SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in 3 bundles.
  Section A (Questions 1 - 3),
  Section B (Questions 4 - 5) and
  Section C (Questions 6 - 7).
- Start each Section in a **NEW** answer booklet.

Total Marks - 84 Marks

- Attempt questions 1- 7
- All questions are of equal value.

Examiner: *R. Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 84
Attempt Questions 1 – 7
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (12 marks) **Marks**

(a) Solve for x: $(x^2-1)(x+5)>0$ **2**

(b) Differentiate $y=\ln\sqrt{x+1}$ for $x>-1$ **2**

(c) Use the Table of Integrals provided to evaluate **2**

$$\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx$$

(d) Find the exact value of $\displaystyle\int_0^{\sqrt{3}} \frac{1}{9+x^2}\,dx$ **2**

(e) 8 people including A and B are to be seated around a circle. **2**

How many arrangements are possible if A and B do not wish to sit together?

(f) Show that $\dfrac{1-\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{1+\cos\theta}=2\tan\dfrac{\theta}{2}$ **2**

Question 2 (12 marks) **Marks**

(a) Differentiate $y = \sin^{-1} 2x$ **2**

(b) Find the domain and range of $y = 3\sin^{-1}\sqrt{1-x^2}$ **2**

(c) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Hence or otherwise, find the general solution for **2**

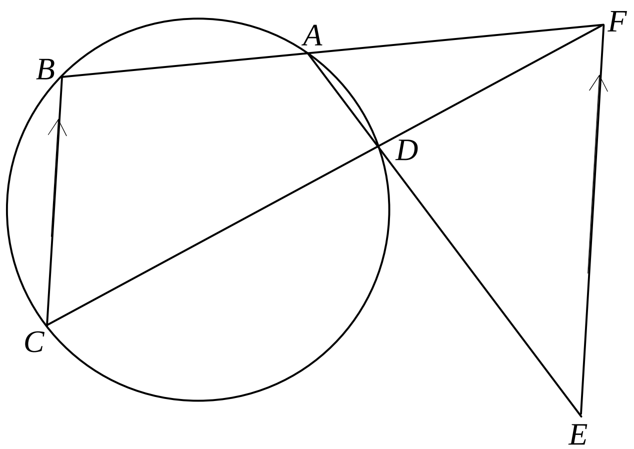
$$\sqrt{3}\cos x - \sin x = 1$$

(d) In the diagram below $ABCD$ is a cyclic quadrilateral.

BA is produced to F.

$BC \parallel FE$

CF and AE meet at D.



Copy or trace the diagram into your answer booklet.

(i) Show that $\Delta DEF \parallel\mid \Delta FEA$ **2**

(ii) Hence show that $(EF)^2 = EA \times ED$ **2**

Section A is continued on page 4

SECTION A continued

Question 3 (12 marks) **Marks**

(a) Use the Principle of Mathematical Induction to show that $2^{3n}-1$ is divisible by 7 for all integers $n \geq 1$. **3**

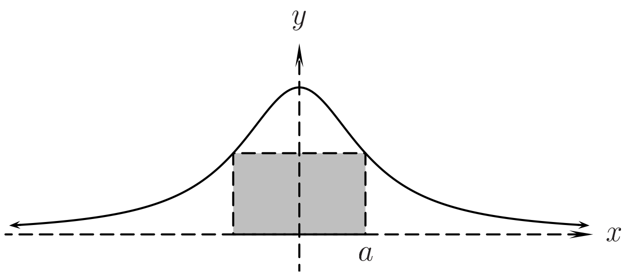
(b) For the curve $y = 1 + 2\cos x - 2\cos^2 x$,

(i) Show that $\dfrac{dy}{dx} = 2\sin x\left(2\cos x - 1\right)$ **1**

(ii) Hence find the stationary point(s) in the interval $-\dfrac{\pi}{6} \leq x \leq \dfrac{\pi}{2}$ **2**

(iii) Sketch the curve and find the greatest and least value of y in $-\dfrac{\pi}{6} \leq x \leq \dfrac{\pi}{2}$ **2**

(c)



A rectangle is inscribed under the curve $y = \dfrac{1}{1+x^2}$, as shown in the diagram above, such that the rectangle is symmetrical about the y axis.

