SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2002

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work.

Total Marks - 84 marks

- All questions are of equal value.

Examiner: *E. Choy*

NOTE: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate Examination Paper for this subject.

Question 1: [12 Marks] **Marks**

(a) Evaluate $\int_{-2}^{2} \frac{dx}{\sqrt{16-x^2}}$, giving your answer in exact form. **2**

(b) If $f(x)=e^{x+1}$ find the inverse function $f^{-1}(x)$ and hence show that **3**

$$f[f^{-1}(x)]=f^{-1}[f(x)]=x$$

(c) Solve the inequality **2**

$$\frac{4-x}{x}\leq 1$$

(d) Find the acute angle between the lines $y=\frac{1}{2}x$ and $x+\sqrt{3}y+1=0$. **2**
Give your answer in radians correct to two decimal places.

(e) $A(10,1)$, $P(8,5)$ and B are points on the number plane. **3**

Point P divides the interval AB externally in the ratio 2: 3.

Find the coordinates of B.

Question 2: [12 Marks] **Marks**

(a) Differentiate $y = \tan^{-1}(\cot x)$ with respect to x. **2**

(b) Show that $\tan^{-1}(x) = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ **2**

(c) The polynomial $p(x) = ax^3 + bx^2 - 8x + 3$ has a factor $(x-1)$. When divided by $(x+2)$ the remainder is 15. **2**

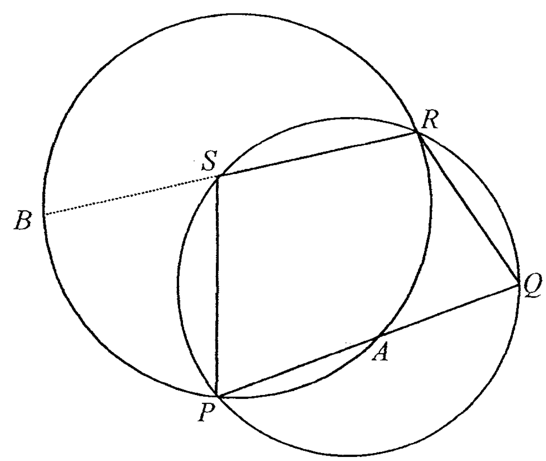
Find the values of a and b.

(d) Find $\frac{d}{dx}\left(\frac{\ln x}{x}\right)$ and hence find the primitive function of $\frac{2 - \ln x}{x^2}$ **2**

(e) The word EQUATION contains all five vowels. How many 3 letter "words" consisting of at least 1 vowel and 1 consonant can be made from the letters of EQUATION? **2**

[NB a "word" is ANY arrangement of the letters without any necessary meaning]

(f) **2**



PQRS is a cyclic quadrilateral and *A* is any point on *PQ*.

A circle through the points *P*, *A* and *R* cuts *RS* produced at *B*.

Prove that $AB \| SQ$

Question 3: [12 Marks] **Marks**

(a) Use mathematical induction to show that for all positive integers n **4**

$$\sum_{r=1}^{n} a^{-r} = \frac{a^n - 1}{(a-1)a^n}$$

(b) The tangent at the point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ cuts the y - axis at T. **4**

The line through the focus S parallel to this tangent cuts the directrix at V.

M is the midpoint of TV.

Find the locus of M as P moves on the parabola.

(c) Show that $f(x) = x - 3 + \ln x$ has a root between $x = 1$ and $x = 3$. **4**
If x_1 is this root, using Newton's method, prove that the second approximation is given by

$$x_2 = \frac{x_1(4 - \ln x_1)}{1 + x_1}$$

If $x_1 = 2$, find the value of x_2 giving your answer correct to two decimal places.

Question 4: [12 Marks] **Marks**

(a) Tidal flow in a harbour is assumed to be simple harmonic motion and water depth x metres at time t hours is given by

$$x = 20 + A\cos(nt + \alpha)$$

where A, n and α are positive constants.

The depth of water is 12 m at low tide and 28 m at high tide which occurs 7 hours later.

