2017 SYDNEY BOYS HIGH SCHOOL
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions
- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided with this paper
- Leave your answers in the simplest exact form, unless otherwise stated
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2–5

10 marks
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–12

60 marks
- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Examiner: E.C.

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

---

1 In what ratio does the point $M(4, 5)$ divide the interval PQ, where P and Q are (1, 2) and (3, 4) respectively?

(A) 2 : 1

(B) 1 : 2

(C) –1 : 3

(D) 3 : –1

2 A particle moves in simple harmonic motion on a horizontal line and its acceleration is

$$\frac{d^2x}{dt^2} = 36 - 4x,$$

where x is the displacement after t seconds.

Where is the centre of motion?

(A) $x = -2$

(B) $x = 2$

(C) $x = -9$

(D) $x = 9$

3 What is the value of $\lim_{x \to 0} \frac{\sin 3x}{\tan 10x}$?

(A) 3

(B) 10

(C) $\frac{3}{10}$

(D) $\frac{10}{3}$

4 Which of the following is an equivalent expression for $\sin(\tan^{-1}x)$?

(A) $\dfrac{x}{\sqrt{1-x^2}}$

(B) $\dfrac{x}{\sqrt{1+x^2}}$

(C) $\dfrac{1}{\sqrt{1-x^2}}$

(D) $\dfrac{1}{\sqrt{1+x^2}}$

5 The velocity, v, of a particle moving in a straight line at position x is given by $v = 2e^{-2x}$. Initially the particle is at the origin. What is the acceleration of the particle at position x?

(A) $a = -4e^{-2x}$

(B) $a = -16e^{-2x}$

(C) $a = -4e^{-4x}$

(D) $a = -8e^{-4x}$

6 Let α, β and γ be the roots of $x^3 + px^2 + q = 0$.
Express $\dfrac{1}{\alpha\beta}+\dfrac{1}{\beta\gamma}+\dfrac{1}{\gamma\alpha}$ in terms of p and q.

(A) pq

(B) $-pq$

(C) $-\dfrac{p}{q}$

(D) $\dfrac{p}{q}$

7 If $f(x)=\dfrac{3+e^{2x}}{5}$, which of the following is $f^{-1}(x)$?

(A) $\ln(5x-3)$

(B) $\frac{1}{2}\ln(5x-3)$

(C) $\ln 5x-\ln 3$

(D) $\frac{1}{2}(\ln 5x-\ln 3)$

8 Tom, Jerry and five other people get on a bus one at a time.
How many ways can the seven get on the bus if Tom gets on the bus after Jerry?

(A) 21

(B) 120

(C) 2520

(D) 5040

9 What is a general solution of $\cos 2\theta=\dfrac{1}{\sqrt{2}}$?

(A) $\theta=\dfrac{\pi}{8}+n\pi$ or $\theta=\dfrac{7\pi}{8}+n\pi$, for $n\in\mathbb{Z}$

(B) $\theta=\dfrac{\pi}{8}+2n\pi$ or $\theta=\dfrac{7\pi}{8}+2n\pi$, for $n\in\mathbb{Z}$

(C) $\theta=\dfrac{\pi}{4}+n\pi$ or $\theta=\dfrac{3\pi}{4}+n\pi$, for $n\in\mathbb{Z}$

(D) $\theta=\dfrac{\pi}{4}+2n\pi$ or $\theta=\dfrac{3\pi}{4}+2n\pi$, for $n\in\mathbb{Z}$

10 The size of a population at time t is given by $P(t) = 100 + 200\,e^{-0.1t}$.
What is the time for the population size to fall to half its initial value?

(A) $10\log_e 2$

(B) $10\log_e 3$

(C) $10\log_e 4$

(D) $10\log_e 5$

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve $\dfrac{x+1}{x-2} \geq 1$ **3**

(b) Find

(i) $\displaystyle\int \frac{2}{\sqrt{1-9x^2}}\,dx$ **1**

(ii) $\displaystyle\int \sin^2\left(\frac{x}{2}\right)dx$ **2**

(c) Using the substitution $u = x + 2$, find $\displaystyle\int \frac{x}{3}\sqrt{x+2}\,dx$. **2**

(d) Sketch the graph of $y = f(x)$, where $f(x) = \dfrac{1}{2}\cos^{-1}(1-3x)$. **2**

(e) (i) Show that $f(x) = e^x - x^3 + 1$ has a zero between $x = 4.4$ and $x = 4.6$ **1**

(ii) Starting at $x = 4.5$, find an approximation for the zero in part (i) using Newton's method.
Express this approximation correct to 2 decimal places. **2**

(f) Prove $\tan^{-1}\dfrac{2}{3} + \cos^{-1}\dfrac{2}{\sqrt{5}} = \tan^{-1}\dfrac{7}{4}$ **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A particle with displacement x and velocity v, is moving in simple harmonic motion such that its acceleration, $\ddot{x}$, is given by $\ddot{x} = -12x$.

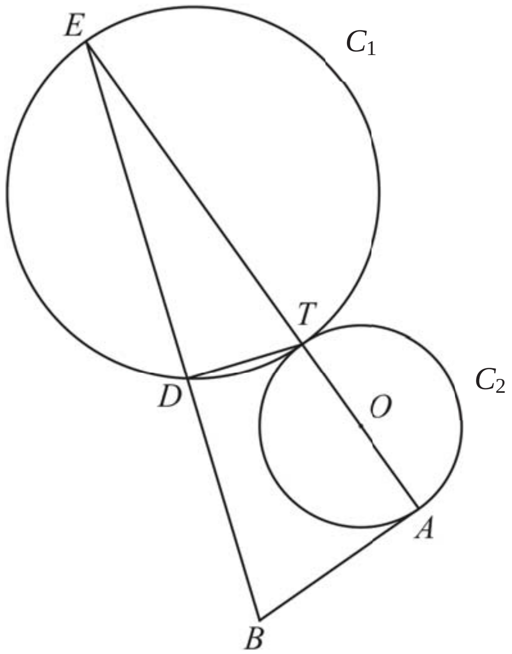
Initially the particle it is at rest at $x = -4$.

(i) State the period of the motion **1**

(ii) Show that $v^2 = 12(16 - x^2)$ **2**

(iii) Find x as a function of time t. **1**

(b)



Two circles C_1 and C_2 touch at T. The line AE passes through O, the centre of C_2, and through T.
The point A lies on C_2 and E lies on C_1.
The line AB is a tangent to C_2 at A, D lies on C_1 and BE passes through D.
The radius of C_1 is R and the radius of C_2 is r.