(i) Show that the area of the rectangle is given by $\dfrac{2a}{1+a^2}$. **1**

(ii) Find the value of a that produces the maximum area of the rectangle and what is this maximum area? **3**

END OF SECTION A

SECTION B (Use a SEPARATE writing booklet)

Question 4 (12 marks) **Marks**

(a) (i) Show that the equation of the tangent at $T\left(-2t, t^2\right)$ on the parabola $y = \frac{1}{4}x^2$ is given by $tx + y + t^2 = 0$. **2**

(ii) $M\left(x, y\right)$ is the midpoint of the interval TA where A is the x intercept of the tangent at T. **2**

Find the equation of the locus of M as T moves on the parabola.

(b) Solve $4x^3 - 12x^2 + 11x - 3 = 0$ if the roots are the terms of an arithmetic series. **3**

(c) (i) Find the points of intersection of the curves $y = 2\cos x$ and $y = \frac{1}{2}\sec x$ in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **2**

(ii) The area enclosed between the two curves listed above is rotated $360°$ about the x axis. **3**

Find the volume of the solid of revolution.
(Leave your answer in exact form.)

Section B is continued on page 6

Question 5 (12 marks) **Marks**

(a) A spherical balloon leaks air such that the radius decreases at a rate of 5 cm/second. **2**

Calculate the rate of change of the volume of the balloon when the radius is 100 mm.

[The volume of a sphere is $V = \frac{4}{3}\pi r^3$]

(b) A particle moves in such a way that its displacement x cm from the origin O after a time t seconds is given by

$$x = 2\cos\left(t + \frac{\pi}{6}\right) \text{ cm}$$

(i) Show that the particle moves in Simple Harmonic Motion. **2**

(ii) Evaluate the period of the motion. **1**

(iii) Find the time at which the particle first passes through the origin on its first oscillation. **1**

(iv) Find the velocity when the particle is 1 cm from the origin on its first oscillation. **2**

(c) Find $\int \sqrt{16 - x^2}\, dx$ using the substitution $x = 4\sin\theta$. **4**

END OF SECTION B

SECTION C (Use a SEPARATE writing booklet)

Question 6 (12 marks) **Marks**

(a) Find a primitive function for $\dfrac{3x}{4+x^2}$ **1**

(b) If $P(x)=8x^3-12x^2+6x+13$,

(i) For what values of x is $P(x)$ increasing? **1**

(ii) Show that $P(x)$ has only one zero, x_1 and that $x_1<0$. **1**

(iii) Taking $x=-1$ as a first approximation to $P(x)=0$, find a second approximation for x_1, using Newton's Method. **2**

[Express your answer correct to 2 decimal places.]

(c) At any time *t*, the rate of cooling of the temperature *T* of a body, when the surrounding temperature is *S*, is given by the differential equation

$$\frac{dT}{dt}=-k(T-S)$$

for some constant *k*.

(i) Show that $T=S+Ae^{-kt}$, for some constant *A*, satisfies this differential equation. **2**

(ii) A metal rod has a temperature of $1390°$ C and cools to $1060°$ C in 10 minutes when the surrounding temperature is $30°$ C. **3**

Find how much *longer* it will take the rod to cool to $110°$ C, giving your answer to the nearest minute.

(iii) Sketch the graph of the function $T=S+Ae^{-kt}$, using the values of *S*, *A* and *k* found above. **2**

Section C continues on page 8

SECTION C continued

Question 7 (12 marks) **Marks**

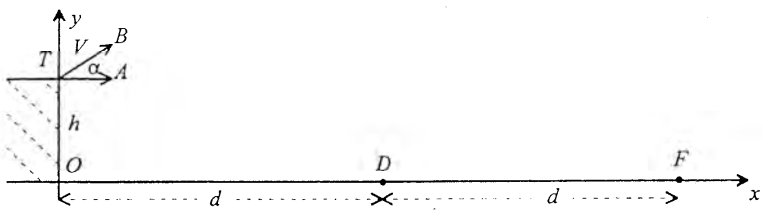
(a) Using the expansion of $(1+x)^n$

(i) Find an expression for $\sum_{r=1}^{n} r\binom{n}{r}$ **2**

(ii) Hence, or otherwise, prove that $\sum_{r=0}^{n}(r+1)\binom{n}{r} = 2^{n-1}(n+2)$ **2**