(i) Evaluate A and n. **3**

(ii) On a day when low tide occurs at 2.00 am, find the first time period during which the water level is greater than 22 m. **3**

(b) The acceleration of a body moving along a straight line is given by

$$\frac{d^2x}{dt^2} = -\frac{24}{x^2}$$

where x is the displacement from the origin after t seconds.
When $t = 0$, the body is 3 metres to the right of the origin with a velocity of 4 m/s.

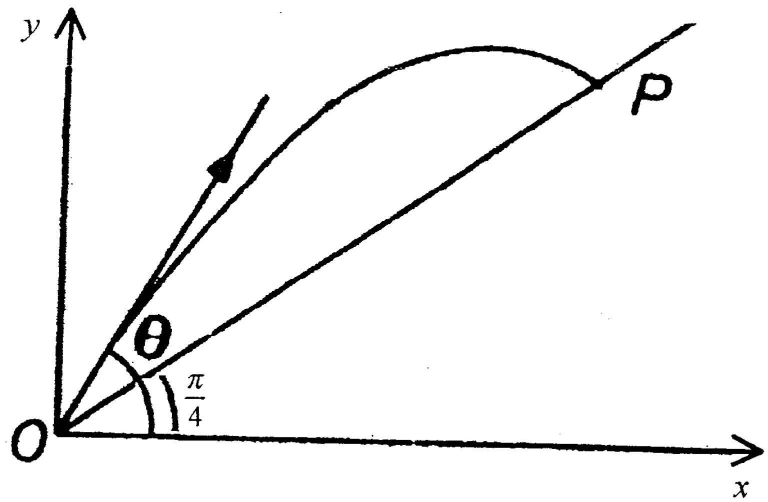
(i) Show that the velocity, v, of the body in terms of x is given by **2**

$$v = \frac{4\sqrt{3}}{\sqrt{x}}$$

(ii) Find an expression for t in terms of x. **2**

(iii) How long does it take for the body to reach a point 10 m to the right of the origin? **2**

Question 5: [12 Marks] **Marks**



A golf ball is hit with a velocity of 5 m/s. It is projected at O, at the bottom of the slope inclined at $\frac{\pi}{4}$ to the horizontal.

The ball is projected at an angle θ to the horizontal, where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

The equations of motion are $\ddot{x} = 0$ and $\ddot{y} = -10$

(i) Use calculus to show that the coordinates of the ball's position at time t seconds are given by **3**

$$x = 5t\cos\theta \text{ and } y = -5t^2 + 5t\sin\theta$$

(ii) The ball lands at P, where the length of $OP = R$ metres. **2**

Show that $x = y = \frac{R}{\sqrt{2}}$.

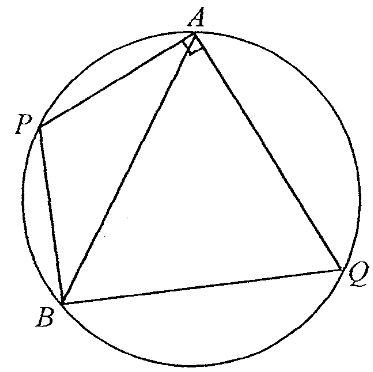
(iii) Show that $R = 5\sqrt{2}\left(\cos\theta\sin\theta - \cos^2\theta\right)$ **3**

(iv) By differentiation, find the exact value of θ (in radians) for the ball to achieve the maximum distance R. **2**

(v) Find the maximum value of R. **2**

Question 6: [12 Marks] **Marks**

(a)



A, P, B, Q are four points on a circle in a horizontal plane.

$\angle AQB = \theta$ and $\angle PAQ = \dfrac{\pi}{2}$

(i) Express $\sin\angle ABQ$ in terms of AB, AQ and θ **2**

(ii) Hence find PQ in terms of AB and θ **3**

(iii) Show that **2**

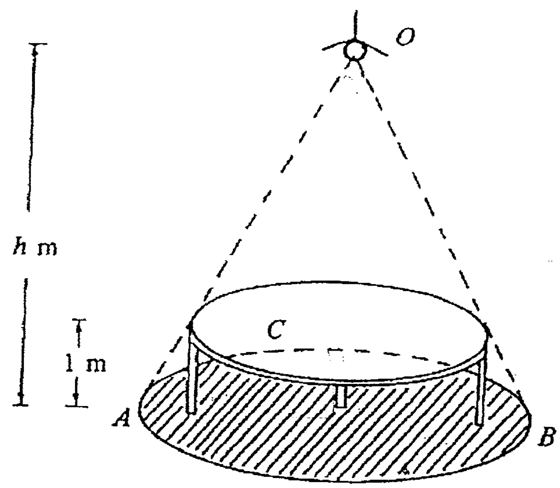
$$PQ = \frac{\sqrt{AP^2 + BP^2 + 2AP \times BP\cos\theta}}{\sin\theta}$$