(i) Find the size of $\angle EDT$, giving reasons. **2**

(ii) If $DE = 2r$ find an expression for the length of EB in terms of r and R. **3**

Question 12 continues on page 9

Question 12 (continued)

(c) (i) Show that the equation of the normal at $P(at^2, 2at)$ on the parabola $y^2 = 4ax$ is given by $tx + y = 2at + at^3$. **2**

(ii) The normal intersects the x-axis at point Q. **2**
Find the coordinates coordinates of Q and hence find the coordinates of R, where R is the midpoint of PQ.

(iii) Hence find the Cartesian equation of the locus of R. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Sketch the curve $y = x + \dfrac{4}{x}$, showing clearly all the stationary points and asymptotes. **3**

Hence find the values of k such that the equation $x + \dfrac{4}{x} = k$ has no real roots.

(b) How many different arrangements can be made using all the letters of PARALLEL? **2**

(c) Find the obtuse angle between the lines $3x - y + 5 = 0$ and $2x + 3y - 1 = 0$. **2**
Give your answer correct to the nearest degree.

(d) Prove by mathematical induction that **4**

$$n \times 1 + (n-1) \times 2 + (n-2) \times 3 + \ldots + 2 \times (n-1) + 1 \times n = \frac{n}{6}(n+1)(n+2)$$

for positive integers n.

(e) A particle moves in a straight line so that its velocity v m/s at a position x metres from the origin is given by

$$v = 9 + 4x^2.$$

It starts at $x = 0$.

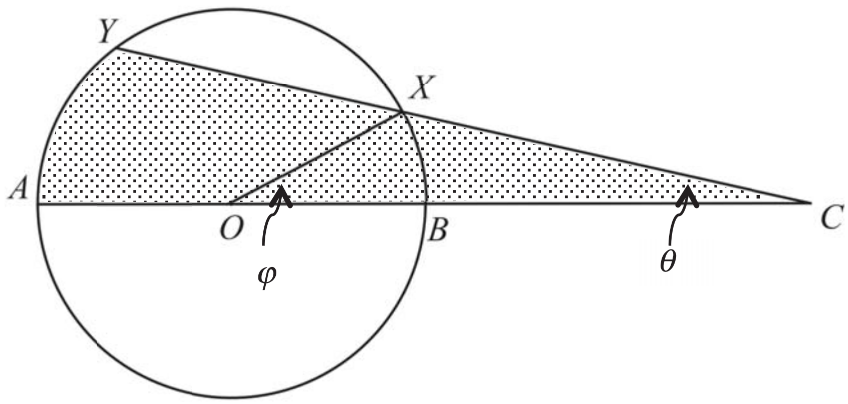
(i) Find its acceleration, $\ddot{x}$, as a function of its displacement x. **1**

(ii) Express its displacement x, as a function of time t. **3**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Differentiate $\cos^{-1}(\sin x)$ with respect to x. **2**

(b)



In the diagram above, AB is a fixed diameter of a circle centre O, radius 10 cm.
AB is produced to C such that BC = 20 cm.
X and Y lie on the circle such that CXY is a straight line.
Also, $\angle BOX = \varphi$ and $\angle ACX = \theta$.

X is free to move around the circle such that $\dfrac{d\varphi}{dt} = 2\pi$ radians per second.

(i) Show that the area, A, of the shaded region is given by **3**

$$A = 50\left[2\theta + \varphi + \sin 2(\theta + \varphi) + 3\sin\varphi\right]$$

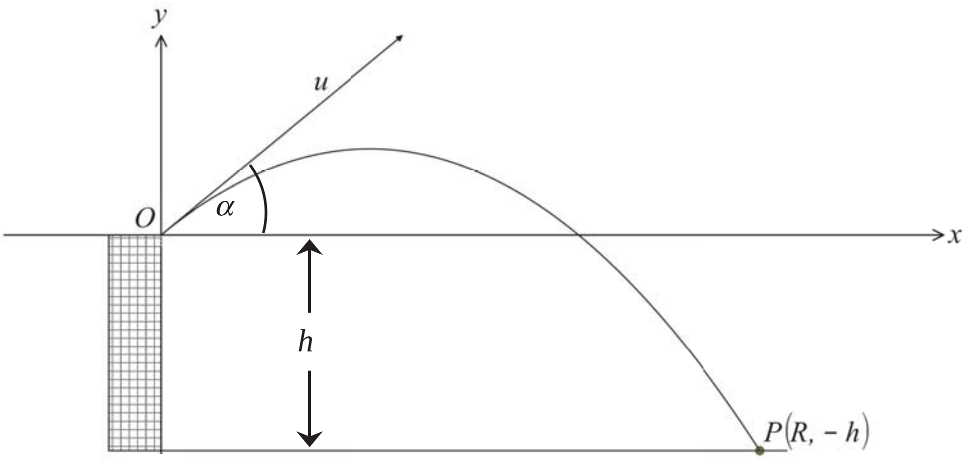
(ii) Find the maximum area of the shaded region. **2**
Leave your answer correct to 1 decimal place.
[Note: Calculus is not required]

(iii) Determine the rate of change of $\angle ACX$ at the instant when CX is a tangent to the circle. **2**

Question 14 continues on page 12

Question 14 (continued)

(c) A particle is projected from the top of a wall of height h with a speed u at an angle α to the horizontal, where $0° < \alpha < 90°$.
It strikes the horizontal ground at a point P which is R metres from the wall.



You may assume that the trajectory of the particle is given by

$$y = x\tan\alpha - \frac{gx^2(1+\tan^2\alpha)}{2u^2} \quad \text{(Do NOT prove)}$$

(i) If $u = \sqrt{\dfrac{4gh}{3}}$ and $R = 2h$, find the two possible values of α. **2**

(ii) If $u = \sqrt{2gh}$, find the maximum value of R in terms of h and also find the corresponding value of α. **4**

[Hint: Form a quadratic equation in terms of $\tan\alpha$.]

End of paper

2017 SYDNEY BOYS HIGH SCHOOL
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Suggested Solutions

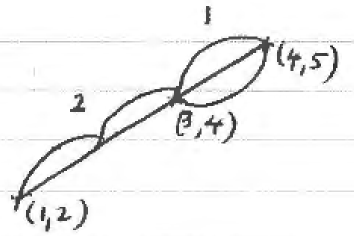
MC Answers

Q1	D
Q2	D
Q3	C
Q4	B
Q5	D
Q6	D
Q7	B
Q8	C
Q9	A
Q10	C

X1 Y12 Assessment THSC 2017 Multiple choice solutions

Mean (out of 10): 8.83

1.



