SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2007

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board-approved calculators maybe used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question.
- Each Question is to be returned in a separate bundle.

Total Marks – 84

- Attempt Questions 1 – 7.
- All questions are of equal value.

Examiner: *A. Fuller*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

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Total marks - 84
Attempt Questions 1-7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\lim_{x \to 0} \frac{\sin 4x}{5x}$. **1**

(b) Calculate the acute angle (to the nearest minute) between the lines $2x + y = 4$ and $x - 3y = 6$. **2**

(c) (i) Show that $x + 1$ is a factor of $x^3 - 4x^2 + x + 6$. **1**

(ii) Hence, or otherwise factorise $x^3 - 4x^2 + x + 6$ fully. **2**

(d) The point $P(5,7)$ divides the interval joining the points $A(-1,1)$ and $B(3,5)$ externally in the ratio $k:1$. Find the value of k. **2**

(e) Find the horizontal asymptote of the function $y = \frac{3x^2 - 4x + 1}{2x^2 - 1}$. **1**

(f) Find a primitive of $\frac{1}{\sqrt{4 - x^2}}$. **1**

(g) Solve the equation $|x + 1|^2 - 4|x + 1| - 5 = 0$. **2**

Marks

Question 2 (12 marks)

(a) Let $f(x)=\frac{1}{2}\cos^{-1}\left(\frac{x}{3}\right)$.

(i) State the domain and range of the function $f(x)$. **2**

(ii) Show that $y=f(x)$ is a decreasing function. **2**

(iii) Find the equation of the tangent to the curve $y=f(x)$ at the point where $x=0$. **2**

(b) Find the derivative of $y=\ln\left(\sin^3 x\right)$. **2**

(c) (i) Write $\cos x-\sqrt{3}\sin x$ in the form $A\cos(x+\alpha)$, where $A>0$ and $0<\alpha<\frac{\pi}{2}$. **2**

(ii) Hence, or otherwise, solve $\cos x-\sqrt{3}\sin x+1=0$ for $0\le x\le 2\pi$. **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Show that the equation $e^x - x - 2 = 0$ has a solution in the interval $1 < x < 2$. **1**

(ii) Taking an initial approximation of $x = 1.5$ use one application of Newton's method to approximate the solution, correct to three decimal places. **2**

(b) The normal at $P\left(2ap, ap^2\right)$ on the parabola $x^2 = 4ay$ cuts the y-axis at Q and is produced to a point R such that $PQ = QR$.

(i) Show that the equation of the normal at P is $x + py = 2ap + ap^3$. **2**

(ii) Find the coordinates of Q. **1**

(iii) Show that R has coordinates $\left(-2ap, ap^2 + 4a\right)$. **1**

(iv) Show that the locus of R is a parabola, and find its vertex. **3**

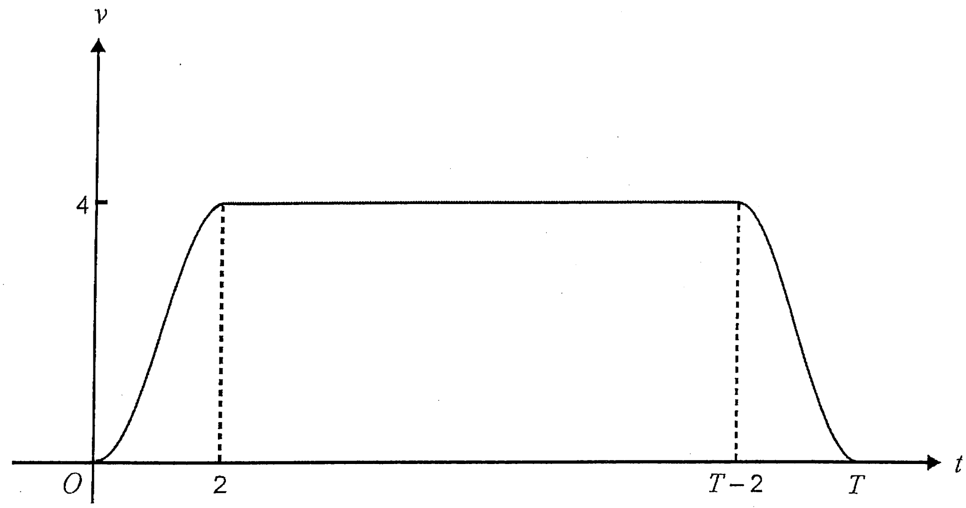
(c) If $\int_1^5 f(x)dx = 3$, find $\int_1^5 (2f(x) + 1)dx$. **2**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Using the substitution $u = e^x$, or otherwise, find $\int e^{\left(e^x + x\right)} dx$ **3**

(b) The velocity-time graph below shows the velocity of a lift as it travels from the first floor to the twentieth floor of a tall building during the T seconds of its motion.



