SYDNEY BOYS HIGH SCHOOL

2019

YEAR 12 TRIAL HSC ASSESSMENT TASK

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using **black** pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11 - 14, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 70

Section I – 10 marks (pages 3 - 7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 9 - 15)

- Attempt Questions 11–14
- Allow about 1 hours and 45 minutes for this section

Examiner: **J.C.**

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 If $\frac{dN}{dt} = 0.1(N - 100)$ and $N = 300$ when $t = 0$, which of the following is an expression for N?

A. $200 + 100e^{0.1t}$

B. $300 + 100e^{0.1t}$

C. $100 + 200e^{0.1t}$

D. $100 + 300e^{0.1t}$

2 The polynomial $P(x) = 5x^3 - 2x^2 + 7x - 3$ has roots α, β and γ.

What is the value of $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$?

A. $\frac{2}{3}$

B. $-\frac{2}{3}$

C. $\frac{3}{2}$

D. $-\frac{3}{2}$

3 A particle moving in simple harmonic motion has acceleration given by $\ddot{x} = 50 - 25x$, where x is the displacement of the particle from the origin after t seconds.

Which of the following is true?

A. The centre of motion is at $x = 0$ and the period is $\frac{2\pi}{25}$ seconds.

B. The centre of motion is at $x = 0$ and the period is $\frac{2\pi}{5}$ seconds.

C. The centre of motion is at $x = 2$ and the period is $\frac{2\pi}{25}$ seconds.

D. The centre of motion is at $x = 2$ and the period is $\frac{2\pi}{5}$ seconds.

4 Which of the following is the derivative of $y = \sin^{-1}\left[2f(x)\right]$?

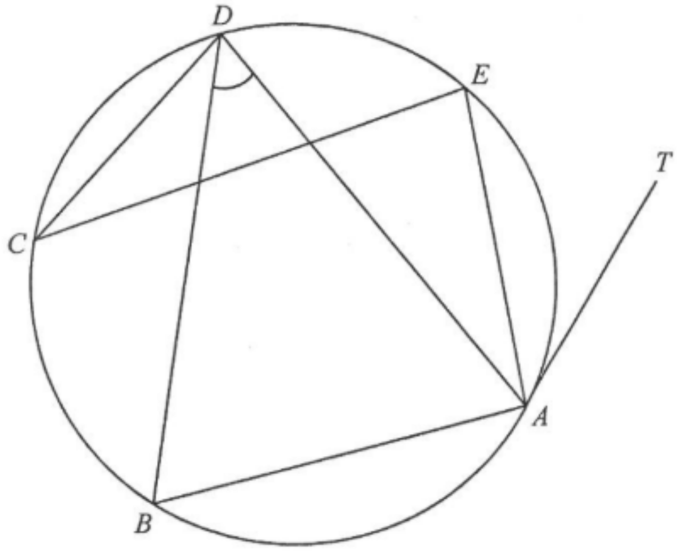
A. $\frac{dy}{dx} = \frac{1}{\sqrt{1-2\left[f(x)\right]^2}}$

B. $\frac{dy}{dx} = \frac{2}{\sqrt{1-4\left[f(x)\right]^2}}$

C. $\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-4\left[f(x)\right]^2}}$

D. $\frac{dy}{dx} = \frac{2f'(x)}{\sqrt{1-4\left[f(x)\right]^2}}$

5 In the diagram, TA is the tangent to the circle $ABCDE$ at the point A. If $\angle BAD = 64°$, $\angle EAT = 38°$ and $\angle DCE = 22°$, then what is the size of $\angle ADB$?



A. 52°

B. 56°

C. 60°

D. 68°

6 The velocity of a particle, in centimetres per second, is given by $v = \frac{1}{2}x^2$, where x is its displacement in centimetres from the origin.

What is the acceleration of the particle at $x = 2$ cm?

A. 2 cm s^{-2}

B. 4 cm s^{-2}

C. 6 cm s^{-2}

D. 8 cm s^{-2}

7 The parametric equations $x = \cos(2t) + 1$ and $y = 3\sin(t) + 2$ correspond to which of the following:

A. $\frac{(x-1)^2}{1} + \frac{(y-2)^2}{9} = 1$

B. $\frac{(x-1)^2}{1} - \frac{(y-2)^2}{9} = 1$

C. $x - 2 = \frac{2}{9}(y-2)^2$

D. $x - 2 = -\frac{2}{9}(y-2)^2$

8 A school committee consists of 6 members and a chairperson. The members are selected from 18 students. The chairperson is selected from 5 teachers. In how many ways could the committee be selected?