External division

$\therefore\ 3:-1$ or $-3:1$ D

A	3
B	0
C	12
D	149

2. $\ddot{x} = -4(9-x)$

$\therefore$ Centre of motion is $x=9$

D

A	1
B	3
C	3
D	157

3. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\tan 10x}$

$= \lim\limits_{3x\to 0} \dfrac{\sin 3x}{3x} \cdot \lim\limits_{10x\to 0} \dfrac{10x}{\tan 10x} \cdot \lim\limits_{x\to 0} \dfrac{3x}{10x}$

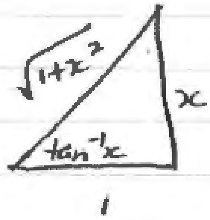
$= 1 \cdot 1 \cdot \frac{3}{10}$

$= \frac{3}{10}$ C

A	3
B	0
C	152
D	9

4. $\sin(\tan^{-1}x)$

$= \dfrac{x}{\sqrt{1+x^2}}$



B

A	2
B	154
C	3
D	5

5. $v = 2e^{-2x}$

$\dfrac{dv}{dx} = -4e^{-2x}$

$v\dfrac{dv}{dx} = -8e^{-4x}$

$\therefore a = -8e^{-4x}$ D

A	25
B	1
C	9
D	128

6. $x^3 + px^2 + q = 0$

$\alpha+\beta+\gamma = -p$

$\alpha\beta+\beta\gamma+\gamma\alpha = 0$

$\alpha\beta\gamma = -q$

$\therefore \dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha}$

$= \dfrac{\gamma+\alpha+\beta}{\alpha\beta\gamma}$

$= \dfrac{-p}{-q}$

$= \dfrac{p}{q}$ D

A	0
B	1
C	24
D	138

7. $f(x) = \frac{3+e^{2x}}{5}$

For inverse $x = \frac{3+e^{2y}}{5}$

$$5x - 3 = e^{2y}$$

$$\therefore 2y = \ln(5x-3)$$

$$y = \frac{\ln(5x-3)}{2}$$

B

A	0
B	160
C	0
D	3

8. Tom gets on the bus after Jerry half the time.

7 people getting on the bus
$\Rightarrow 7! = 5040$ arrangements.

$\therefore$ Tom gets on after Jerry in 2520 arrangements

C

A	1
B	9
C	148
D	5

9. $\cos 2\theta = \frac{1}{\sqrt{2}}$

$$\therefore 2\theta = \frac{\pi}{4} + 2n\pi \text{ OR } -\frac{\pi}{4} + 2n\pi$$

$$\therefore \theta = \frac{\pi}{8} + n\pi \text{ OR } -\frac{\pi}{8} + n\pi$$

A

A	139
B	18
C	3
D	3

10. $P(t) = 100 + 200e^{-0.1t}$

Inital size $= 100 + 200$
$= 300$

$$\therefore 150 = 100 + 200e^{-0.1t}$$

$$\therefore 50 = 200e^{-0.1t}$$

$$\therefore \frac{50}{200} = e^{-0.1t}$$

$$\therefore 4 = e^{0.1t}$$

$$\therefore \ln 4 = 0.1t$$

$$\therefore t = 10\ln 4$$

C

A	4
B	0
C	158
D	1

Question 11

a) $\dfrac{x+1}{x-2} \geq 1$

$\times (x-2)^2 \qquad \times (x-2)^2$ ①

$(x+1)(x-2) \geq (x-2)^2$

$x^2 - x - 2 \geq x^2 - 4x + 4$

$3x \geq 6$

$x \geq 2$ ①

But $x \neq 2$ ① (Undefined in the original inequality)

$\therefore x > 2$

Marker's Comments

Most candidates were able to solve the inequality correctly. However, candidates lost a mark if they did not consider the denominator in the original inequation.

Candidates whom wrote $x \neq 2$ in their first line of working were less likely to make the error of $x \geq 2$.

b) i) $\displaystyle\int \frac{2}{\sqrt{1-9x^2}}\,dx$

$= 2\displaystyle\int \frac{1}{\sqrt{9(\frac{1}{9}-x^2)}}\,dx$

$= \dfrac{2}{3}\displaystyle\int \frac{1}{\sqrt{(\frac{1}{3})^2-x^2}}\,dx$

$= \dfrac{2}{3}\sin^{-1}\left(\dfrac{x}{\frac{1}{3}}\right) + C$

$= \dfrac{2}{3}\sin^{-1}(3x) + C$ ①

Marker's Comments

Most candidates did well in this question; however, a few did not have a denominator of three in their final answer.

(ii) $\displaystyle\int \sin^2\left(\frac{x}{2}\right) dx$

$= 2\displaystyle\int \sin^2 u \, du$ $\qquad$ let $u = \dfrac{x}{2}$

$= 2\displaystyle\int \frac{1-\cos 2u}{2}\,du$ $\qquad$ $\dfrac{du}{dx} = \dfrac{1}{2}$

$= \displaystyle\int 1 - \cos 2u \, du$ ① $\qquad$ $2\,du = dx$

$= u - \dfrac{1}{2}\sin 2u + C$

$= \dfrac{x}{2} - \dfrac{1}{2}\sin x + C$ ①

Marker's Comments

Most candidates did well in this question; however, a few silly errors were made in this question especially with the substitution of u = 2x or equivalent.

c) $\int \frac{x}{3}\sqrt{x+2}\,dx$

let $u = x+2$

$\frac{du}{dx} = 1$

$du = dx$

$u - 2 = x$

$= \int \frac{u-2}{3} \times \sqrt{u}\,du$

$= \frac{1}{3}\int (u-2) \times u^{1/2}\,du$

$= \frac{1}{3}\int u^{3/2} - 2u^{1/2}\,du$

$= \frac{1}{3}\left[\frac{u^{5/2}}{5/2} - \frac{2u^{3/2}}{3/2}\right] + c$

$= \frac{1}{3}\left[\frac{2u^{5/2}}{5} - \frac{4u^{3/2}}{3}\right] + c$

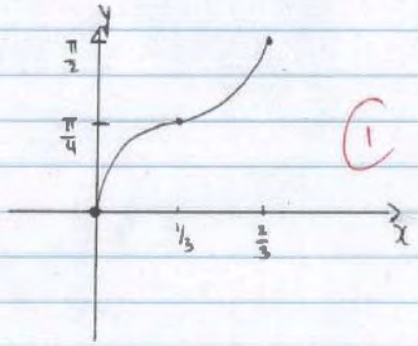
$= \frac{2}{15}(x+2)^{5/2} - \frac{4(x+2)^{3/2}}{9} + c$ 1

1

Marker's Comments

- Candidates lost half a mark for not substituting the *x + 2* back into *u* after integration.

d)



1

D: $-1 \leq 1 - 3x \leq 1$

$-2 \leq -3x \leq 0$

$\frac{2}{3} \geq x \geq 0$

$\therefore 0 \leq x \leq \frac{2}{3}$

1

R: $0 \leq y \leq \frac{\pi}{2}$

Marker's Comments

- One mark is given for the right shape of the graph and one mark for the correct domain and range.
- A significant number of candidates lost one mark, due to not satisfying one of the criteria above.

f(x) is a continuous function

e) (i) $f(x) = e^x - x^3 + 1$

$f(4.4) = e^{4.4} - (4.4)^3 + 1$

$= -2.733131\ldots < 0$

$f(4.6) = e^{4.6} - (4.6)^3 + 1$ 1

$= 3.148315642 > 0$

Since there is a sign change, a zero exist between $x = 4.4$ and $x = 4.6$.