The velocity v m/s at time t s for $0 \le t \le 2$ is given by $v = t^2(3-t)$. After the First two seconds, the lift moves with a constant velocity of 4 m/s for a time, and then decelerates to rest in the final two seconds.

The velocity-time graph is symmetrical about $t = \frac{1}{2}T$.

(i) Express the acceleration in terms of t for the first two seconds of the motion of the lift. **1**

(ii) Hence, find the maximum acceleration of the lift during the first two seconds of its motion. **2**

(iii) Given that the total distance travelled by the lift during its journey is 41 metres, find the exact value of T. **2**

(c) A solid is formed by rotating about the y-axis the region bounded by the curve $y = \cos^{-1} x$, the x-axis and the y-axis.

(i) Show that the volume of the solid is given by $V = \pi \int_0^{\frac{\pi}{2}} \cos^2 y\, dy$. **1**

(ii) Calculate the volume of this solid. **3**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to prove that $\sum_{r=1}^{n} r \times r! = (n+1)! - 1$. **3**

(b) In the expansion of $\left(2x + \dfrac{1}{x^2}\right)^{15}$, determine the coefficient of the term that is independent of x. **3**

(c) The acceleration of a particle P is given by the equation $a = 8x\left(x^2 + 1\right)$, where x is the displacement of P from the origin in metres after t seconds, with movement being in a straight line.
Initially, the particle is projected from the origin with a velocity of 2 metres per second in the negative direction.

(i) Show that the velocity of the particle can be expressed as $v = 2\left(x^2 + 1\right)$. **2**

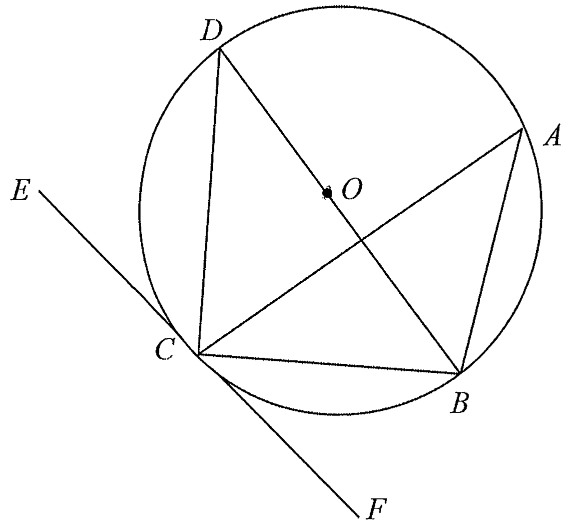
(ii) Hence, show that the equation describing the displacement of the particle at time t is given by $x = \tan 2t$. **2**

(iii) Determine the velocity of the particle after $\dfrac{\pi}{8}$ seconds. **2**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a)



A, B, C and D are points on the circumference of a circle with centre O. EF is a tangent to the circle at C and the angle ECD is $60°$. **3**

Find the value of $\angle BAC$ giving reasons.

(b) (i) By considering the expansion of $(1+x)^n$ in ascending powers of x, where n is a positive integer, and differentiating, show that **1**

$$\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+.....+n\binom{n}{n}=n\left(2^{n-1}\right).$$

(ii) Hence, find an expression for $2\binom{n}{1}+3\binom{n}{2}+4\binom{n}{3}+.....+(n+1)\binom{n}{n}$. **2**

(c) If $f(x+2)=x^2+2$, find $f(x)$. **2**

(d) At a particular dinner, each rectangular table has nine seats, five facing the stage and four with their backs to the stage.
In how many ways can 9 people be seated at the table if