A. ${}^{18}C_6 + {}^{5}C_1$

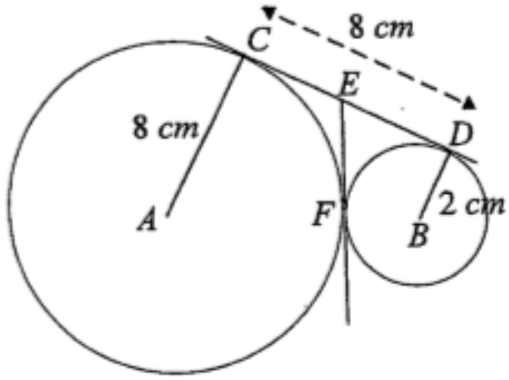
B. ${}^{18}P_6 + {}^{5}P_1$

C. ${}^{18}C_6 \times {}^{5}C_1$

D. ${}^{18}P_6 \times {}^{5}P_1$

9 Two circles, one with centre A and radius 8 cm, the other with centre B and radius 2 cm, touch externally at F.

C is a point on the larger circle and D is a point on the smaller circle such that CD, of length 8 cm, is common tangent to the two circles. The common tangent to the two circles at F meets CD at E. What is the length of FE?



NOT TO SCALE

A. 6 cm

B. 5 cm

C. 4 cm

D. 3 cm

10 Consider the function f with the rule $f(x) = \dfrac{1}{\sqrt{\sin^{-1}(cx+d)}}$, where $c, d \in \mathbb{R}$ and $c > 0$.

What is the domain of f?

A. $x > -\dfrac{d}{c}$

B. $-\dfrac{d}{c} < x \leq \dfrac{1-d}{c}$

C. $\dfrac{-1-d}{c} \leq x \leq \dfrac{1-d}{c}, x \neq -\dfrac{d}{c}$

D. $x \in \mathbb{R}, x \neq -\dfrac{d}{c}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE Writing Booklet.

(a) Solve $\dfrac{5x}{x-3} > x+4$. **3**

(b) The point P (11, 7) divides AB externally in the ratio 3 : 1. If B is (6, 5), find the coordinates of A. **2**

(c) The graphs of $y = \ln(x+1)$ and $y = \ln(7-2x)$ intersect at $(2, \ln 3)$. **2**

Find the exact value of the tangent of the acute angle between the two graphs at $(2, \ln 3)$.

(d) When a polynomial is divided by $x^2 + 2x - 3$ the remainder is $2x - 1$. **2**

What is the remainder when the polynomial is divided by $x + 3$?

(e) Find the value of $\lim\limits_{h \to 0} \dfrac{2\sin\frac{h}{2}\cos(2h)}{h}$. **1**

Question 11 continues on page 10

(f) The surface area of a sphere ($SA = 4\pi r^2$) of radius r metres is decreasing at the rate of $0.4\ \text{m}^2/\text{s}$ at an instant when $r = 0.25$. **2**

Calculate the rate of decrease, at this instant, of the radius of the sphere.

(g) Use the substitution $u = \tan x$ to show that, for $n \neq -1$, **3**

$$\int_0^{\frac{\pi}{4}} \left(\tan^{n+2} x + \tan^n x\right) dx = \frac{1}{n+1}.$$

End of Question 11

Question 12 (15 marks) Use a SEPARATE Writing Booklet.

(a) Consider the function $f(x) = \sin(e^x)$. **2**

By taking $x = 5$ as the first approximation to a root of the function, use Newton's method to find a second approximation.

Give your answer correct to two decimal places.

(b) The parametric equations of a curve are

$$x = 1 + 2\sin^2\theta, \qquad y = 4\tan\theta$$

(i) Show that $\dfrac{dy}{dx} = \dfrac{1}{\sin\theta\cos^3\theta}$. **2**

(ii) Find the equation of the tangent to the curve at the point where $\theta = \dfrac{\pi}{4}$. **2**

(c) It is known that 36% of the customers of a certain supermarket will bring their own shopping bags. There are 3 cashiers and there are 5 customers on each queue.

(i) What is the probability that all the customers in one of the queues brought a bag and no other customers brought bags? **2**

(ii) What is the probability that amongst all fifteen customers at least three people have brought their shopping bags? **2**

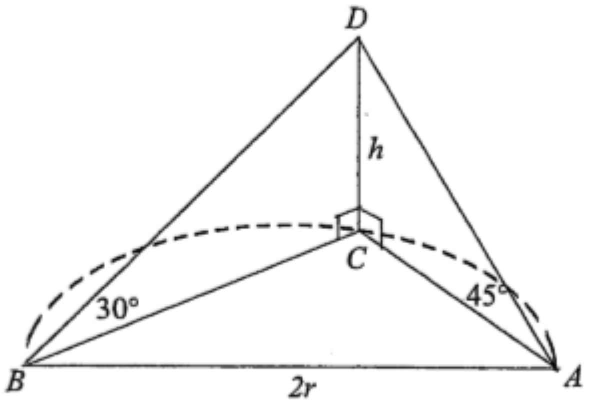
(d) (i) Express $\cos\theta + 3\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of $\tan\alpha$. **2**

(ii) Hence, solve the equation $\cos 2\theta + 3\sin 2\theta = 2$ for $0° < \theta < 90°$. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE Writing Booklet.