(b) T is the top of a building, h metres high. The points O, D and F are in the same line on flat level ground.
O is the base of the building.
D is d metres from O, and F is a further d metres from D.
At time $t=0$, two particles A and B are projected with the same initial velocity V m/s from T.
Particle A is projected horizontally and particle B is projected in the same direction, but at an angle α, $\alpha > 0$, to the horizontal.
The equations of motion of both particles are

$$\ddot{x} = 0 \text{ and } \ddot{y} = -g$$



(i) Assuming that the position of particle A at time t is given by **1**

$$x = Vt,\ y = -\frac{1}{2}gt^2 + h$$

show that the Cartesian equation of the trajectory is given by

$$y = h - \frac{g}{2V^2}x^2$$

(ii) Assuming that the position of particle B at time t is given by **1**

$$x = Vt\cos\alpha \text{ and } y = -\frac{1}{2}gt^2 + Vt\sin\alpha + h$$

show that the Cartesian equation of the trajectory is given by

$$y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha + h$$

(iii) If A lands at D show that $h = \dfrac{gd^2}{2V^2}$ **1**

(iv) If both A and B land at D show that $\tan\alpha = \dfrac{d}{h}$ **2**

(v) If A lands at D and B lands at F show that $d \geq 2h\sqrt{3}$ **3**

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

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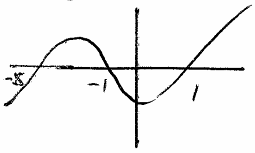
Mathematics Extension 1

Sample Solutions

Section	Marker
A	Mr Dunn
B	Ms Nesbitt
C	Mr Bigelow

Section A

1 a) $(x^2-1)(x+5)>0$



$x>1$

AND

$-5<x<-1$ (2 marks)

b) $y=\ln\sqrt{x+1}$

$=\frac{1}{2}\ln(x+1)$

$y'=\frac{1}{2(x+1)}$ (2 marks)

c) $\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx$

$=\frac{1}{2}\sec 2x\Big]_0^{\pi/6}$

$=\frac{1}{2}\sec\frac{\pi}{3}-\frac{1}{2}\sec 0$

$=\frac{1}{2}\times 2-\frac{1}{2}\times 1$

$=\frac{1}{2}$ (2 marks)

d) $\int_0^{\sqrt{3}} \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\frac{x}{3}\Big]_0^{\sqrt{3}}$

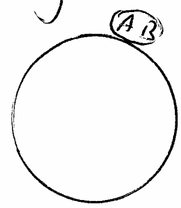
$=\left[\frac{1}{3}\tan^{-1}\frac{\sqrt{3}}{3}\right]$

$=\frac{1}{3}\tan^{-1}\frac{1}{\sqrt{3}}$

$=\frac{1}{3}\times\frac{\pi}{6}$

$=\frac{\pi}{18}$ (2 marks)

e) Total number of arrangements $=7!$



If A and B are together

Then $2\times 6!$

Hence not together

$=7!-2\times 6!$

$=6!(7-2)$

$=5\times 6!$

$=3600$ (2 marks)

f) LHS $=\frac{1-\cos\theta}{\sin\theta}+\frac{\sin\theta}{1+\cos\theta}$

$=\frac{1-\cos^2\theta+\sin^2\theta}{\sin\theta(1+\cos\theta)}$

$=\frac{2\sin^2\theta}{\sin\theta(1+\cos\theta)}$

$=\frac{2\sin\theta}{1+\cos\theta}$

Let $t=\tan\frac{\theta}{2}$

$=\frac{2\times\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$

$=\frac{\frac{4t}{1+t^2}}{\frac{1+t^2+1-t^2}{}}=\frac{4t}{2}=2t$

$=2\tan\frac{\theta}{2}=$ RHS (2 marks)

QUESTION TWO

a) $y = \sin^{-1} 2x$

Let $u = 2x$

Then $y = \sin^{-1} u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \times 2$$

$$= \frac{2}{\sqrt{1-4x^2}}$$ (2 marks)

b) $y = 3 \sin^{-1} \sqrt{1-x^2}$

Consider $\sqrt{1-x^2}$

$-1 \leq x \leq 1$ ~~Range~~ Domain

Then

$y = 3\sin^{-1} 0$ to $3 \sin^{-1} 1$

ie $0 \leq y \leq \frac{3\pi}{2}$ Range

(2 marks)

c) $\sqrt{3} \cos x - \sin x = R\cos(x+\alpha)$

$= R\cos x \cos\alpha - R\sin x \sin\alpha$

$R\cos\alpha = \sqrt{3}$

$R\sin\alpha = 1$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$R^2(\cos^2\alpha + \sin^2\alpha) = 3+1$