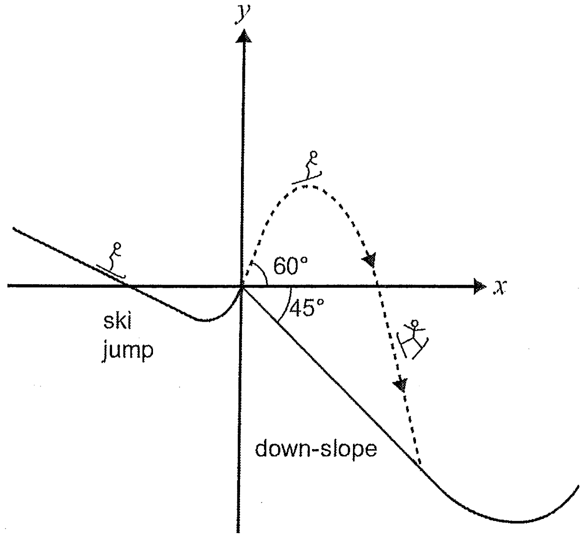
(i) John and Mary sit on the same side? **2**

(ii) John and Mary sit on opposite sides? **2**

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) A skier accelerates down a slope and then skis up a short jump (see diagram). The skier leaves the jump at a speed of 12 m/s and at an angle of 60° to the horizontal. The skier performs various gymnastic twists and lands on a straight line section of the 45° down-slope T seconds after leaving the jump.
Let the origin O of a Cartesian coordinate system be at the point where the skier leaves the jump. Displacements are measured in metres and time in seconds.
Let $g = 10ms^{-2}$ and neglect air resistance.

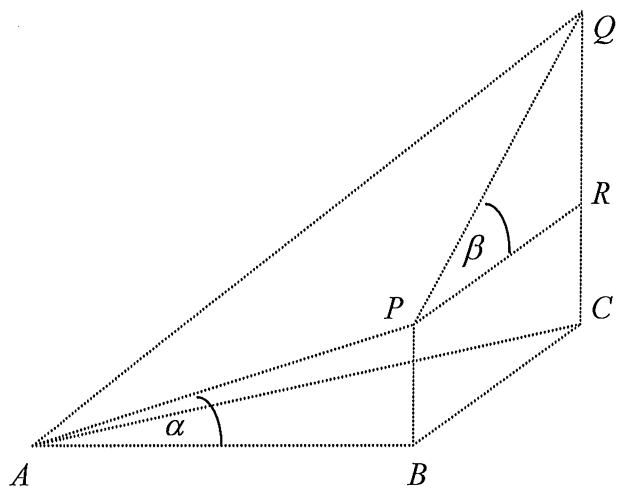


(i) Derive the cartesian equation of the skiers flight as a function of y in terms of x. **3**

(ii) Show that $T = \frac{6}{5}\left(\sqrt{3}+1\right)$. **3**

(iii) At what speed, in metres per second does the skier land on the down-slope? Give your answer correct to one decimal place. **2**

(b)



ABC is a horizontal, right-angled, isosceles triangle where $AB = BC$ and $\angle ABC = 90°$. P is vertically above B; Q is vertically above C. The angle of elevation of P from A, and Q from P are α and β respectively.

(i) If the angle of elevation of Q from A is θ, prove that **2**

$$\tan\theta = \frac{\tan\alpha + \tan\beta}{\sqrt{2}}.$$

(ii) If $\angle APQ = \phi$, prove that $\cos\phi = -\sin\alpha\sin\beta$. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

2007 THSC Mathematics Extension 1: **Solutions— Question 1**

1. (a) Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 4x}{5x}$. [1]

Solution:
$$\begin{aligned}\lim_{x\to 0}\frac{\sin 4x}{5x} &= \lim_{x\to 0}\frac{\sin 4x}{4x}\times\frac{4}{5},\\ &= \frac{4}{5}\times\lim_{x\to 0}\frac{\sin 4x}{4x},\\ &= \frac{4}{5}.\end{aligned}$$

(b) Calculate the acute angle (to the nearest minute) between the lines $2x + y = 4$ and $x - 3y = 6$. [2]

Solution:
$$\begin{aligned}\tan\alpha &= \frac{|-2-{}^1\!/_3|}{1+(-2)\times({}^1\!/_3)},\\ &= 7.\\ \therefore\ \alpha &= \tan^{-1}7,\\ &= 81.86989765^\circ \text{ by calculator},\\ &= 81^\circ 52'.\end{aligned}$$