(a) AB is the diameter of a semicircular piece of horizontal ground with radius r metres. **3**
CD is a vertical flagpole of height h metres standing with its base C on the arc AB.
From A and B, the angles of elevation of the top D of the flagpole are 45° and 30° respectively.
Show that $h = r$.



Not to scale

(b) Suppose $r+\sqrt{r}$ is a root of the cubic equation $x^3+ax+b=0$, where a, b, r are rational numbers and $\sqrt{r}$ is an irrational number.

(i) Show that $r^3+3r^2+ar+b=0$ and $3r^2+r+a=0$. **2**

(ii) Hence or otherwise, show that $r-\sqrt{r}$ is also a root of the equation. **2**

Question 13 continues on page 13

Question 13 (continued)

(c) Given that $y = \sin^{-1}(\cos x)$

(i) Find $\frac{dy}{dx}$. **2**

(ii) Sketch $y = \sin^{-1}(\cos x)$ for $-\pi \le x \le \pi$. **2**

(d) Consider the function $f_n(x) = (\cos 2x)(\cos 4x)...(\cos 2^n x)$ for $n \ge 1$, and n an integer.

(i) Determine whether f_n is an odd or even function. Justify your answer. **1**

(ii) By using mathematical induction, prove that **3**

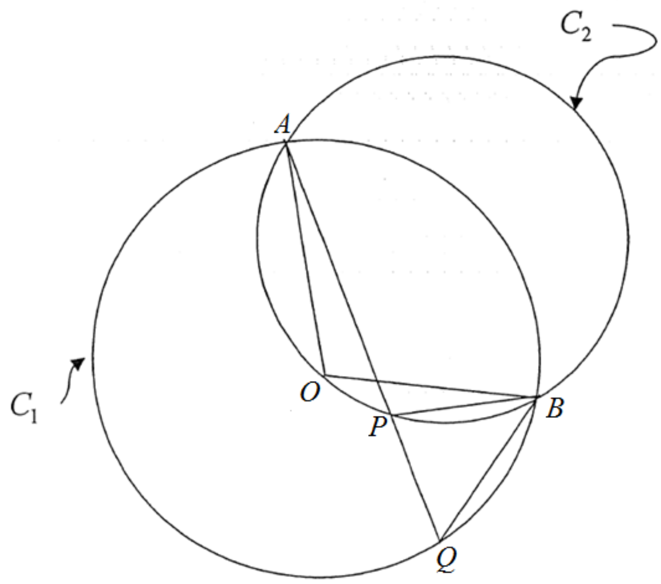
$$f_n(x) = \frac{\sin 2^{n+1} x}{2^n \sin 2x} \text{ for } x \ne \frac{m\pi}{2},$$

where m is an integer.

End of Question 13

Question 14 (15 marks) Use a SEPARATE Writing Booklet.

(a) O is the centre of the circle C_1.



(i) If $\angle AQB = x$, express $\angle AOB$ in the terms of x, giving reason(s) for your answer. **1**

(ii) Hence, or otherwise, show that $PB = PQ$. **2**

(b) (i) By considering the coefficient of x^r in the expansion of $(1+x)(1+x)^n$, **2**

Prove that for $1 \le r \le n$,

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

(ii) The numbers $B_0, B_1, B_2, \ldots$ is defined by **3**

$$B_{2m} = \sum_{j=0}^{m} \binom{2m-j}{j} \text{ and } B_{2m+1} = \sum_{k=0}^{m} \binom{2m+1-k}{k}$$

Show that $B_{2n+3} - B_{2n+2} = B_{2n+1}$ for $n \ge 0$.

Question 14 continues on page 15

Question 14 (continued)

(c) When an object is projected from the origin with an initial velocity of V at an angle θ to the horizontal, the equations of motion of the object are:

$$x(t) = Vt\cos\theta \text{ and } y(t) = Vt\sin\theta - \frac{1}{2}gt^2. \text{ (Do NOT prove these)}$$

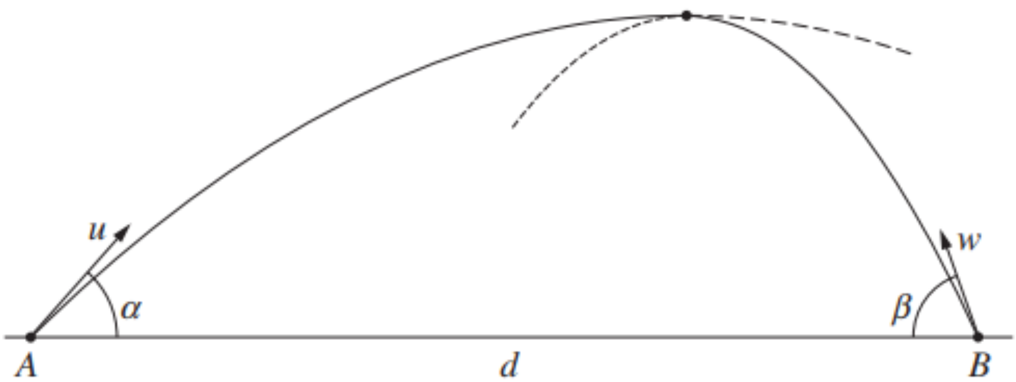
Two particles, A and B, are projected simultaneously towards each other from two points which are d distance apart in a horizontal plane.