Marker's Comments

- Candidates should mention f(x) is a continuous function as well, however no mark was penalised.

(ii) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$f(x) = e^x - x^3 + 1$

$f'(x) = e^x - 3x^2$

$\therefore x_2 = \dfrac{e^{4.5} - (4.5)^3 + 1}{e^{4.5} - 3(4.5)^2}$ 1

$= 4.50368566\ldots$

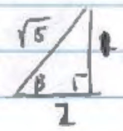
≈ 4.50 (2d.p) 1

Marker's Comments

- Candidates should substitute x = 4.5 into Newton's Method rather state the answer after writing the first line.

f) let $\alpha = \tan^{-1}\frac{2}{3}$ let $\beta = \cos^{-1}\frac{2}{\sqrt{5}}$

$\tan\alpha = \frac{2}{3}$ $\cos\beta = \frac{2}{\sqrt{5}}$



$\therefore \tan\beta = \frac{1}{2}$ 1

$\therefore \alpha + \beta = \tan^{-1}\frac{7}{4}$

Take tan of both sides

$\tan(\alpha+\beta) = \tan(\tan^{-1}\frac{7}{4})$

$= \frac{7}{4}$

$LHS = \tan(\alpha+\beta)$

$= \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{\frac{2}{3} + \frac{1}{2}}{1 - \frac{2}{3}\times\frac{1}{2}}$ 1

$= \dfrac{\frac{7}{6}}{\frac{2}{3}}$

$= \frac{7}{6}\times\frac{3}{2} = \frac{7}{4}$

$= RHS.$

$\therefore \tan^{-1}\frac{2}{3} + \cos^{-1}\frac{2}{\sqrt{5}} = \tan^{-1}\frac{7}{4}$

Marker's Comments

- Few candidates did not PROVE LHS is equal to RHS, rather SHOWING by typing the results into their calculator. Candidates were penalised for this.
- Candidates to need to show LHS = RHS by some trigonometric identity and use right angled triangles to find side ratios to be successful in gaining full marks for this question.

Solutions and Comments.

Question 12 (a)

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2x$
$= -12x$.

$n^2 = 12 \Rightarrow n = 2\sqrt{3}$

$T = \frac{2\pi}{n} = \frac{\pi}{\sqrt{3}} = \frac{\sqrt{3}\pi}{3}$.

Comment:

Answered very well

(ii) $\frac{1}{2}v^2 = -12\int x\,dx$

$= -6x^2 + c$.

When $x = -4$, $v = 0$

$c = 96$

$\therefore \frac{1}{2}v^2 = -6x^2 + 96$

$v^2 = 12(16 - x^2)$.

Comment:

Some students made too significant a jump from

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -12x$ to

$v^2 = 12(16 - x^2)$

they just quote the formula

$v^2 = n^2(a^2 - x^2)$.

Marks were awarded as follows:

- 1st mark for obtaining $\frac{1}{2}v^2 = -6x^2 + c$ OR correct substitution of initial conditions into incorrect function
- 2nd mark for correct answer.

(iii) Since the motion starts at the 'negative' end of the path of motion we use the general form

$x = -a\cos nt$

The particle is stationary at $x = -4$

Since $\ddot{x} = -12x$ the centre of motion is at $x = 0$

So $a = 4$.

$\therefore x = -4\cos(2\sqrt{3}t)$

Comment

Marks were awarded as follows:

- ½ mark for finding a and n correctly

OR

- giving correct general form for x with reasoning like c.o.m. $= x \Rightarrow a = 4$

- 1 mark was awarded for starting from DE and integrated (by using separating the variables)

$$\frac{dt}{dx} = \frac{1}{2\sqrt{3}\sqrt{4^2 - x^2}}$$

$$t = \frac{1}{2\sqrt{3}} \sin^{-1}\left(\frac{x}{4}\right) + c$$

When $t = 0$, $x = -4$

$\sin^{-1}(-x) = -\sin^{-1} x$

$\therefore 0 = -\frac{1}{2\sqrt{3}} \frac{\pi}{2} + c$

$\Rightarrow c = \frac{\pi}{4\sqrt{3}}$

$\therefore t = \frac{1}{2\sqrt{3}} \sin^{-1} \frac{x}{4} + \frac{\pi}{4\sqrt{3}}$

$t - \frac{\pi}{4\sqrt{3}} = \frac{1}{2\sqrt{3}} \sin^{-1} \frac{x}{4}$

$2\sqrt{3}\, t - \frac{\pi}{2} = \sin^{-1} \frac{x}{4}$

$4 \sin\left(2\sqrt{3}\, t - \frac{\pi}{2}\right) = x.$

$4 \sin\left[-\left(\frac{\pi}{2} - 2\sqrt{3}\, t\right)\right] = x$

$\sin(-x) = -\sin x$

and $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

$\therefore x = -4 \sin\left(\frac{\pi}{2} - 2\sqrt{3}\, t\right)$

i.e. $x = -4 \cos 2\sqrt{3}\, t$

- OR equivalent merit

(b) C_1 and C_2 touch at T, AE passes

(i) through O the centre of C_2, and through T, then AE also passes through the centre of C_1. So TE is a diameter of C_1

(When circles touch, the line through the centres passes through the point of contact.)

$\therefore \angle EDT = 90^\circ$

(Angles in a semi-circle)

Comment:

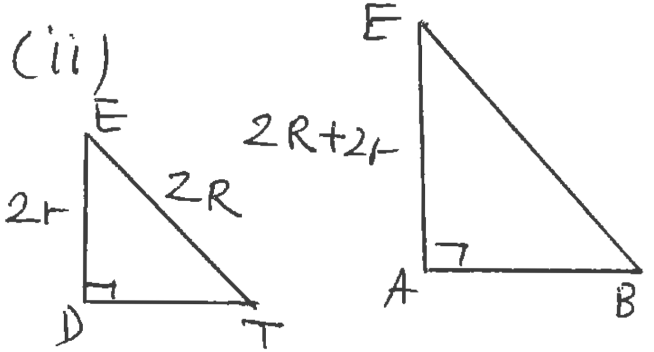
- Majority of students answered this poorly with incorrect reasoning

  Most could not substantiate the argument using their own words of reasoning.

- Quite a few students drew a common tangent and proved DTAB

Is a cyclic quad, without proving $\angle OAB = 90^\circ$ first (tgt $\perp$ radius at the contact .