(b) (i) Prove that $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$ **3**

(ii) Hence, or otherwise, obtain a value for $\cot 67\tfrac{1}{2}^\circ$ **2**

Question 7: [12 Marks]

Marks



A small lamp O is placed h m above the ground, where $1 < h \leq 5$.

Vertically below the lamp is the centre of a round table of radius 2 m and height 1 m.

The lamp casts a shadow ABC of the table on the ground.

Let S m^2 be the area of the shadow.

(i) Show that $S = \dfrac{4\pi h^2}{(h-1)^2}$ **3**

(ii) If the lamp is lowered vertically at a constant rate of $\dfrac{1}{8}$ m/s, find the rate of change of S with respect to time when $h = 2$. **4**

Let V m^3 be the volume of the cone $OABC$.

(iii) Show that $V = \dfrac{4\pi h^3}{3(h-1)^2}$ **1**

(iv) Find the minimum value of V as h varies. **4**

Does S attain a minimum when V attains its minimum? Explain your answer.

THIS IS THE END OF THE EXAMINATION

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Mathematics Extension 1

Sample Solutions

Question 1

(a) $\displaystyle\int_{-2}^{2}\frac{dx}{\sqrt{16-x^2}} = \left[\sin^{-1}\frac{x}{4}\right]_{-2}^{2}$ (1)

$= \sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1}\left(-\tfrac{1}{2}\right)$

$= \frac{\pi}{6} + \frac{\pi}{6}$ (1)

$= \frac{\pi}{3}$

(b) let $y = e^{x+1}$

inverse $x = e^{y+1}$

$\Rightarrow \log_e x = y+1$

ie $y = \log_e x - 1$ (1)

$\therefore f^{-1}(x) = \log_e x - 1$

Now $f(x) = e^{x+1}$

$\Rightarrow f\left[f^{-1}(x)\right] = e^{f^{-1}(x)+1}$

$= e^{\log_e x - 1 + 1}$

$= e^{\log_e x}$ (1)

$= x$

and

$f^{-1}(x) = \log_e x - 1$

$\Rightarrow f^{-1}(f(x)) = \log_e\left[f(x)\right] - 1$ (1)

$= \log_e\left[e^{x+1}\right] - 1$

$= (x+1)\log_e e - 1$

$= x+1 - 1$

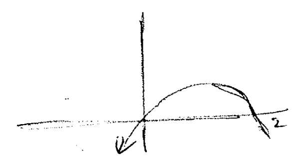
$= x$

(c) $\frac{4-x}{x} \le 1$

$x(4-x) \le x^2$

$4x - 2x^2 \le 0$ (2)

$2x - x^2 \le 0$



$\therefore$ ~~$x\le0$ or $x\ge2$~~

$x \le 0$ or $x \ge 2$

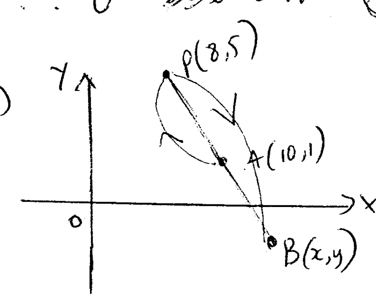
(d) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{\frac{1}{2} - \left(-\frac{1}{\sqrt{3}}\right)}{1 + \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)}\right|$ (1)

$= \left|\frac{\sqrt{3}+2}{2\sqrt{3}-1}\right|$

$\therefore \theta =$ ~~$20°$~~ 0.99^c (1)

(e)