(c) i. Show that $x + 1$ is a factor of $x^3 - 4x^2 + x + 6$. [1]

Solution: Putting
$$\begin{aligned}P(x) &= x^3 - 4x^2 + x + 6;\\ P(-1) &= -1 - 4 - 1 + 6,\\ &= 0.\end{aligned}$$
$\therefore\ x + 1$ is a factor.

ii. Hence or otherwise factorise $x^3 - 4x^2 + x + 6$ fully. [2]

Solution: Possible factors of 6 are 1, 2, 3 or 1, -2, -3.
$$\begin{aligned}P(-2) &= -8 - 16 - 2 + 6 \neq 0,\\ P(2) &= 8 - 16 + 2 + 6,\\ &= 0.\end{aligned}$$
$\therefore\ x^3 - 4x^2 + x + 6 = (x + 1)(x - 2)(x - 3).$

(d) The point $P(5,\,7)$ divides the interval joining the points $A(-1,\,1)$ and $B(3,\,5)$ externally in the ratio $k:1$. Find the value of k. 2

Solution:
$$\begin{aligned} \frac{5-{}^{-}1}{5-3} &= \frac{k}{1}, \\ 6 &= 2k, \\ k &= 3. \end{aligned}$$

(e) Find the horizontal asymptote of the function $y = \dfrac{3x^2-4x+1}{2x^2-1}$. 1

Solution: $\displaystyle\lim_{x\to\pm\infty} \frac{3-{}^4\!/_x+{}^1\!/_{x^2}}{2-{}^1\!/_{x^2}} = \frac{3}{2}.$

$\therefore\ y = {}^3\!/_2$ is the horizontal asymptote.

(f) Find a primitive of $\dfrac{1}{\sqrt{4-x^2}}$. 1

Solution: From the table of standard integrals,

$$\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2} + c.$$

(g) Solve the equation $|x+1|^2 - 4|x+1| - 5 = 0$. 2

Solution:
$$\begin{aligned} \text{Putting } y &= |x+1|; \\ y^2-4y-5 &= 0, \\ (y-5)(y+1) &= 0, \\ \therefore\ y &= 5 \text{ or } -1. \\ \text{But } |x+1| &\geq 0, \\ \text{hence } x+1 &= 5 \text{ or } x+1=-5, \\ \text{so } x &= 4,\ -6. \end{aligned}$$

(a)(i) $f(x) = \frac{1}{2}\cos^{-1}\left(\frac{x}{3}\right)$

$y = \cos^{-1}x$ has $-1 \le x \le 1$ D
$0 \le y \le \pi$ R

Domain $-1 \le x \le 1$
$-1 \le \frac{x}{3} \le 1$
$-3 \le x \le 3$ ①

Range: $0 \le \cos^{-1}x \le \pi$

$\frac{1}{2}\times 0 \le \frac{1}{2}\cos^{-1}\left(\frac{x}{3}\right) \le \frac{1}{2}\times\pi$

$0 \le f(x) \le \frac{\pi}{2}$ ①

(ii) $f'(x) = \frac{1}{2} \times \frac{-1}{\sqrt{1-\frac{x^2}{9}}} \times \frac{1}{3}$

$= -\frac{1}{6} \times \frac{1}{\sqrt{\frac{9-x^2}{9}}} = -\frac{1}{6} \times \frac{3}{\sqrt{9-x^2}} = -\frac{1}{2} \cdot \frac{1}{\sqrt{9-x^2}}$

So for $-3 < x < 3$, $f'(x) < 0$ always. ②

(iii) when $x=0$, $f(x) = \frac{1}{2}\cos^{-1}\left(\frac{0}{3}\right)$
$= \frac{1}{2}\cos^{-1}(0)$
$= \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}$

$f'(x) = \frac{-1}{2\sqrt{9-x^2}}$

at $x=0$, $m = \frac{-1}{2\times 3} = -\frac{1}{6}$

$(y-y_1) = m(x-x_1)$
$\left(y-\frac{\pi}{4}\right) = -\frac{1}{6}(x-0)$
$y = -\frac{1}{6}x + \frac{\pi}{4}$
or $\frac{1}{6}x + y - \frac{\pi}{4} = 0$
or $12\times\frac{1}{6}x + 12y - 12\times\frac{\pi}{4} = 0$
$2x + 12y - 3\pi = 0$ ②

(b) $y = \ln(\sin^3 x)$

$y' = \frac{1}{\sin^3 x} \times 3\sin^2 x \times \cos x \times 1$

$= \frac{3\sin^2 x\cos x}{\sin^3 x}$

$= \frac{3\cos x}{\sin x}$

$= 3\cot x$ ②

(c) (i) $1\cos x - \sqrt{3}\sin x$ $\qquad A = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$.