(ii)



$DE = 2r$, $ET = 2R$ and $EA = 2R+2r$

$\angle EAB = 90^\circ$

(The tgt to a circle is perpendicular to the radius drawn to the point of contact).

In $\triangle$s ABE and DTE

$\angle EDT = \angle EAB = 90^\circ$ (from part (i) and from above)

$\angle DET$ is common

$\angle ETD = \angle EBA$

(Angles in a triangle add to give 180°)

$\therefore \triangle ABE \;|||\; \triangle DTE$ (equiangular).

$$\therefore \frac{EB}{ET} = \frac{EA}{ED}.$$

$$EB = 2R\left(\frac{2R+2r}{2r}\right)$$

$$= \underline{\frac{2R(R+r)}{r}}$$

Comment

This question caused a lot of grief for students

Quite a few did not stated proper reasoning, and marks were lost as a result.

Marks were awarded as follows

1st mark stating that $\angle EAB = 90^\circ$ (with proper reasoning)

2nd mark Showing $\triangle ABE \;|||\; \triangle DTE$ with proper reasoning

3rd mark Correct answer including showing that $\triangle ABE \;|||\; \triangle DTE$

(c)

$x = at^2,\ y = 2at$

(i) $\frac{dx}{dt} = 2at,$

$\frac{dy}{dt} = 2a.$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{1}{t}$

$\therefore m_T = \frac{1}{t}$

$m_N = -t$

$y - 2at = -t(x - at^2)$

$\therefore tx + y = 2at + at^3.$

2 marks Correct Answer

1 mark obtaining the correct gradient of the normal

Comment

(i) Answered very well by cohort. Carless mistake was make by using m_T instead of m_N.

At Q $y = 0$,

(ii) $tx = 2at + at^3$

$\therefore x = a(2 + t^2)$

$\therefore Q\ [a(2+t^2), 0]$

$R = \left(\frac{a(2+t^2) + at^2}{2}, \frac{0 + 2at}{2}\right)$

$\therefore R = (a(1+t^2), at)$

2 marks Correct answer

1 mark obtaining one correct coordinate of R, OR find the correct coordinates of Q.

$x = a(1+t^2)$

$y = at$

$x = a\left(1 + \frac{y^2}{a^2}\right)$

$t = \frac{y}{a}$

$\frac{y^2}{a^2} = \frac{x}{a} - 1$

$y^2 = ax - a^2$

$\therefore y^2 = a(x - a)$

2 marks correct answer.

1 mark : Correct substitution of $t = y/a$ into $x = a(1+t^2)$

Q13.

(a) $y = x + \frac{4}{x}$

Intercepts $x \neq 0$

If $y = 0$, $x + \frac{4}{x} = 0$

$x^2 + 4 = 0$

No solns.

$\therefore$ no intercepts.

Stat. Pts

$y' = 1 - \frac{4}{x^2}$

$y' = 0$ for st pts

$\Rightarrow 1 - \frac{4}{x^2} = 0$

$\frac{4}{x^2} = 1$

$x^2 = 4$

$x = \pm 2$

St pts at $(-2, -4)$ and $(2, 4)$

Type $y'' = \frac{8}{x^3}$

$y''(-2) = -1 < 0 \Rightarrow$ max at $(-2, -4)$

$y''(2) = 1 > 0 \Rightarrow$ min at $(2, 4)$

Note Vertical Asymptote at $x = 0$.

x	-1	0	1
y'	-3	ND	-3

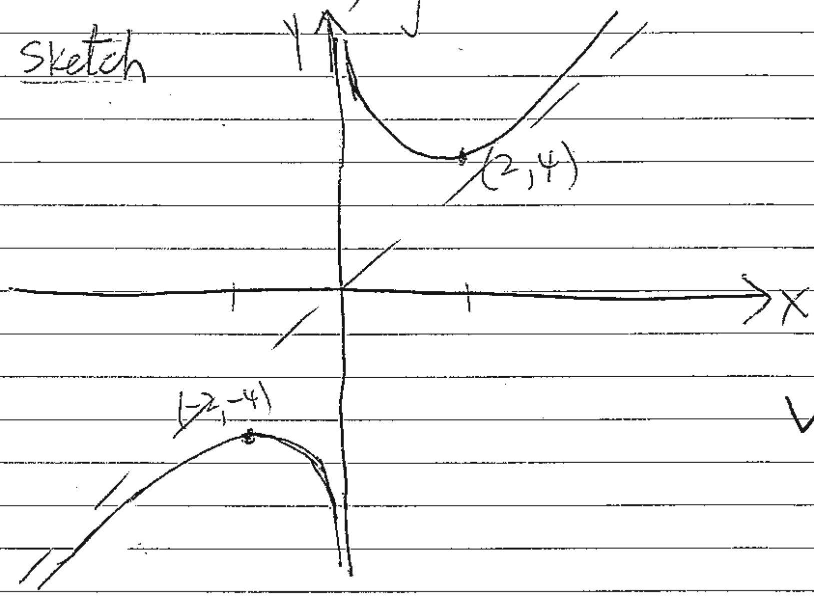
$\Rightarrow$ ← Behaviour near $x = 0$

3(a) (cont)

As $x \to \infty$, $y \to \infty$ along the line $y = x$

$x \to -\infty$, $y \to -\infty$ " " " $y = x$

Sketch



Values of k for which no solution

$y = x + \frac{4}{x}$

or $y = K$

No solution for $x = 0$ or $y = x$

$\therefore$ If $x + \frac{4}{x} = K$

$\Rightarrow x^2 + 4 = Kx$

$x^2 - Kx + 4 = 0$

$\Delta = K^2 - 16$

No soln if $K^2 - 16 < 0$

$K^2 < 16$

$\Rightarrow -4 < K < 4 \Rightarrow$ no real roots

This question was done well. For values of K students needed to realise that k is simply the same as y. 1 mark was given for turning points, 1 mark for sketch and 1 mark for values of k.

[redacted]

(b) · PARALLEL

1 P
2 A
1 R
3 L
1 E

8

No. arrangements $= \dfrac{8!}{3!\,2!}$

$= 3360$

Done well. 1 mark for numerator and 1 mark for denominator.

2

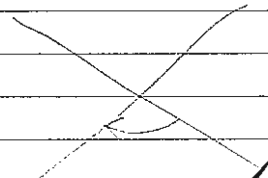
(c) $3x - y + 5 = 0 \Rightarrow y = 3x + 5 \Rightarrow m_1 = 3$

$2x + 3y - 1 = 0 \Rightarrow y = -\frac{2}{3}x + \frac{1}{3} \Rightarrow m_2 = -\frac{2}{3}$

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\dfrac{3 + \frac{2}{3}}{1 + 3 \times -\frac{2}{3}}\right|$

$= \left|\dfrac{\frac{11}{3}}{-1}\right|$



$\tan\theta = \left|-\frac{11}{3}\right|$

$\theta = |-74°45'|$

$\therefore$ obtuse angle $= 105°\ 15'$

2

Done well. 1 mark for acute angle and 1 mark for obtuse angle.