$AP : PB = 2:3$

$A(x_1, y_1)\ B(x_2, y_2)$ $m:n$

ie $A(10,1)\ B(x_2, y_2)$ (1) $-2:3$

(1) $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right) \equiv (8,5)$ (1)

ie $\left(\frac{-2x_2+30}{1}, \frac{-2y_2+3}{1}\right) \equiv (8,5)$ $\therefore B(-11,-1)$

Question 2

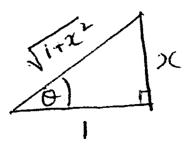
(a) $y = \tan^{-1}(\cot x)$

$$\frac{dy}{dx} = \frac{1}{1+(\cot^2 x)} \cdot \frac{d(\cot x)}{dx} \quad 1$$

$$= \frac{1}{1+\cot^2 x} \cdot \frac{-1}{\sin^2 x} \quad 1$$

$$= \frac{-\operatorname{cosec}^2 x}{1+\cot^2 x} = \frac{-\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} = -1$$

(b) let $\tan^{-1} x = \theta \Rightarrow \tan\theta = x$



2

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}$$

$$\therefore \theta = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \tan^{-1} x$$

(c) $P(x) = ax^3 + bx^2 - 8x + 3$

$P(1) = a + b - 8 + 3 = 0$ ½

$\therefore a + b = 5$ —(1)

$P(-2) = -8a + 4b + 16 + 3 = 15$ ½

$\therefore -8a + 4b = -4$ —(2)

$\Rightarrow a = 2 \qquad b = 3$ 1

(d) $$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$$

$$\therefore \frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2} \quad 1$$

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) + \frac{1}{x^2} = \frac{2-\ln x}{x^2}$$

ie $$\frac{d}{dx}\left[\frac{\ln x}{x} - \frac{1}{x}\right] = \frac{2-\ln x}{x^2}$$

$\therefore$ primitive of $\frac{2-\ln x}{x^2}$ is 1

$$\frac{\ln x}{x} - \frac{1}{x} \quad \text{or} \quad \frac{\ln x - 1}{x}$$

(e) $${}^5C_1 \times {}^3C_1 \times {}^6C_1 \times 3! = 540 \quad 2$$

(f) ½ $S\hat{R}P = S\hat{Q}P$ (angles on same arc SP at cir.

½ $B\hat{R}P = B\hat{A}P$ (angles on sam arc BP at circ

$\Rightarrow S\hat{Q}P = B\hat{A}P$ 1 which are corresp. angles formed by lines SQ and BA, transversal QA. Since corresp. angles equal, lines SQ and BA must be parallel

Question 3

$p(n): \sum_{r=1}^{n} a^{-r} = \frac{a^n - 1}{(a-1)a^n}$

$p(1)$: Test for $n=1$

LHS $= \frac{1}{a}$ RHS $= \frac{a-1}{(a-1)a}$

$= \frac{1}{a}$

$=$ LHS

$\therefore p(1)$ is true

$p(k)$: Assume $p(n)$ is true when $n=k$ ($k \in J^+$).

ie $\sum_{r=1}^{k} a^{-r} = \frac{a^k - 1}{(a-1)a^k}$

$p(k+1)$: Required to prove that $p(k) \rightarrow p(k+1)$.

ie $\sum_{r=1}^{k+1} a^{-r} = \frac{a^{k+1} - 1}{(a-1)a^{k+1}}$

LHS $= \sum_{r=1}^{k} a^{-r} + a^{-(k+1)}$

$= \frac{a^k - 1}{(a-1)a^k} + a^{-(k+1)}$ (by assumption)

$= \frac{a(a^k - 1)}{(a-1)a^{k+1}} + \frac{1}{a^{k+1}}$

$= \frac{a^{k+1} - a + (a-1)}{(a-1)a^{k+1}}$

$= \frac{a^{k+1} - 1}{(a-1)a^{k+1}}$

$=$ RHS

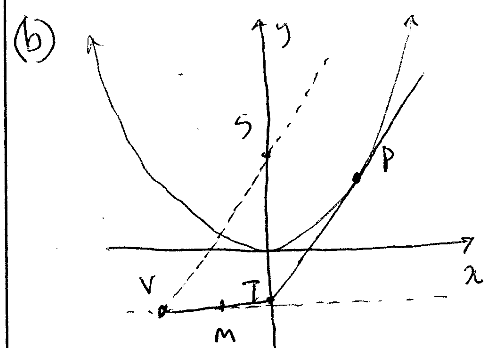
$\therefore p(k) \rightarrow p(k+1)$.