$R = 2$

c) continued

$2\cos(x + \frac{\pi}{6}) = 1$ (2 marks)

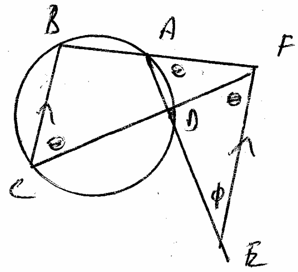
$\cos(x + \frac{\pi}{6}) = \frac{1}{2}$

$x + \frac{\pi}{6} = \pm \frac{\pi}{3}$

$x = 2k\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$

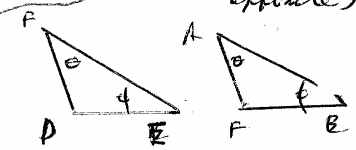
$x = 2k\pi + \frac{\pi}{6}$ or $2k\pi - \frac{\pi}{2}$ (2 marks)

d)



$F\hat{A}E = F\hat{B}C$ (angle in alternate segment)

$B\hat{C}F = C\hat{F}E$ (alternate opposite)



Hence $\Delta DEF \;|||\; \Delta FEA$ (2 marks)

$$\frac{EF}{EA} = \frac{ED}{EF}$$

$EF^2 = EA \times ED$

(2 marks)

QUESTION THREE

a) Prove $2^{3n}-1$ is divisible by 7 for $n \geq 1$ (integers)

Let $n=1$ Then $2^3-1=7$

ie True for $n=1$

Assume

$2^{3k}-1=7K$ where K is an integer

Try to prove

$2^{3k+3}-1=7N$ where N is an integer

$LHS = 2^3 . 2^{3k} - 1$

$= 8(7K+1)-1$ from assumption.

$= 56K+7$

$= 7(8K+1)$

$= 7N$

True for $n=1$

$n=1+1=2$

$n=2+1=3$

$\vdots$

All integers $n \geq 1$ (3marks)

b) i) $y = 1 + 2\cos x - 2\cos^2 x$

$y' = -2\sin x + 4\cos x \sin x$

$= 2\sin x\,(2\cos x - 1)$ (1mark)

ii) $y' = 0$ when $\sin x = 0$

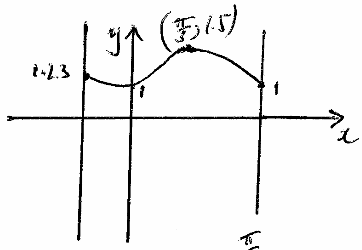
$\cos x = 1/2$

ie $x = 0, \frac{\pi}{3}$

When $x=0$, $y=1$

$x=\frac{\pi}{3}$, $y=\frac{3}{2}$ $\Big\}$ 2marks

iii)



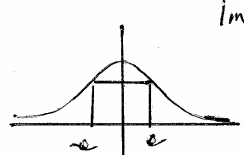
(1marks)

Max of 1.5 at $x=\frac{\pi}{3}$

Minimum of 1 at $x=0$ or $x=\frac{\pi}{2}$

1mark

c)

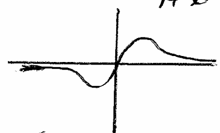


Area $= lb$

$= 2a \times \frac{1}{1+a^2} = \frac{2a}{1+a^2}$

(1mark)

Consider $y = \frac{2x}{1+x^2}$



$y' = \frac{(x^2+1)2 - 2.2x}{(1+x^2)^2}$

$= \frac{2-2x^2}{(1+x^2)^2}$

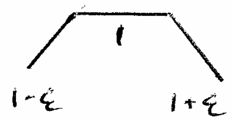
$y' = 0$ when $x = \pm 1$

THREE

c ii) continued.

