i.e. $y=0$) at same horizontal height as cannon.
[Too many variables otherwise]
Formulae quoted without proof:

(i)



$$D = \frac{v^2 \sin 2\theta}{g} = \frac{150^2 \sin 60}{10} = \frac{150^2 \sin 120^\circ}{10}$$
$$= 1125\sqrt{3}$$

(ii) $\theta = 60$ is the other angle.

(iii) $$T = \frac{2V\sin\theta}{g}$$

$\therefore$ time elapse $= \frac{2V}{g}(\sin 60 - \sin 30)$

$$= \frac{2\times150}{10}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$
$$= 15(\sqrt{3}-1)$$
$\doteqdot$ 11.0 secs difference