Since $p(1)$ is true,

$p(1) \rightarrow p(2) \rightarrow p(3) \rightarrow$

By Principle of Mathematical induction, $p(n)$ is true for positive integral n.

(b)



At $P(2ap, ap^2)$ $\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$

$= 2ap \cdot \frac{1}{2a}$

$= p$

$\therefore$ Tgt at P: $y - ap^2 = p(x - 2ap)$

$\therefore px - y - ap^2 = 0$

This line cuts y-axis when $x=0$

$\therefore y = -ap^2$

$\therefore T$ is $(0, -ap^2)$

The line thro' S ∥ PT is

$y - a = p(x - 0)$

$px - y + a = 0$

This line cuts $y = -a$

$px + a + a = 0$

$x = -\frac{2a}{p}$

Q3/2)

$\therefore V$ is $\left(\frac{-2a}{p}, -a\right)$

$\therefore$ For M: $x = \dfrac{\frac{-2a}{p} + 0}{2}$

$\therefore x = -\dfrac{a}{p}$ — ①

$y = \dfrac{-a + -ap^2}{2}$

$= \dfrac{-a(1+p^2)}{2}$ — ②

For locus, eliminate p.

From equation ① $p = -\dfrac{a}{x}$

$\therefore y = \dfrac{-a\left(1 + \left(\frac{-a}{x}\right)^2\right)}{2}$

$2y = -a - a \times \dfrac{a^2}{x^2}$

$2y = -a\left(1 + \dfrac{a^2}{x^2}\right)$

$y = -\dfrac{a}{2}\left(1 + \dfrac{a^2}{x^2}\right)$

(c) $f(x) = x - 3 + \ln x$

$f(1) = 1 - 3 + 0$
$= -2$

$f(3) = 3 - 3 + \ln 3$
$= \ln 3$

Since the sign of $f(x)$ changes between 1 and 3 and it is continuous in the domain, there must be at least one root.

Newton's method state

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Here $f'(x) = 1 + \dfrac{1}{x}$

$\therefore x_2 = x_1 - \dfrac{f(x_1)}{1 + \frac{1}{x_1}}$

$= x_1 - \dfrac{x_1 - 3 + \ln x_1}{1 + \frac{1}{x_1}}$

$= x_1 - \dfrac{x_1^2 - 3x_1 + x_1 \ln x_1}{1 + x_1}$

$= \dfrac{x_1(1 + x_1) - x_1(x_1 - 3 + \ln x_1)}{1 + x_1}$

$= \dfrac{x_1(4 - \ln x_1)}{1 + x_1}$

Now if $x_1 = 2$

$x_2 = \dfrac{2(4 - \ln 2)}{1 + 2}$

$= \dfrac{8 - 2\ln 2}{3}$

$\doteqdot 2.20$

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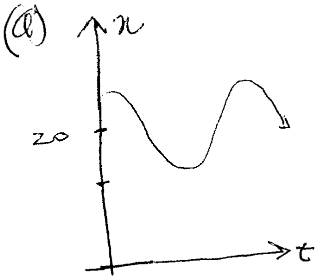
Question 4

(a)



$x = 20 + A\cos(nt + \alpha)$

Trough to crest $= 28 - 12$

$= 16$

$\therefore A = 8$

Period $= 2 \times$ (Trough to crest)

$= 2 \times 7$

$= 14$

$\therefore 14 = \frac{2\pi}{n}$

$\therefore n = \frac{\pi}{7}$

Let $t=0$ at 2:00pm.

We seek.