$= 2\left(\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right)$

$= A(\cos x \cos\alpha - \sin x \sin\alpha)$

$\Rightarrow A = 2$. (1)

$\left.\begin{array}{l}\cos\alpha = \frac{1}{2} \\ \sin\alpha = \frac{\sqrt{3}}{2}\end{array}\right\}$ $\alpha = 60^\circ = \frac{\pi}{3}$ (1) (1^{st} quad $0 < \alpha < \frac{\pi}{2}$)

So $\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$.

(ii) Now $\cos x - \sqrt{3}\sin x + 1 = 0$

$\cos x - \sqrt{3}\sin x = -1$

$2\cos\left(x + \frac{\pi}{3}\right) = -1$

$\cos\left(x + \frac{\pi}{3}\right) = -\frac{1}{2}$. $\qquad \cos\frac{\pi}{3} = \frac{1}{2}$

$2^{nd}/3^{rd}$ quad.

now $x + \frac{\pi}{3} = \pi - \frac{\pi}{3}$ $\quad 2^{nd}$ quad.

and $x + \frac{\pi}{3} = \pi + \frac{\pi}{3}$ $\quad 3^{rd}$ quad.

So $x = \pi - \frac{2\pi}{3} = \frac{3\pi - 2\pi}{3} = \pi/3$ ✓

and $x = \pi + \frac{\pi}{3} - \frac{\pi}{3} = \pi$. ✓ (2)

Q3. (a) (i) let $f(x) = e^x - x - 2$.

now $f(1) = e - 1 - 2 = e - 3 < 0$. (~ -0.28)

& $f(2) = e^2 - 2 - 2 = e^2 - 4 > 0$. (~ 3.38).

Since $f(x)$ changes sign in $1 < x < 2$ (✓)

$f(x) = 0$ has a solution in $1 < x < 2$.

(ii) now $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ & $f(x) = e^x - x - 2$.

$f'(x) = e^x - 1$.

$$\therefore x_2 = 1.5 - \frac{f(1.5)}{f'(1.5)}$$

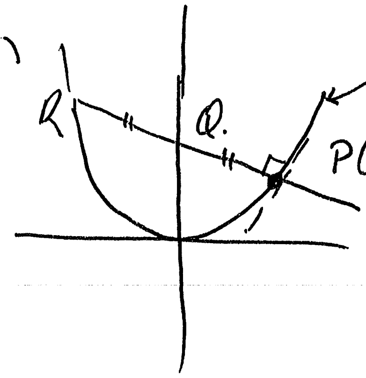
$$= 1.5 - \frac{(e^{1.5} - 1.5 - 2)}{e^{1.5} - 1}.$$

$$\doteqdot 1.5 - \frac{0.98168\ldots}{3.481689\ldots}$$

$$\doteqdot 1.5 - 0.281955\ldots$$ (✓✓)

$$\doteqdot \underline{1.218}$$

(b)



(i) $y = \frac{1}{4a}x^2$

$y' = \frac{1}{2a}x$.

$\therefore m_T = \frac{2ap}{2a} = p$

$\therefore m_N = -\frac{1}{p}$

N: $\frac{y - ap^2}{x - 2ap} = -\frac{1}{p}$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$. (✓✓)

(ii) Co-ords of Q. $x = 0$ $\therefore py = 2ap + ap^3$

$y = 2a + ap^2$

$\therefore Q\ (0, 2a + ap^2)$ (✓).

(iii) Q is the mid-pt of PR.

let R be (x_1, y_1)

$\therefore \frac{x_1 + 2ap}{2} = 0$

$\therefore x_1 = -2ap$.