Particle A has mass m and is projected at speed u at angle α above the horizontal.

Particle B has a mass M and is projected at speed w at angle β above the horizontal.

The trajectories of the two particles lie in the same vertical plane. The particles collide directly when each is at its point of greatest height above the plane.

As a result of the collision $m\cot\alpha = M\cot\beta$. (Do NOT prove this result)



(i) Show that $u\sin\alpha = w\sin\beta$. **2**

(ii) Show that the collision occurs at a point which is a horizontal distance, b metres, from the point of projection of A where $b = \dfrac{Md}{m+M}$. **3**

(iii) Show that the height, h, above the horizontal plane at which the collision occurs is given by **2**

$$h = \frac{b\tan\alpha}{2}.$$

End of Paper

SYDNEY
BOYS
HIGH
SCHOOL

2019

YEAR 12 TRIAL HSC
ASSESSMENT TASK

Mathematics Extension 1

SUGGESTED SOLUTIONS

MC QUICK ANSWERS

1. C
2. A
3. D
4. D
5. B
6. B
7. D
8. C
9. C
10. B

X1 Y12 THSC 2019 Multiple choice solutions

Mean (out of 10): 8.56

1. $\frac{dN}{dt} = 0.1(N-100)$

$\int \frac{dN}{N-100} = 0.1\int dt$

$\ln(N-100) = 0.1t + c$

$\therefore N-100 = e^{0.1t+c}$

$N = 100 + Ae^{0.1t}$.

When $t=0$: $300 = 100 + A.1$

$\therefore A = 200$

$\therefore N = 100 + 200e^{0.1t}$ C

A	11
B	2
C	152
D	0

2. $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$

$= \frac{\gamma+\beta+\alpha}{\alpha\beta\gamma}$

$= \frac{\frac{2}{5}}{\frac{3}{5}}$

$= \frac{2}{3}$ A

A	162
B	2
C	1
D	0

3. $\ddot{x} = 50 - 25x$

$= -25(x-2)$

$\therefore n = 5$

$\therefore$ Centre of motion: $x = 2$

Period $= \frac{2\pi}{5}$ D

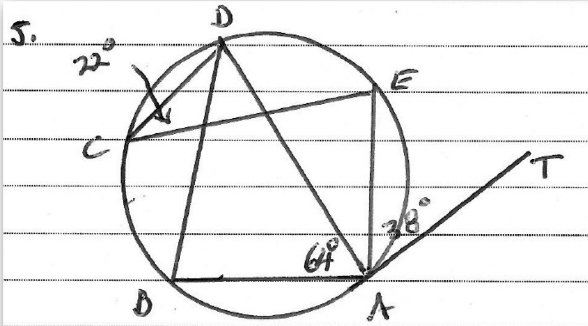
A	0
B	10
C	6
D	149

4. $y' = \frac{1}{\sqrt{1-4(f(x))^2}} \times 2f'(x)$

$= \frac{2f'(x)}{\sqrt{1-4(f(x))^2}}$ D

A	1
B	2
C	4
D	158

5.



$\angle DAE = \angle DCE = 22^\circ$ (angles at circumference)

$\angle DBA = \angle DAT = 60^\circ$ (alternate segment theorem)

$\therefore \angle ADB = 56^\circ$ (angle sum of $\triangle ADB$)

B

A	4
B	143
C	15
D	3

6. $\ddot{x} = v\dfrac{dv}{dx}$

$= \frac{1}{2}x^2 \cdot x$

$= \frac{1}{2}x^3$

When $x = 2$, $\ddot{x} = \frac{1}{2} \times 8$

$= 4 \text{ cm s}^{-2}$ B

A	19
B	142
C	0
D	4

7. $x = \cos 2t + 1$

$= (1 - 2\sin^2 t) + 1$

$= 2 - 2\sin^2 t$

$y = 3\sin t + 2$

$\therefore \sin t = \dfrac{y-2}{3}$

$\therefore x = 2 - 2\left(\dfrac{y-2}{3}\right)^2$

$\therefore x - 2 = -\frac{2}{9}(y-2)^2$ D

A	19
B	5
C	13
D	128

8. No of ways to form committee

$= {}^{18}C_6 \times {}^{5}C_1$

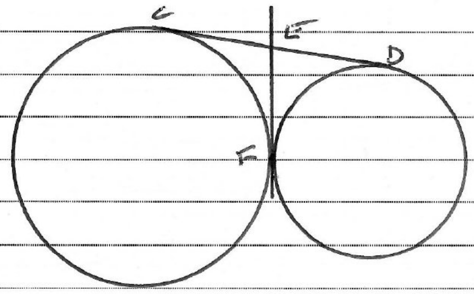
choose 6 students from 18

choose 1 teacher from 5

C

A	13
B	1
C	150
D	1

9.