[redacted]

HSC ME 1 q13(b)

(d) Prove by mathematical induction that **4**

$$n \times 1 + (n-1) \times 2 + (n-2) \times 3 + \ldots + 2 \times (n-1) + 1 \times n = \frac{n}{6}(n+1)(n+2)$$

for positive integers n.

Test $n = 1$:

LHS $= 1 \times 1 = 1$

RHS $= \frac{1}{6}(1+1)(1+2) = 1$

$\therefore$ true for $n = 1$

Assume true for $n = k$

i.e. $k \times 1 + (k-1) \times 2 + (k-2) \times 3 + \ldots + 2 \times (k-1) + 1 \times k = \frac{k}{6}(k+1)(k+2)$

Need to prove true for $n = k + 1$:

i.e. $(k+1) \times 1 + k \times 2 + (k-1) \times 3 + \ldots + 2 \times k + 1 \times (k+1) = \frac{k+1}{6}(k+2)(k+3)$

Method 1:

Consider the LHS of the assumption.

If you add 1 to the first term, 2 to the 2^{nd} term and continuing in this fashion so that you add k to the last term then an extra $k + 1$, then you get the LHS of the expression that needs to be proved

$\therefore$ add $1 + 2 + \ldots + k + k + 1$ to the RHS of the assumption

$$\frac{k}{6}(k+1)(k+2) + 1 + 2 + \ldots + k + 1 = \frac{k}{6}(k+1)(k+2) + \frac{(k+1)(k+2)}{2}$$

$$= \frac{k}{6}(k+1)(k+2) + \frac{3(k+1)(k+2)}{6}$$

$$= \frac{(k+1)(k+2)}{6} \times (k+3)$$

$$= \frac{(k+1)(k+2)(k+3)}{6}$$

This is the RHS of the expression that needs to be proved.

So assuming true for $n = k$ means that the statement is true for $n = k + 1$.

Method 2:

Let $S_k = k\times 1 + (k-1)\times 2 + (k-2)\times 3 + \ldots + 2\times(k-1) + 1\times k$

$\therefore S_{k+1} = (k+1)\times 1 + k\times 2 + (k-1)\times 3 + \ldots + 2\times k + 1\times(k+1)$

$$\begin{array}{lcccccc} S_{k+1} = & (k+1)\times 1 & k\times 2 & (k-1)\times 3 & \ldots & 2\times k & 1\times(k+1) \\ \hline S_k = & & k\times 1 & (k-1)\times 2 & \ldots & 2\times(k-1) & 1\times k \end{array}$$

Consider $S_{k+1} - S_k$:

$S_{k+1} - S_k = (k+1)\times 1 + k + (k-1) + \ldots + 2 + 1$

$= \dfrac{(k+1)(k+2)}{2}$

$\therefore S_{k+1} = S_k + \dfrac{(k+1)(k+2)}{2}$

$\therefore S_{k+1} = \dfrac{k}{6}(k+1)(k+2) + \dfrac{(k+1)(k+2)}{2}$ (by assumption)

This now follows the path of Method 1

Method 3

$LHS = k\times 1 + 1 + (k-1)\times 2 + 2 + (k-2)\times 3 + 3 + \ldots$
$+ 1\times k + k + k+1$

$= k\times 1 + (k-1)\times 2 + (k-2)\times 3 + \ldots + 1\times k + \underbrace{1+2+3+\ldots+k+k+1}_{\uparrow \text{ sum of AP.}}$

$= \dfrac{k}{6}(k+1)(k+2) + \dfrac{k+1}{2}(1+k+1)$ from assumption

$= \dfrac{k+1}{6}(k+2)[k+3]$

$= RHS$

Most students did not get more than 2 marks for this - these 2 marks were given for the case n=1 and assumption n=k. Final 2 marks were for proving true for n=k+1. Many students tried to fudge the answer.

3(e) $v = 9 + 4x^2$ At $x = 0$, $v = 9$.

(i) $\ddot{x} = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx}$

$= \dfrac{d}{dx}\left(\frac{1}{2}(9+4x^2)^2\right)$

$\ddot{x} = 8x(9+4x^2)$. or $\ddot{x} = 72x + 32x^3$

Some students had derivative as 8x. But this is not v as a function of t. Most did it well.

1

(ii) $v = 9 + 4x^2$

$\Rightarrow \dfrac{dx}{dt} = 9 + 4x^2$

$\dfrac{dt}{dx} = \dfrac{1}{9+4x^2} = \dfrac{1}{4}\left(\dfrac{1}{\frac{9}{4}+x^2}\right)$

$t = \dfrac{1}{4}\times\dfrac{2}{3}\tan^{-1}\dfrac{x}{\left(\frac{3}{2}\right)} + C$

$t = \dfrac{1}{6}\tan^{-1}\dfrac{2x}{3} + C$

At $x = 0$, $t = 0 \Rightarrow 0 = \frac{1}{6}\tan^{-1}0 + C$

$\Rightarrow C = 0$

$\therefore t = \dfrac{1}{6}\tan^{-1}\dfrac{2x}{3}$

$6t = \tan^{-1}\left(\dfrac{2x}{3}\right)$

$\tan 6t = \dfrac{2x}{3}$

$\Rightarrow x = \dfrac{3}{2}\tan 6t$

3

Many students forgot to multiply by 1/(3/2) which = 2/3 when differentiating.

[redacted]

(a) Differentiate $\cos^{-1}(\sin x)$ with respect to x. **2**

$$\frac{d}{dx}\left[\cos^{-1}(\sin x)\right] = -\frac{\cos x}{\sqrt{1-\sin^2 x}}$$

$$= -\frac{\cos x}{\sqrt{\cos^2 x}}$$

$$= -\frac{\cos x}{|\cos x|}$$

$$= \begin{cases} 1 & \text{if } \cos x < 0 \\ -1 & \text{if } \cos x > 0 \end{cases}$$

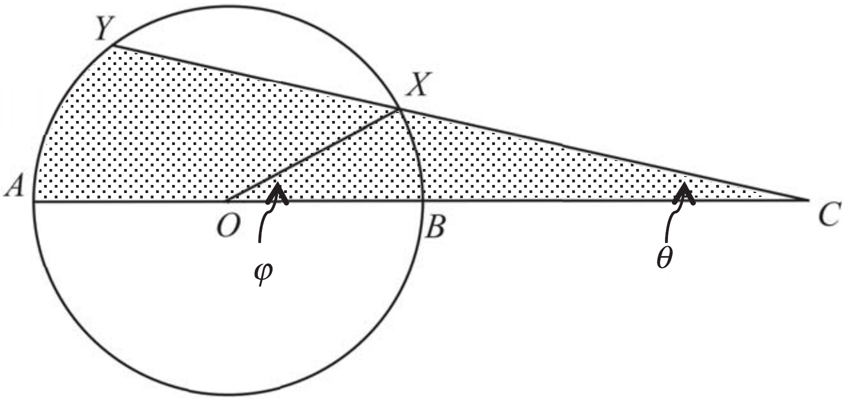
Note $\frac{d}{dx}\left[\cos^{-1}(\sin x)\right]$ is undefined if $\cos x = 0$

Comment

Too many students stopped at the first line of the solutions and also many made the mistake of cancelling leaving the answer as –1.