When $x = 1+\epsilon$ $\quad y' < 0$

$x = 1-\epsilon$ $\quad y' > 0$



Hence $x = 1$ produces maximum

$$\text{Area} = \frac{2}{1+1} = 2 \text{ square units}$$ (3 marks)

OR

$$y'' = \frac{(1+x^2)^2(-4x) - (2-2x^2)\,4x(1+x^2)}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)\left[1+x^2+(2-2x^2)\right]}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)(3-x^2)}{(1+x^2)^4}$$

When $x = 1$ $\quad y'' = \dfrac{-4 \times 2 \times 2}{2^4}$

$y'' < 0$ Hence maximum.

QUESTION 4

i) $x=-2t$, $t=-\frac{x}{2}$

$y=\frac{1}{4}x^2$

$y' = \frac{x}{2} = -t$

eqn of tangent $y-t^2=-t(x+2t)$

$y-t^2+tx+2t^2=0$

$tx+y+t^2=0$

ii) $tx+y+t^2=0$

at A , $y=0$

$tx+t^2=0$

$t(x+t)=0$, $x=-t$

A $(-t,0)$ $\quad$ T$(-2t,t^2)$

Midpoint M $\frac{-t-2t}{2}, \left(\frac{0+t^2}{2}\right)$

$M = \left(-\frac{3t}{2}, \frac{t^2}{2}\right)$

$x=-\frac{3t}{2}$, $t=-\frac{2x}{3}$

$y=\frac{t^2}{2}$

$=\frac{1}{2}\left(-\frac{2x}{3}\right)^2$

$y=\frac{2x^2}{9}$

Locus of M $\quad x^2=\frac{9}{2}y$

$4x^3-12x^2+11x-3=0$

roots $\alpha-d, \alpha, \alpha+d$ (arith series)

Sum of roots $= 3\alpha = -\frac{b}{a} = 3$

$\alpha = 3$

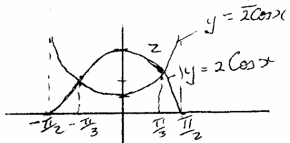
product $1(1-d)+1(1+d)+(1-d)(1+d)=\frac{c}{a}$

$3-d^2=\frac{11}{4}$

$d^2=\frac{1}{4}$

$d=\pm\frac{1}{2}$

roots $\frac{1}{2}, 1, 1\frac{1}{2}$.



$2\cos x=\frac{1}{2\cos x}$

$4\cos^2 x=1$

$\cos x=\frac{1}{2}$ or $\cos x=-\frac{1}{2}$

$x=-\frac{\pi}{3}, \frac{\pi}{3}$ or no sol'n in domain

$V=\pi\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left(4\cos^2 x-\frac{1}{4}\sec^2 x\right)dx$

$2\cos^2 x = \cos 2x+1$

$V=2\pi\int_0^{\frac{\pi}{3}}\left(2\cos 2x+2-\frac{1}{4}\sec^2 x\right)dx$

$=2\pi\left[\sin 2x+2x-\frac{1}{4}\tan x\right]_0^{\frac{\pi}{3}}$

$=2\pi\left(\frac{\sqrt{3}}{2}+\frac{2\pi}{3}-\frac{\sqrt{3}}{4}\right)-0$

$V=\left(\frac{4\pi^2}{3}+\frac{\sqrt{3}}{2}\right)u^3$

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5 (a) Find $\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$

$\frac{dr}{dt}=-5$ cm/s $\quad V=\frac{4}{3}\pi r^3$

$\frac{dV}{dr}=4\pi r^2$

$r=10$ cm

$\frac{dV}{dt} = -5\times 4\times\pi\times 100$

$= -2000\pi\ cm^3/s$

(b) $x=2\cos\left(t+\frac{\pi}{6}\right)$

$\dot{x}=-2\sin\left(t+\frac{\pi}{6}\right)$

$\ddot{x}=-2\cos\left(t+\frac{\pi}{6}\right)$

$\ddot{x}=-1^2x$, in the form $-n^2x$, $n=1$

$\therefore$ motion is SHM

(ii) Period $=\frac{2\pi}{n}=2\pi$

(iii) $x=2\cos\left(t+\frac{\pi}{6}\right)=0$

$t+\frac{\pi}{6}=\frac{\pi}{2}+2n\pi$

$t=\frac{\pi}{3}$ sec (1st osc.)

(iv) $2\cos\left(t+\frac{\pi}{6}\right)=1$

$t+\frac{\pi}{6}=\pm\frac{\pi}{3}+2n\pi$

$t=\frac{\pi}{6}$ (1st osc.)