CE = EF (tangents from external point equal)

DE = EF (" ")

∴ CD = CE + DE = 2EF

∴ EF = 4 cm C

A	6
B	15
C	141
D	3

10. $\sin^{-1}(cx+d) > 0$

$\therefore \quad 0 < cx+d \le 1$

$\therefore \quad -d < cx \le 1-d$

$\therefore \quad -\frac{d}{c} < x \le \frac{1-d}{c}$ B

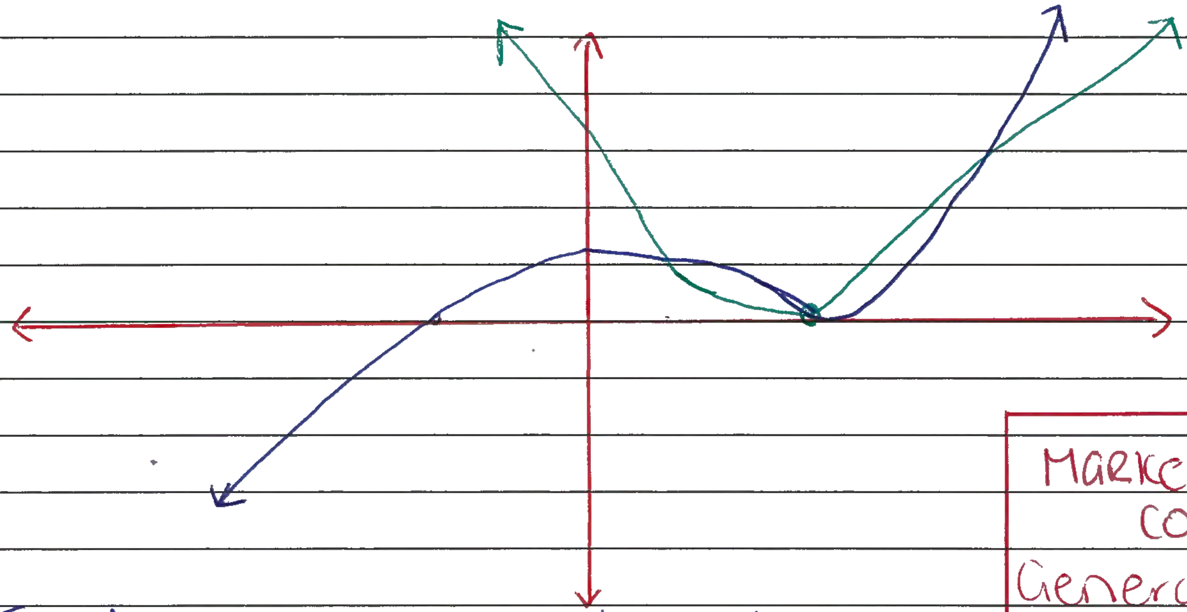
A	22
B	87
C	54
D	2

Question 11.

a. $\frac{5x}{x-3} \times (x-3)^2 > (x+4)(x-3)^2 \qquad x \neq 3$

$5x(x-3) > (x+4)(x-3)^2$

Sketch $5x(x-3)$ and $(x+4)(x-3)^2$ on the same axis.



Find points of intersection:

Markers comments.
Generally well done.

$5x(x-3) = (x+4)(x-3)^2$

$0 = (x+4)(x-3)^2 - 5x(x-3)$

$= (x-3)\left[(x+4)(x-3) - 5x\right]$

$\therefore x-3=0, \quad (x+4)(x-3)-5x = 0$

$x = 3 \qquad x^2+x-12-5x = 0$

$x^2-4x-12 = 0$

$(x-6)(x+2) = 0$

$\therefore x = 6, -2$

$\frac{5x}{x-3} > x+4$ for $x < -2, \quad 3 < x < 6$

b $P(11, 7)$ AB externally $\overset{m}{3} : \overset{n}{1}$

B $(\underset{x_2}{6}, \underset{y_2}{5})$

$$(11, 7) = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

$$11 = \frac{3 \cdot 6 - x_1}{3-1} \qquad 7 = \frac{3 \cdot 5 - y_1}{3-1}$$

$$22 = 18 - x_1 \qquad 14 = 15 - y_1$$

$$x_1 = -4 \qquad y_1 = 1$$

$$\therefore A = (-4, 1)$$

Comments
very well done.