$\frac{y_1 + ap^2}{2} = 2a + ap^2$

$y_1 + ap^2 = 4a + 2ap^2$

$y_1 = ap^2 + 4a$.

$\therefore R\ (-2ap, ap^2 + 4a)$ (✓)

(iii) To find the locus of R. $x = -2ap$ $\therefore p = \frac{x}{-2a}$

$\therefore y = a\left(\frac{x}{-2a}\right)^2 + 4a$.

$y = \frac{x^2}{4a} + 4a$.

$$4ay = x^2 + 16a^2$$

$$x^2 = 4ay - 16a^2$$

$$x^2 = 4a(y - 4a)$$

PARABOLA VERTEX
(0, 4a).

(✓✓✓)

(c) $$\int_1^5 (2f(x)+1)\,dx = 2\int_1^5 f(x)\,dx + \int_1^5 1.\,dx.$$

$$= 2 \times 3 + [x]_1^5$$

$$= 6 + (5-1).$$

$$= 10.$$

(✓✓)

Question (4)

(a) Let $u = e^{x}$.

$du = e^{x}\,dx$ 1

$\int e^{(e^{x}+x)}\,dx$ (3)

$= \int e^{e^{x}} \cdot e^{x}\,dx$ 1

$= \int e^{u}\,du = e^{u} + c$

$\Rightarrow$ $e^{e^{x}} + c$ 1

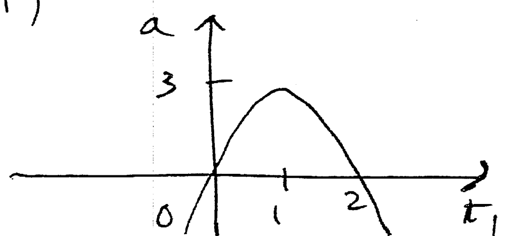
(b) For $0 \le t \le 2$.

(i) $v = 3t^2 - t^3$ (1)

$a = \frac{dv}{dt} = 6t - 3t^2$

$= 3t(2-t)$

(ii)



1

From the graph of a versus t.

The max. acceleration occurs when $t = 1$ (2)

$\therefore a_{max} = 3 \times 1 = 3\ m/s^2$

(iii) Let $d(t)$ be the total distance travelled (which is 41 m).

$\therefore d(t) = 2\int_0^2 v(t)\,dt + \int_2^{T-2} 4\,dt$

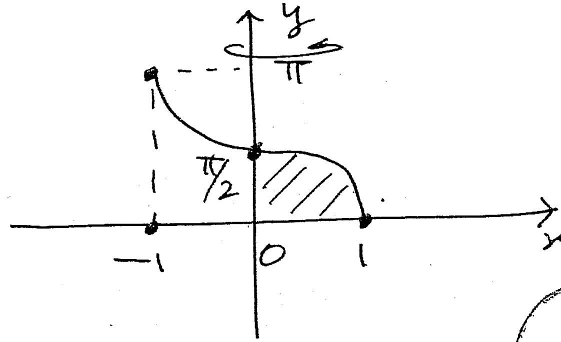
$\therefore 41 = 2\int_0^2 (3t^2 - t^3)\,dt + \Big[4t\Big]_2^{T-2}$ (2)

$\therefore 41 = 2\left[t^3 - \frac{t^4}{4}\right]_0^2 + 4(T-2) - 8$

$\therefore 41 = 2(8-4) + 4T - 16$.

$41 = 4T - 8 \Rightarrow T = 12\frac{1}{4}$.

(c)



(1)

$y = \cos^{-1} x$.

$\Rightarrow x = \cos y, \begin{cases} -1 \le x \le 1 \\ 0 \le y \le \pi \end{cases}$

$\therefore V = \pi \int_0^{\pi/2} x^2\,dy$ ✓

$\therefore x = \cos y,\ V = \pi \int_0^{\pi/2} \cos^2 y\,dy$

(ii) $V = \pi \int_0^{\pi/2} \cos^2 y\,dy$. (3)

but $\cos^2 y = \left(\frac{1 + \cos 2y}{2}\right)$ 1

$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2y)\,dy$

$= \frac{\pi}{2}\left[y + \frac{\sin 2y}{2}\right]_0^{\pi/2}$

$= \frac{\pi}{2}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$ 1

EX1 QUESTION 5

(a) if $n=1$, $1\times 1! = (1+1)! - 1$

$\therefore P(1)$ is true

Assume $P(k)$ is true $1\times 1! + 2\times 2! ... + k\times k! = (k+1)! - 1$

if $P(k+1)$ is $1\times 1! + 2\times 2! ... + k\times k! + (k+1)(k+1)! = (k+2)! - 1$