Question 14 (continued)

(b)

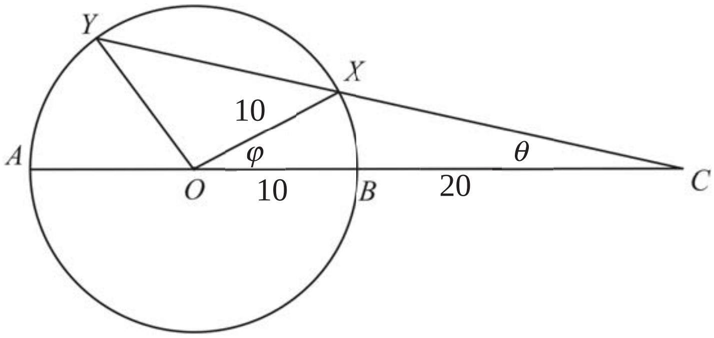


In the diagram above, AB is a fixed diameter of a circle centre O, radius 10 cm.
AB is produced to C such that $BC = 20$ cm.
X and Y lie on the circle such that CXY is a straight line.
Also, $\angle BOX = \varphi$ and $\angle ACX = \theta$.

X is free to move around the circle such that $\dfrac{d\varphi}{dt} = 2\pi$ radians per second.

(i) Show that the area, A, of the shaded region is given by **3**

$$A = 50\left[2\theta + \varphi + \sin 2(\theta + \varphi) + 3\sin\varphi\right]$$



$\angle YXO = \varphi + \theta$ (exterior angle ΔOXC)
$\angle YOX = \pi - 2(\varphi + \theta)$ (angle sum ΔOXC)
$\angle YOA = 2\theta + \varphi$ (straight angle $\angle AOB$)

Area ΔOXC: $OC = OB + BC = 30$
Area $= \frac{1}{2} \times 30 \times 10 \sin\varphi = 150\sin\varphi$

Area ΔOYX: Area $= \frac{1}{2} \times 10 \times 10 \times \sin\left[\pi - 2(\theta + \varphi)\right]$
$= 50\sin 2(\varphi + \theta)$

Area sector OYA: Area $= \frac{1}{2} \times 10 \times 10 \times (2\theta + \varphi)$
$= 50(2\theta + \varphi)$

$\therefore A = 50\left[2\theta + \varphi + \sin 2(\theta + \varphi) + 3\sin\varphi\right]$

Comment

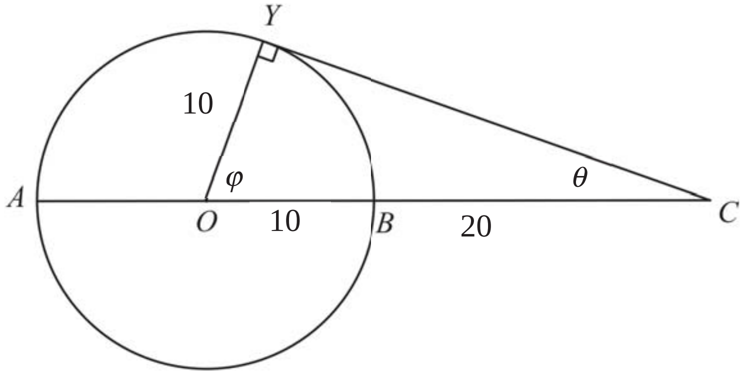
Generally done well, but it is a “Show that” question and so detail, which might seem obvious, was required to achieve full marks.

Question 14 (continued)

(a)	(ii)	Find the maximum area of the shaded region. Leave your answer correct to 1 decimal place. [Note: Calculus is not required]	**2**

The maximum area will occur when Y and X are coincident

$\therefore$ *YXC* is a tangent i.e. $\theta + \varphi = \dfrac{\pi}{2}$.



$$\cos\varphi = \frac{1}{3} \Rightarrow \sin\varphi = \frac{2\sqrt{2}}{3}$$

$$A = 50\left[2\left(\frac{\pi}{2} - \varphi\right) + \varphi + \sin\pi + 3\sin\varphi\right]$$

$$= 50\left[\pi - \cos^{-1}\left(\frac{1}{3}\right) + 2\sqrt{2}\right] \text{ u}^2$$

$$\doteqdot 237.0 \text{ u}^2$$

Comment

Generally done well, though many made the mistake of taking $\varphi = 90°$, as long as this was the only mistake they were only penalised 1 mark.
Part (iii) was not done well at all.

	(iii)	Determine the rate of change of $\angle ACX$ at the instant when *CX* is a tangent to the circle.	**2**

When X is at B then $\theta = 0$, as X continues anti-clockwise it increases until *YXC* is a tangent and then starts to decrease again after tangency.
This means that when *YXC* is a tangent that θ is a maximum or $\dot{\theta} = 0$.

Calculus Proof: From diagram in part (i) and using the Sine rule

$$\frac{\sin\theta}{10} = \frac{\sin(\theta+\varphi)}{30} \Rightarrow 3\sin\theta = \sin(\theta+\varphi)$$

Differentiating wrt t: $3\cos\theta \times \dot{\theta} = \cos(\theta+\varphi) \times (\dot{\theta}+\dot{\varphi})$

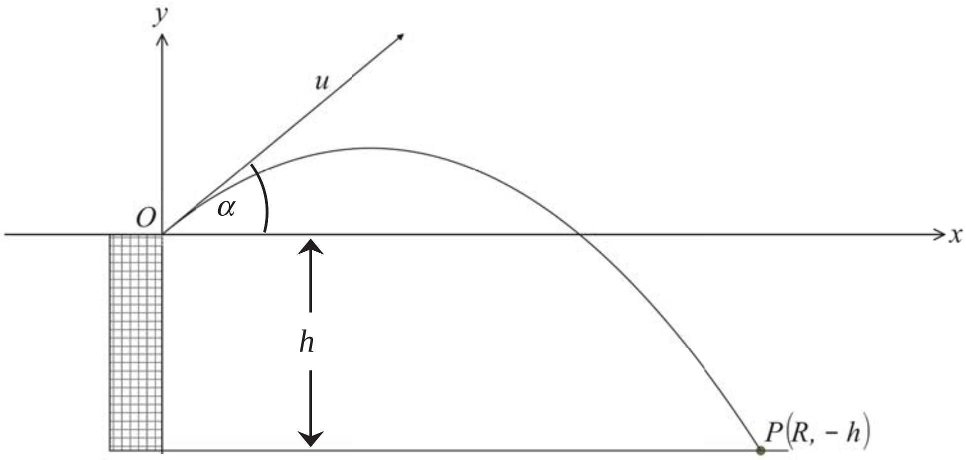
With $\dot{\varphi} = 2\pi$ and $\theta + \varphi = \dfrac{\pi}{2}$ then $3\cos\theta \times \dot{\theta} = 0 \times (\dot{\theta} + 2\pi)$