C $y = \ln(x+1)$ $y = \ln(7-2x)$

Intersect at $(2, \ln 3)$

$$y' = \frac{1}{x+1} \qquad y' = \frac{-2}{7-2x}$$

at $x = 2$ at $x = 2$

$$y' = \frac{1}{3} \qquad y' = \frac{-2}{7-4}$$

$$= \frac{-2}{3}$$

$$\tan\alpha = \frac{1/3 + 2/3}{1 + 1/3 \times -2/3}$$

$$= \frac{9}{7}$$

$$\therefore \alpha = \tan^{-1}(9/7)$$

Comments:
well done.
common error
not simplifying
the equation
correctly

d. $P(x) = (x^2+2x-3)\cdot Qx + (2x-1)$

~~$x^2+2x-3 = (x+3)(x-1)$~~

$x^2+2x-3+2x-1 = x^2+4x-4$

$$\begin{array}{r}
x+1 \\
x+3\overline{)\,x^2+4x-4} \\
\underline{x^2+3x} \\
x-4 \\
\underline{x+3} \\
\underline{\underline{-7}}
\end{array}$$

comments: majority of Ss had no troubles with this Q.

the remainder is -7.

e $\lim_{h\to 0} \dfrac{2\sin h/2 \cos 2h}{h} \times \dfrac{1/2}{1/2}$

$= \lim_{h\to 0} \dfrac{\sin h/2 \cos 2h}{h/2}$

comments: no 1/2 marks here

$= \lim_{h\to 0} \cos 2h$

$= \cos 0$

$= 1.$

f. $\dfrac{dA}{dt} = 0.4$ $\quad$ $\dfrac{dA}{dr} = 8\pi r$

$\dfrac{dr}{dt} = \dfrac{dA}{dt} \cdot \dfrac{dr}{dA}$

$= 0.4 \times \dfrac{1}{8\pi r}$

$$= \frac{1}{20\pi r}$$

when $r = 0.25$

$$\frac{dr}{dt} = \frac{1}{5\pi}$$

$$\doteqdot 0.06\text{m/s (2dp)}$$

MARKERS COMMENTS:
Too many Ss are writing $\frac{0.2}{\pi}$
Decimals + fractions DO NOT MIX!

9. $\int_0^{\pi/4} (\tan^{n+2} x + \tan^n x)\, dx$ = ~~RHS~~ LHS

$$= \int_0^{\pi/4} \tan^n x(\tan^2 x + 1)\, dx$$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$\boxed{dx = \frac{du}{\sec^2 x}}$

$$= \int_0^{\pi/4} \tan^n x \cdot \sec^2 x\, dx$$

$$= \int_0^1 u^n \cdot \sec^2 x \cdot \frac{du}{\sec^2 x}$$

$x = \pi/4$

$u = \tan \pi/4$

$\boxed{u = 1}$

$x = 0$

$u = \tan 0$

$\boxed{u = 0}$

$$= \int_0^1 u^n\, du$$

$$= \left[\frac{u^{n+1}}{n+1}\right]_0^1$$

$$= \left[\frac{1^{n+1}}{n+1} - 0\right]$$

$$= \frac{1}{n+1}$$

$$= RHS$$

Markers Comments
Generally well done

Question 12

a) $f(x) = \sin(e^x)$

$f'(x) = e^x \cos(e^x)$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 5 - \frac{\sin(e^5)}{e^5 \cos(e^5)}$$

$$= 4.99$$

Comments

- Well done.
- Some students were unable to differentiate $f(x)$ correctly.
- MANY students had their calculator in DEG mode, and therefore got 5.00 as their result. They were penalised $\frac{1}{2}$ mark.
- A few students unnecessarily did more than 1 application. Waste of time!

b) i) $\frac{dx}{d\theta} = 4\sin\theta\cos\theta$

$$\frac{dy}{d\theta} = 4\sec^2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{4}{\cos^2\theta} \times \frac{1}{4\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos^3\theta} \text{ as required.}$$

Comments

- Well done.
- Some poor answers tried to get a Cartesian equation to eliminate θ.

b) ii) Where $\theta = \frac{\pi}{4}$,

$$\frac{dy}{dx} = \left(\sin\frac{\pi}{4}\cos^3\frac{\pi}{4}\right)^{-1}$$
$$= \left(\frac{1}{4}\right)^{-1}$$
$$= 4$$
$$x = 1 + 2\sin^2\frac{\pi}{4}$$
$$= 2$$
$$y = 4\tan\frac{\pi}{4}$$
$$= 4$$
$$y - 4 = 4(x-2)$$
$$= 4x - 8$$
$$y = 4x - 4$$
$$4x - y - 4 = 0$$

Comments

- Well done, and students who weren't able to answer part i) still often answered ii)
- Many trivial errors in substituting $\theta = \frac{\pi}{4}$
  eg. $\frac{dy}{dx} = \frac{1}{4}$, $x = 1$ or $\frac{3}{2}$
- A large number didn't evaluate x and y values for $\theta = \frac{\pi}{4}$, and left their answer in the form $y = 4x - 4 + 4\tan\theta + 8\sin^2\theta$.

c) i) P(all in 1^{st} queue, none in 2^{nd} or 3^{rd}) = $0.36^5 \times 0.64^{10}$