LHS is $(k+1)! - 1 + (k+1)(k+1)!$ using assumption

$= (k+1)!(1+k+1) - 1$

$= (k+1)!(k+2) - 1 = (k+2)! - 1 = RHS$

$\therefore P(k+1)$ is true if $P(k)$ is true. $P(1)$ is true + by Mathematical induction $\sum_{r=1}^{n} r\times r! = (n+1)! - 1$

(b)

$T_{k+1} = {}^{15}C_k (2x)^{n-k} (x^{-2})^k$

for term independent of x $n-k-2k=0$, $k=5$

Cf ${}^{15}C_5 \times 2^{10} = 3075072$

(c)(i) $\frac{d(\frac{1}{2}v^2)}{dx} = 8x(x^2+1) = 8x^3+8x$

$\frac{1}{2}v^2 = 2x^4 + 4x^2 + c$

$v = -2$ $x = 0$ $c = 2$

$v^2 = 4x^4 + 8x^2 + 4 = 4(x^4 + 2x^2 + 1)$

$v = \pm 2(x^2+1)^2$

(ii) if $\frac{dx}{dt} = 2(x^2+1)$

$\frac{dt}{dx} = \frac{1}{2}\left(\frac{1}{x^2+1}\right)$

$t = \frac{1}{2}\tan^{-1}x + c$

$t = 0$ $x = 0$, $c = 0$

$2t = \tan^{-1}x$

$x = \tan 2t$

(iii)

$t = \frac{\pi}{8}$, $x = \tan\frac{\pi}{4}$

$= 1$

$v = 2(1+1)$ from (i)

$v = 4$ m/s

Sydney Boys' High School
Trial HSC 2007 – Mathematics Extension 1

Question 6

(a)

$\angle CBD = 60^\circ$ (alternate segment theorem)

$\angle BCD = 90^\circ$ (angle in semicircle)

$\therefore \angle CDB = 30^\circ$ (angle sum of triangle)

$\therefore \angle CAB = 30^\circ$ (angles at circumference on same arc)

(b) (i) $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_{n-1} x^{n-1} + x^n$

Differentiating with respect to x:

$n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2 x + 3\,{}^nC_3 x^2 + \ldots + n\,{}^nC_n x^{n-1}$

Let $x = 1$:

$n2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \ldots + n\,{}^nC_n$

QED

(ii) Multiplying $(1+x)^n$ by x:

$x(1+x)^n = {}^nC_0 x + {}^nC_1 x^2 + {}^nC_2 x^3 + \ldots + {}^nC_n x^{n+1}$

Differentiating with respect to x:

$xn(1+x)^{n-1} + (1+x)^n = {}^nC_0 + 2\,{}^nC_1 x + 3\,{}^nC_2 x^2 + \ldots + (n+1)\,{}^nC_n x^n$

Let $x = 1$:

$n(2)^{n-1} + (2)^n = 1 + 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n$

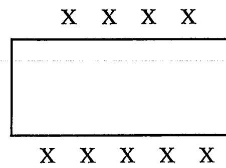
Thus

$$\begin{aligned} 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n &= n(2)^{n-1} + (2)^n - 1 \\ &= (n+2)2^{n-1} - 1 \end{aligned}$$

(c)

$$\begin{aligned} f(x+2) &= x^2 + 2 \\ f(x) &= (x-2)^2 + 2 \\ &= x^2 - 4x + 4 + 2 \\ &= x^2 - 4x + 6 \end{aligned}$$

(d)



(i) If J&M sit on the short side, they can be arranged in 12 ways, and the other guests in 7! ways. Thus $12\times7!$ ways.
If J&M sit on the long side they can be arranged in 20 ways, and the other guests in 7! Ways. Thus $20\times7!$

Hence there are $32\times7!=161\ 280$ ways.

(ii) If John sits on the short side he has four seats available, and Mary (on the long side) has 5, thus $20\times7!$

But Mary may be the one on the short side.