$22 = 20 + 8\cos\left(\frac{\pi t}{7} + \alpha\right)$

Now $12 = 20 + 8\cos(0 + \alpha)$

$-8 = 8\cos\alpha$

$\cos\alpha = -1$

$\alpha = \pi$

$\therefore 22 = 20 + 8\cos\left(\frac{\pi t}{7} + \pi\right)$

$\frac{1}{4} = \cos\left(\frac{\pi}{7}t + \pi\right)$

$\frac{\pi}{7}t + \pi = \cos^{-1}\left(\frac{1}{4}\right) + k\pi$

$\frac{\pi}{7}t = \cos^{-1}\frac{1}{4} - \pi + k\pi$

$t = \frac{7}{\pi}\left(\cos^{-1}\frac{1}{4} + (k-1)\pi\right)$

$= 9.936$ when $k=$

$\therefore$ Time 2:00p + 9hr 56' 13"

= 11:56:13 pm until

$= 16.936$ when $k=3$

$\therefore$ Tide remains above 22m from 11:56pm till 4:56am.

(b) $\frac{d^2x}{dt^2} = -\frac{24}{x^2}$

When $t=0, x=3, v=4$

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{24}{x^2}$

$\int \frac{d}{dx}\left(\frac{1}{2}v^2\right)dx = -24\int \frac{dx}{x^2} + C$

$\frac{1}{2}v^2 = \frac{24}{x} + C$

$v^2 = \frac{48}{x} + C'$

When $x=3, v=4$

$16 = 16 + C'$

$C' = 0$

$\therefore v^2 = \frac{48}{x}$

Now $v = \pm \frac{\sqrt{48}}{\sqrt{x}}$

Since $v > 0$ initially, we choose the positive root.

$\therefore v = \frac{4\sqrt{3}}{\sqrt{x}}$

(ii) $\frac{dx}{dt} = \frac{4\sqrt{3}}{\sqrt{x}}$

$\therefore \frac{dt}{dx} = \frac{\sqrt{x}}{4\sqrt{3}}$

So $\int \frac{dt}{dx} dx = \int \frac{\sqrt{x}}{4\sqrt{3}} dx + D$

$t = \frac{x^{3/2}}{3/2 \times 4\sqrt{3}} + D$

$t = \frac{x\sqrt{x}}{6\sqrt{3}} + D$

When $t=0$, $x=3$

$0 = \frac{3\sqrt{3}}{6\sqrt{3}} + D$

$0 = \frac{1}{2} + D$

$\therefore D = -\frac{1}{2}$

$t = \frac{x\sqrt{x}}{6\sqrt{3}} - \frac{1}{2}$

(iii) When $x = 10$

$t = \frac{10\sqrt{10}}{6\sqrt{3}} - \frac{1}{2}$

$= 2.543$ sec

(5)

(i)

$\ddot{x} = 0$

$\dot{x} = c_1 \quad t=0, \dot{x} = 5\cos\theta.$

$\dot{x} = 5\cos\theta$

$x = 5t\cos\theta + c_2$

$\boxed{x = 5t\cos\theta}$

$c_2 = 0$ as $x = 0$ when $t = 0$

$\ddot{y} = -10$

$\dot{y} = -10t + c_3$ clearly $t=0, \dot{y} = 5\sin\theta$

$\dot{y} = -10t + 5\sin\theta$

$y = -5t^2 + 5t\sin\theta + c_4$

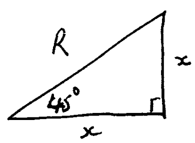
clearly when $t=0, y=0$

$\therefore \boxed{y = -5t^2 + 5t\sin\theta}$

(3)

($2\frac{1}{2}$ marks if not evaluating c_2 etc.)

(ii)



clearly $\frac{y}{x} = \tan 45° = 1$

$\therefore y = x.$

and $\frac{x}{R} = \sin 45°$

$x = \frac{R}{\sqrt{2}}.$

$\therefore \boxed{y = x = \frac{R}{\sqrt{2}}}$

(2)

(iii) If $x = y$. $\quad 5t\cos\theta = -5t^2 + 5t\sin\theta$

$5t^2 + 5t\cos\theta - 5t\sin\theta = 0$

$5t(t + \cos\theta - \sin\theta) = 0$

$t = 0, \sin\theta - \cos\theta.$

$\therefore \quad x = 5(\sin\theta - \cos\theta)\cos\theta$

$\therefore \boxed{R = 5\sqrt{2}(\sin\theta\cos\theta - \cos^2\theta)}$

(3)