$\therefore$ P(all in any queue, no others) = $3 \times 0.36^5 \times 0.64^{10}$ by symmetry

$\approx \cancel{\$}\ 2.09 \times 10^{-4}$

Comments

- Many mistakes. Many students calculated as ${}^{15}C_5 \times 0.36^5 \times 0.64^{10}$, which only gets probability of ANY 5 having bags.
- Was marked leniently, with only $\frac{1}{2}$ mark removed for some substantial errors. Remember that the HSC only gives whole marks.

c) ii) $P(\text{at least 3 bags}) = 1 - P(2\text{ bags}) - P(1\text{ bag}) - P(0\text{ bags})$

$$(*) \quad = 1 - {}^{15}C_2 \times 0.36^2 \times 0.64^{13} - {}^{15}C_1 \times 0.36 \times 0.64^{14} - {}^{15}C_0 \times 0.64^{15}$$

$$= 0.9472$$

$$= 94.7\%$$

Comment:

- Done better than part i)
- Common mistakes included forgetting P(0 bags), or also subtracting P(3), and missing the ${}^{15}C_1$ and ${}^{15}C_2$ from the relevant terms. Again, it was marked leniently, where you'd likely get only 1 in the HSC.
- MANY students stopped their working at the line marked (*). Although in this paper they were awarded full marks, you should not presume that is how the HSC will be marked. You may be missing easy marks if you do not follow a question to completion.

d) i) $R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$$= \cos\theta \qquad + 3\sin\theta$$

equating coefficients gives

$$R\cos\alpha = 1$$

$$R\sin\alpha = 3$$

$$\tan\alpha = \frac{R\sin\alpha}{R\cos\alpha} = 3$$

$$R^2 = R^2\cos^2\alpha + R^2\sin^2\alpha$$

$$= 1^2 + 3^2$$

$$\therefore \quad R = \sqrt{10} \quad (\text{since } R > 0)$$

Comments

- Very well done. Most students awarded full marks.

d) ii) Since $\cos(x) + 3\sin(x) = \sqrt{10}\cos(x - \tan^{-1}(3))$,

let $2\theta = x$

If $\cos 2\theta + 3\sin 2\theta = 2$

then $\sqrt{10}\cos(2\theta - \tan^{-1}(3)) = 2$

$$\cos(2\theta - \tan^{-1}(3)) = \frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}$$

$$2\theta - \tan^{-1}(3) = \pm\cos^{-1}\left(\frac{2}{\sqrt{10}}\right) \qquad (◎)$$

$$2\theta = \tan^{-1}(3) \pm \cos^{-1}\left(\frac{2}{\sqrt{10}}\right)$$

$$\theta = \frac{1}{2}\left[\tan^{-1}(3) \pm \cos^{-1}\left(\frac{2}{\sqrt{10}}\right)\right]$$

$= 61°10', 10°24'$ to the nearest minute.

Explanation of (◎)

Since $0° < \theta < 90°$,

$0° < 2\theta < 180°$

$-71°34' < 2\theta - \tan^{-1}(3) < 108°26'$

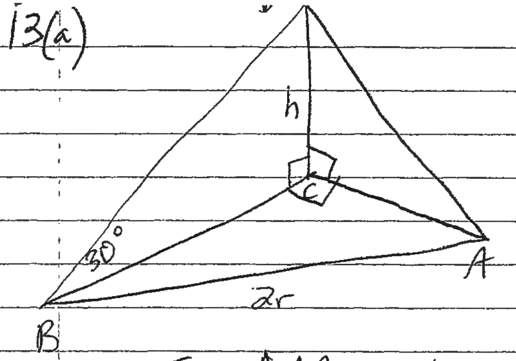
Because $\cos^{-1}\left(\frac{2}{\sqrt{10}}\right) \approx 50°46'$ then $\cos(-50°46') \approx \frac{2}{\sqrt{10}}$ too,

since $\cos(-x) = \cos(x)$

Comments

- Done well for finding 61°10'.
- Most students didn't find the 2nd solution.
- Again, marking was lenient in expressing your answer, but the instruction $0° < \theta < 90°$ should tell you to express your answer in degrees, to whatever rounding is appropriate.

13(a)



$\angle ACB = 90^\circ$ ($\angle$ in a semi-circle)

In ΔABC $\tan 30 = \frac{h}{BC}$

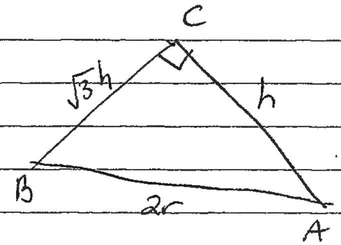
$BC = \frac{h}{\tan 30^\circ} = \sqrt{3}h$

In ΔDAC $\tan 45 = \frac{AC}{h}$

$h = AC \tan 45$

$h = AC$

Then in ΔABC



$h^2 + 3h^2 = (2r)^2$

$4r^2 = 4h^2$

$r^2 = h^2$

$\Rightarrow r = h$

This question was done extremely well. Most got full marks.