Thus the total is $40\times7!=201\ 600$

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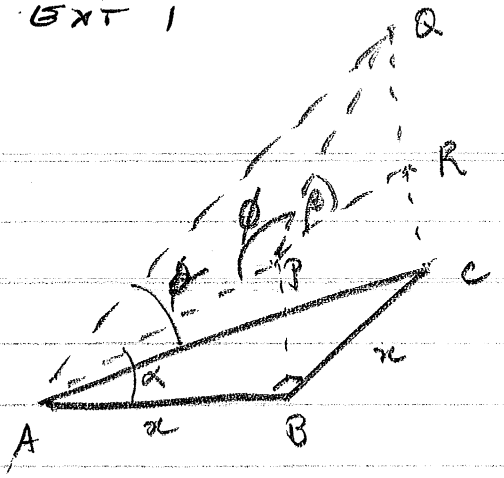
Q7



(i) $\tan\alpha = \frac{BP}{x} = \frac{BP}{AB}$

$\therefore BP = x\tan\alpha$

$\tan\beta = \frac{QR}{x} = \frac{QR}{PR}$

$\therefore QR = x\tan\beta$

$QC = x\tan\alpha + x\tan\beta$

$= x(\tan\alpha + \tan\beta)$

$\tan\theta = \frac{QC}{AC}$

$= \frac{x(\tan\alpha + \tan\beta)}{\sqrt{2}\,x}$

$= \frac{\tan\alpha + \tan\beta}{\sqrt{2}}$ ← GIVEN (2)

(ii) $\cos\phi = \frac{AP^2 + PQ^2 - AQ^2}{2AP \cdot PQ}$

$= \frac{\frac{x^2}{\cos^2\alpha} + \frac{x^2}{\cos^2\beta} - \frac{2x^2}{\cos^2\theta}}{2 \cdot \frac{x}{\cos\alpha} \cdot \frac{x}{\cos\beta}}$

$= \frac{\sec^2\alpha + \sec^2\beta - 2\left(1 + \frac{(\tan\alpha + \tan\beta)^2}{2}\right)}{2\sec\alpha\sec\beta}$

$= \frac{\sec^2\alpha + \sec^2\beta - (2 + \tan^2\alpha + 2\tan\alpha\tan\beta + \tan^2\beta)}{2\sec\alpha\sec\beta}$

$= \frac{\sec^2\alpha + \sec^2\beta - \sec^2\alpha - 2\tan\alpha\tan\beta - \sec^2\beta}{2\sec\alpha\sec\beta}$

$= -\sin\alpha\sin\beta$ ← GIVEN (2).

$\ddot{x} = 0 \qquad \ddot{y} = -10$

$\dot{x} = 12\cos 60^\circ \qquad \dot{y} = -10t + 12\sin 60^\circ$

$= 6 \qquad\qquad = -10t + 6\sqrt{3}$

$x = 6t \qquad y = -5t^2 + 6\sqrt{3}t.$

$\therefore y = -5.\left(\frac{x}{6}\right)^2 + 6\sqrt{3} \times \frac{x}{6}$

$= \frac{-5x^2}{36} + \sqrt{3}x.$ (3)

If $x = -y$, $-x = \frac{-5x^2}{36} + \sqrt{3}x.$

$\therefore \frac{5x^2}{36} - (\sqrt{3}+1)x = 0$

$\therefore x(5x - 36(\sqrt{3}+1)) = 0$

$\therefore x = 0$ or $x = \frac{36(\sqrt{3}+1)}{5}$

$\therefore T = \frac{6(\sqrt{3}+1)}{5}$ ← GIVEN (3)

$\dot{x} = 6 \qquad \dot{y} = -10 \times \frac{6(\sqrt{3}+1)}{5} + 6\sqrt{3}$

$= -12(\sqrt{3}+1) + 6\sqrt{3} \qquad = -6\sqrt{3} - 12$

Speed $= \sqrt{36 + (-6\sqrt{3} - 12)^2}$

$= \sqrt{36 + 108 + 144 + 144\sqrt{3}}$

$= \sqrt{288 + 144\sqrt{3}}$

$= 12\sqrt{2 + \sqrt{3}}$

$= 23.2$ (2)