(iv) $R' = 5\sqrt{2}\left[\sin\theta \times -\sin\theta + \cos\theta . \cos\theta - 2\cos\theta . -\sin\theta\right]$

$= 5\sqrt{2}\left[\cos^2\theta - \sin^2\theta + \sin 2\theta\right]$

$= 5\sqrt{2}\left[\cos 2\theta + \sin 2\theta\right]$

If $R' = 0$

$\cos 2\theta + \sin 2\theta = 0$

$\sin 2\theta = -\cos 2\theta$

$\tan 2\theta = -1.$

$2\theta = \frac{3\pi}{4}$

$R'' = 5\sqrt{2}\left[-2\sin 2\theta + 2\cos 2\theta\right]$

$= +10\sqrt{2}\left[\cos 2\theta - \sin 2\theta\right]$

(2)

($1\frac{1}{2}$ if not verifying max)

$\boxed{\theta = \frac{3\pi}{8}}$, $R'' = +10\sqrt{2}\left[-\frac{1}{\sqrt{2}} - \ldots\right]$

$= -20 < 0$

$\therefore$ MAX

(v) $R = 5\sqrt{2}\left(\cos\frac{3\pi}{8}\sin\frac{3\pi}{8} - \cos^2\frac{3\pi}{8}\right)$

$= 5\sqrt{2}\left(\frac{1}{2\sqrt{2}} - \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)\right)$

$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)$

$= 5\sqrt{2}\left(\frac{2-\sqrt{2}}{2\sqrt{2}}\right)$

$= \boxed{\frac{5}{2}(2-\sqrt{2})}.$ (2 for exact answer, 1½ for approximation)

(2)

NB $\cos\frac{3\pi}{4} = 2\cos^2\frac{3\pi}{8} - 1$

$2\cos^2\frac{3\pi}{8} = 1 + \cos\frac{3\pi}{4}$

$\cos^2\frac{3\pi}{8} = \frac{1}{2} + \frac{1}{2}\cos\frac{3\pi}{4}$

$= \frac{1}{2} + \frac{1}{2} \times -\frac{1}{\sqrt{2}}.$

$= \frac{1}{2} - \frac{1}{2\sqrt{2}}$

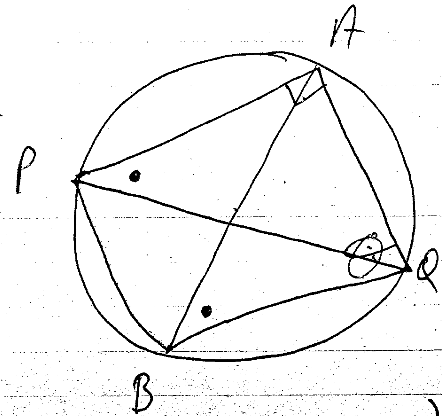
also $\cos\frac{3\pi}{8}\sin\frac{3\pi}{8} = \frac{1}{2}\sin\frac{3\pi}{4}$

$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$

$= \frac{1}{2\sqrt{2}}.$

NB $\frac{5}{2}(2-\sqrt{2}) \doteqdot 1.464.$

Q6



i) $\frac{AQ}{\sin A\hat{B}Q} = \frac{AB}{\sin\theta}$ (2)

So $\sin A\hat{B}Q = \frac{AQ}{AB}\sin\theta$

ii) Now $\sin A\hat{B}Q = \sin A\hat{P}Q$ (equal angles on same arc)

$\therefore \frac{AQ}{AB}\sin\theta = \frac{AQ}{PQ}$ (3)

$\therefore \frac{\sin\theta}{AB} = \frac{1}{PQ}$

$\therefore PQ = \frac{AB}{\sin\theta}$

(iii) Now $APB = 180 - \theta$ ($\angle$s of cyclic quad)

In $\triangle APB$: $AB^2 = BP^2 + PA^2 - 2BP\,PA\cos(180-\theta)$

$AB^2 = AP^2 + BP^2 + 2AP \times BP\cos\theta$ (2)