$\therefore 3\cos\theta \times \dot{\theta} = 0$

$\therefore \dot{\theta} = 0 \qquad \left[\cos\theta = \frac{2\sqrt{2}}{3}\right]$

Question 14 (continued)

(c) A particle is projected from the top of a wall of height h with a speed u at an angle α to the horizontal, where $0° < \alpha < 90°$.
It strikes the horizontal ground at a point P which is R metres from the wall.



You may assume that the trajectory of the particle is given by

$$y = x\tan\alpha - \frac{gx^2(1+\tan^2\alpha)}{2u^2} \quad \text{(Do NOT prove)}$$

(i) If $u = \sqrt{\frac{4gh}{3}}$ and $R = 2h$, find the two possible values of α. **2**

Using $y = x\tan\alpha - \frac{gx^2(1+\tan^2\alpha)}{2u^2}$ and substituting $x = R = 2h$, $y = -h$

and $u = \sqrt{\frac{4gh}{3}}$.

$$-h = 2h\tan\alpha - \frac{g(2h)^2(1+\tan^2\alpha)}{2\times\frac{4gh}{3}}$$

$$-h = 2h\tan\alpha - \frac{3h(1+\tan^2\alpha)}{2} \qquad [\div h\,(\neq 0)]$$

$$\therefore -2 = 4\tan\alpha - 3(1+\tan^2\alpha)$$

$$\therefore 3\tan^2\alpha - 4\tan\alpha + 1 = 0$$

$$\therefore (3\tan\alpha - 1)(\tan\alpha - 1) = 0$$

$$\therefore \tan\alpha = \tfrac{1}{3}, 1$$

$$\therefore \alpha = \tan^{-1}\tfrac{1}{3}, \tfrac{\pi}{4}$$

Comment

It was very hard to get marks if the correct quadratic was not found.
Students need to substitute for u, x and y and simplify initially.
Too many students made the problem harder for themselves by not doing this early on and correctly.
Students who found a negative angle, generally tried to fudge it rather than fix it.

Question 14 (continued)

(c)	(ii)	If $u = \sqrt{2gh}$, find the maximum value of R in terms of h and also find the corresponding value of α. [Hint: Form a quadratic equation in terms of $\tan\alpha$.]	**4**

Use $y = x\tan\alpha - \dfrac{gx^2(1+\tan^2\alpha)}{2u^2}$ with $x = R$ and $y = -h$.

$$-h = R\tan\alpha - \frac{gR^2(1+\tan^2\alpha)}{2\times(2gh)}$$

$$\therefore -4gh^2 = 4ghR\tan\alpha - gR^2(1+\tan^2\alpha) \qquad [\div g]$$

$$\therefore R^2\tan^2\alpha - 4hR\tan\alpha + R^2 - 4h^2 = 0$$

$$\begin{aligned}\Delta &= (4hR)^2 - 4(R^2)(R^2-4h^2)\\ &= 16h^2R^2 - 4R^4 + 16h^2R^2\\ &= 32h^2R^2 - 4R^4\\ &= 4R^2(8h^2-R^2)\end{aligned}$$

For there to be any values of α then $\Delta \geq 0$

$$\therefore 4R^2(8h^2-R^2) \geq 0$$

$$\therefore 8h^2 - R^2 \geq 0$$

$$\therefore R^2 \leq 8h^2$$

$$\therefore 0 < R \leq 2\sqrt{2}h$$

$$\therefore R_{\max} = 2\sqrt{2}h$$

Note: For $R_{\max} = 2\sqrt{2}h$, $\Delta = 0$.

Solving $R^2\tan^2\alpha - 4hR\tan\alpha + R^2 - 4h^2 = 0$

$$\begin{aligned}\tan\alpha &= \frac{4hR}{2R^2}\\ &= \frac{2h}{R}\\ &= \frac{2h}{2\sqrt{2}h}\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

$\therefore$ the angle that gives maximum range is $\tan^{-1}(\frac{1}{\sqrt{2}}) \doteqdot 35°52'$

Alternative on following page

Question 14 (continued)

(ii) (continued)

Consider $R^2 \tan^2\alpha - 4hR\tan\alpha + R^2 - 4h^2 = 0$

The range reaches a maximum as α increases, since $R = 0$ at $\alpha = 0$ and $\alpha = \dfrac{\pi}{2}$,

but for what angle.

Find $\dfrac{dR}{d\alpha}$ i.e. differentiate wrt α.

Note h is a constant.

$$R^2 \times \frac{d}{d\alpha}(\tan^2\alpha) + \frac{d}{d\alpha}(R^2) \times \tan^2\alpha - 4h\left[\frac{d}{d\alpha}(R) \times \tan\alpha + R \times \frac{d}{d\alpha}(\tan\alpha)\right] + \frac{d}{d\alpha}(R^2) = 0$$

$$\therefore 2R^2 \tan\alpha \sec^2\alpha + 2\tan^2\alpha \frac{dR}{d\alpha} - 4h\tan\alpha \frac{dR}{d\alpha} - 4hR\sec^2\alpha + 2R\frac{dR}{d\alpha} = 0$$

Now $R_{\max}$ will occur when $\dfrac{dR}{d\alpha} = 0$

$$\therefore 2R_{\max}^2 \tan\alpha \sec^2\alpha - 4hR_{\max}\sec^2\alpha = 0 \qquad \left[\div R_{\max}\sec^2\alpha\ (\neq 0)\right]$$

$$\therefore 2R_{\max}\tan\alpha - 4h = 0$$

$$\therefore \tan\alpha = \frac{2h}{R_{\max}}$$

This now follows the previous solution.

Comment

As for part (i), it was very hard to get marks if the correct quadratic was not found.
Students need to substitute for u, x and y and simplify initially.
Too many students made the problem harder for themselves by not doing this early on and correctly.
The hint for the most part was ignored or at least did not remind students that this is ultimately a quadratic problem and hence the need to examine Δ (discriminant).

Students simply stating that for $R_{\max}$ then $\Delta = 0$ and then obtaining the correct answers did not score full marks.

Many students are still convinced that the maximum range always occurs when a projectile is fired at 45°.