$\dot{x}=-2\sin\frac{\pi}{3}$

$V=-2\times\frac{\sqrt{3}}{2}$

$V=-\sqrt{3}$ cm/s

QUESTION 5(c)

$\int \sqrt{16-x^2}\, dx$ $\qquad x = 4\sin\theta$

$= \int \sqrt{16-16\sin^2\theta}\; 4\cos\theta\, d\theta$ $\qquad \frac{dx}{d\theta} = 4\cos\theta$

$dx = 4\cos\theta\, d\theta$

$\int \sqrt{16\cos^2\theta} \cdot 4\cos\theta\, d\theta$

$\int 4\cos\theta \cdot 4\cos\theta\, d\theta$

$16\int \cos^2\theta\, d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

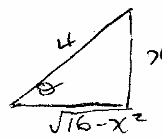
$2\cos^2\theta = \cos 2\theta + 1$

$8\int (\cos 2\theta + 1)\, d\theta$

$8\left(\frac{1}{2}\sin 2\theta + \theta\right)$

$4\sin 2\theta + 8\theta + c$

$4 \cdot 2\sin\theta\cos\theta + 8\theta$



$4 \times 2 \cdot \frac{x}{4} \frac{\sqrt{16-x^2}}{4} + 8\sin^{-1}\frac{x}{4}$ $\qquad \theta = \sin^{-1}\frac{x}{4}$

$= \frac{x}{2}\sqrt{16-x^2} + 8\sin^{-1}\frac{x}{4} + c$

QUESTION 6.

(a) If $y' = \frac{3x}{4+x^2}$

$\boxed{y = \frac{3}{2}\ln(4+x^2)+c.}$ ✓

(b). $P(x) = 8x^3 - 12x^2 + 6x + 13$

$P'(x) = 24x^2 - 24x + 6.$

$= 6(2x-1)^2$

(i) $P(x)$ is increasing where $P'(x) > 0$.

ie $6(2x-1)^2 > 0$

$\therefore$ all Reals, except $x = \frac{1}{2}$. ✓

(ii) Since $P(x) \to -\infty$ as $x \to -\infty$, $P(0) = 13$.
and $P(x)$ is increasing for all $x \neq \frac{1}{2}$.
it follows that there must be a root x_1 where $x_1 < 0$. ✓

(iii) $a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$

if $a_1 = -1$ then $a_2 = -1 - \frac{-8-12-6+13}{24+24+6}$.

$= -1 - \frac{-13}{54}$

$= \frac{-41}{54}$

$= \boxed{-0.76}$ (2.D.P). ✓✓

(c) (i) $T = S + Ae^{-kt}$ — (A)

$\therefore \frac{dT}{dt} = -kAe^{-kt}$

$= -k(T-S)$ from (A)

(ii) when $t=0$, $T=1390$ and $S=30$ (constant)

$\therefore 1390 = 30 + Ae^{0}$

$\therefore A = 1360$

$\therefore T = 30 + 1360e^{-kt}$

when $t=10$, $T = 1060$

$\therefore 1060 = 30 + 1360e^{-10k}$

$\frac{1030}{1360} = e^{-10k}$

$-10k = \ln \frac{103}{136}$

$k \doteqdot 0.0278$

$\therefore T = 30 + 1360e^{-0.0278t}$

Let $T = 110$

$110 = 30 + 1360e^{-0.0278t}$

$\frac{80}{1360} = e^{-0.0278t}$

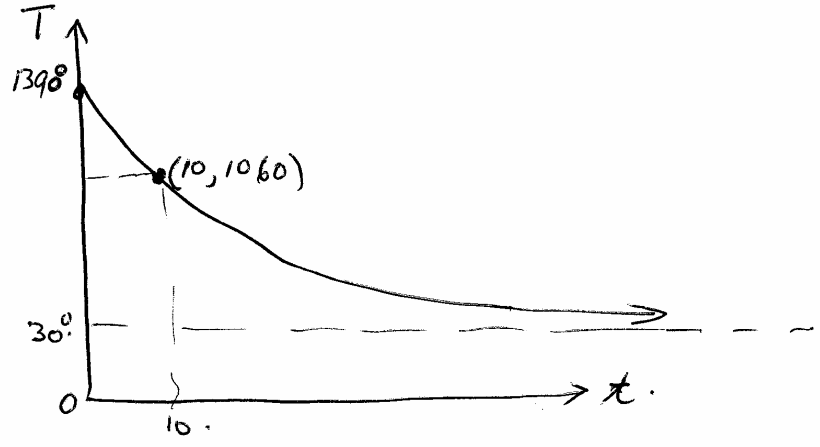
$\therefore \ln \frac{8}{136} = -0.0278t$