$AB = \sqrt{AP^2 + BP^2 + 2AP\,BP\cos\theta}$

but $AB = PQ\sin\theta$ $\therefore PQ = \frac{\sqrt{AP^2+BP^2+2AP\,BP\cos\theta}}{\sin\theta}$

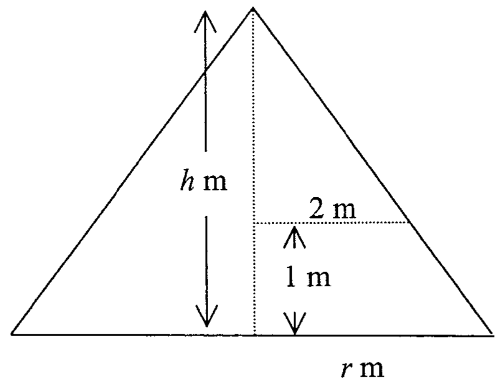
b i) $\frac{\sin 2x}{1-\cos 2x} = \cot x \Rightarrow$ LHS $= \frac{2\sin x\cos x}{1-(1-2\sin^2 x)} = \frac{2\sin x\cos x}{2\sin^2 x}$

$= \frac{\cos x}{\sin x} = \cot x.$ (3)

ii) $\cot 67\tfrac{1}{2}^\circ = \frac{\sin 135^\circ}{1-\cos 135^\circ} = \frac{\sqrt{2}}{2} \div \left(1+\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} \times \frac{2}{2+\sqrt{2}}$ (2)

$= \frac{\sqrt{2}}{2+\sqrt{2}} = \sqrt{2}-1$ or $\frac{1}{\sqrt{2}+1}$

(7)



(i)

$$S = \pi r^2$$

$$\frac{r}{h} = \frac{2}{h-1} \text{(similar triangles)}$$

$$\therefore r = \frac{2h}{h-1}$$

$$\therefore S = \frac{4\pi h^2}{(h-1)^2}$$

(ii)

$$\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = -\frac{1}{8}}$$

$$S = \frac{4\pi h^2}{(h-1)^2}$$

$$\frac{dS}{dh} = \frac{(8\pi h) \times (h-1)^2 - 4\pi h^2 \times 2(h-1)}{(h-1)^4}$$

$$= \frac{8\pi h(h-1)\left[(h-1)-h\right]}{(h-1)^4}$$

$$= -\frac{8\pi h}{(h-1)^3}$$

$$\frac{dS}{dt} = -\frac{8\pi h}{(h-1)^3} \times -\frac{1}{8} = \frac{\pi h}{(h-1)^2}$$

$$= 2\pi \text{ m}^2/\text{s when } h = 2$$

(iii)

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi \times \frac{4h^2}{(h-1)^2} \times h = \frac{4\pi h^3}{3(h-1)^2}$$

(iv)

$$V = \frac{4\pi h^3}{3(h-1)^2}$$

$$\frac{dV}{dh} = \frac{3(h-1)^2 \times 12\pi h^2 - 4\pi h^3 \times 6(h-1)}{9(h-1)^4}$$

$$= \frac{12\pi h^2 (h-1)\left[3(h-1) - 2h\right]}{9(h-1)^4}$$

$$= \frac{4\pi h^2 (h-3)}{9(h-1)^3}$$

Minimum when $\frac{dV}{dh} = 0$

$\frac{dV}{dh} = \frac{4\pi h^2 (h-3)}{9(h-1)^3} = 0 \Rightarrow h = 0, 3$

$\because 1 < h \leq 5 \Rightarrow h = 3$

h	2	3	4
$\frac{dV}{dh}$	- 1	0	$\frac{1}{8}$

NB We only need to test $\frac{(h-3)}{(h-1)^3}$ $\because \frac{4\pi h^2}{9} > 0$

So there is a *relative* minimum at $h = 3$

$V = 9\pi$

Testing end points $h = 5$, $V = \frac{125}{12}\pi$

So the minimum value of V is 9π, when $h = 3$

Note that $\frac{dS}{dh} \neq 0$ for $1 < h \leq 5$

So the minimum value of S will occur when $h = 5$, so the two minimums don't coincide for the same value of h.