$t = \frac{\ln \frac{1}{17}}{-0.0278}$

$\doteqdot 102$ mins

$\therefore$ $\boxed{\text{it takes 92 mins longer.}}$

(III)



QUESTION 7.

(a) now

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{r}x^r + \cdots$$
$$\cdots + \binom{n}{n}x^n \quad \text{— (A)}$$

(i) Differentiate both sides of (A) above.

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + r\binom{n}{r}x^{r-1} + \cdots$$
$$\cdots + n\binom{n}{n}x^{n-1}.$$

let $x = 1$

$$n \cdot 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + r\binom{n}{r} + \cdots + n\binom{n}{n}$$

i.e. $\boxed{\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}}$ ✓✓ (NB This is equivalent to $\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}$.)

(ii) R.T.P. $\sum_{r=0}^{n} (r+1)\binom{n}{r} = 2^{n-1}(n+2)$

$$LHS = \sum_{r=0}^{n} r\binom{n}{r} + \sum_{r=0}^{n}\binom{n}{r}$$

$$= n2^{n-1} + 2^n \qquad \left(\text{if we let } x=1 \text{ in (A)} \quad 2^n = \sum_{r=0}^{n}\binom{n}{r}\right)$$

$$= \boxed{2^{n-1}(n+2)} \quad ✓✓$$

$= RHS.$

(b) (i) $x = Vt \Rightarrow t = \frac{x}{V}$.

$\therefore y = -\frac{1}{2}gt^2 + h$. becomes

$y = -\frac{1}{2}g\left(\frac{x}{V}\right)^2 + h$.

$\boxed{y = h - \frac{1}{2}\frac{g x^2}{V^2}}$ ✓

(ii) $x = Vt\cos\alpha \Rightarrow t = \frac{x}{V\cos\alpha}$ $\therefore y = -\frac{1}{2}gt^2 + Vt\sin\alpha + h$.

becomes $y = -\frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 + V\frac{x}{V\cos\alpha}\sin\alpha + h$.

ie $\boxed{y = x\tan\alpha - \frac{g x^2 \sec^2\alpha}{2V^2} + h}$ ✓

(iii) Substitute $(d, 0)$ in (i) $0 = h - \frac{gd^2}{2V^2}$

$\therefore \boxed{h = \frac{gd^2}{2V^2}}$ ✓

(iv) Substitute $(d, 0)$ in (ii)

$0 = d\tan\alpha - \frac{gd^2}{2V^2}\sec^2\alpha + h$.

$0 = d\tan\alpha - h(1 + \tan^2\alpha) + h$ $\left(h = \frac{gd^2}{2V^2}\right)$

$\therefore h\tan^2\alpha - d\tan\alpha = 0$

$\tan\alpha(h\tan\alpha - d) = 0$

$\therefore \tan\alpha = 0$ or $\tan\alpha = \frac{d}{h}$

Clearly $\tan\alpha \neq 0$ $\therefore \boxed{\tan\alpha = \frac{d}{h}.}$ ✓✓

(v). Substitute $(2d, 0)$ into (ii).

$$2d\tan\alpha - \frac{g \cdot 4d^2}{2v^2}\sec^2\alpha + h = 0.$$

$$2d\tan\alpha - 4h\sec^2\alpha + h = 0.$$

$$2d\tan\alpha - 4h(1+\tan^2\alpha) + h = 0$$

$$2d\tan\alpha - 4h - 4h\tan^2\alpha + h = 0$$

$$4h\tan^2\alpha - 2d\tan\alpha + 3h = 0$$

For $\tan\alpha$ to be real $4d^2 - 4 \times 4h \times 3h \geq 0.$

ie $4d^2 - 48h^2 \geq 0.$

$$4d^2 \geq 48h^2$$

$$d^2 \geq 12h^2$$

$$\boxed{d \geq 2h\sqrt{3}}$$