3

13(b)(i) $(r+\sqrt{r})$ is a root of $x^3+ax+b=0$

Show $r^3+3r^2+ar+b=0$ and $3r^2+r+a=0$.

$(r+\sqrt{r})^3 + a(r+\sqrt{r})+b=0$

$r^3+3r^2\sqrt{r}+3r^2+(\sqrt{r})^3+ar+a\sqrt{r}+b=0$

$\Rightarrow r^3+3r^2(\sqrt{r}+1)+a\sqrt{r}+ar+b=0$

$r^3+3r^2+ar+b+\sqrt{r}\,(3r^2+a+r)=0$

Rational + irrational parts both $=0$

$\Rightarrow r^3+3r^2+ar+b=0$ and $3r^2+a+r=0$.

2

(ii) Show $(r-\sqrt{r})$ is a root

$(r-\sqrt{r})^3+a(r-\sqrt{r})+b=0$

$r^3-3r^2\sqrt{r}+3r^2-r\sqrt{r}+ar-a\sqrt{r}+b=0$

$\Rightarrow (r^3+3r^2+ar+b)-\sqrt{r}(3r^2+r+a)=0$

$\Rightarrow 0 - \sqrt{r}(0) = 0$ from (i)

$\therefore (r-\sqrt{r})$ is also a root.

2

One mark for expanding after substituting in the root and then one mark for equating rational and irrational parts to zero.

Some students incorrectly quoted the conjugate root theorem for complex numbers. These are not complex roots. However if they stated that the irrational roots are in conjugate pairs since the coefficients of the polynomial are not irrarional then they got the marks for part (ii)

13(c) (i) $y = \sin^{-1}(\cos x)$

$y = \sin^{-1} u$ where $u = \cos x$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \times -\sin x$$

$$= \frac{-\sin x}{\sqrt{1-\cos^2 x}} = \frac{-\sin x}{\sqrt{\sin^2 x}}$$

$\frac{dy}{dx} = -1$ if $\sin x > 0$

but $\frac{dy}{dx} = 1$ if $\sin x < 0$

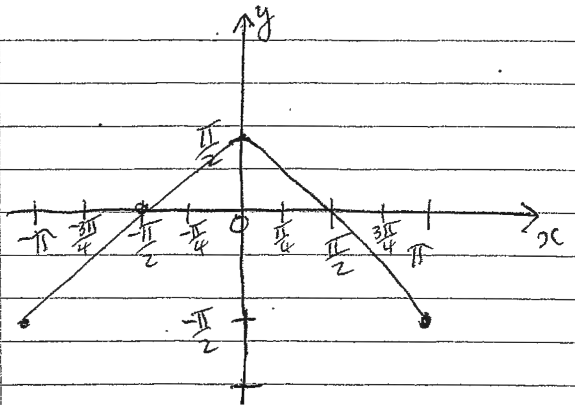
2

Only one mark for solution of -1.
One and a half marks for +/- 1.
For 2 marks students must say when solution is +/-1.

3(c)(ii) $y = \sin^{-1}(\cos x)$ $(-\pi \le x \le \pi)$

Table of values Note EVEN function

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0
$\cos x$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1
$y=\sin^{-1}(\cos x)$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$



2

Generally done well.

13(d) $f_n(x) = (\cos 2x)(\cos 4x)\ldots(\cos 2^n x)$ $n \geq 1$

(i) $f_n(-x) = \cos(-2x)\cos(-4x)\ldots\cos(-2^n x)$

But $\cos(-\theta) = \cos\theta$

$\Rightarrow f_n(-x) = \cos(2x)(\cos 4x)\ldots(\cos 2^n x)$

$\therefore$ even

Done well.

1

(ii) Prove $f_n(x) = \frac{\sin 2^{n+1} x}{2^n \sin 2x}$, $x \neq \frac{m\pi}{2}$, $m \in Z$

Let $n=1$

$LHS = \cos 2x$

$RHS = \frac{\sin 4x}{2\sin 2x} = \frac{2\sin 2x\cos 2x}{2\sin 2x} = \cos 2x$

$\therefore$ true for $n=1$.

Assume true for $n=k$

ie $f_k(x) = (\cos 2x)(\cos 4x)\ldots(\cos 2^k x) = \frac{\sin 2^{k+1} x}{2^k \sin 2x}$

Consider $n = k+1$

Must show

$(\cos 2x)(\cos 4x)\ldots(\cos 2^k x)(\cos 2^{k+1} x) = \frac{\sin 2^{k+2} x}{2^{k+1}\sin 2x}$

13 (d) (ii) (cont)

$$LHS = (\cos 2x)(\cos 4x)\ldots(\cos 2^{k}x)(\cos 2^{k+1}x)$$

$$= \frac{\sin 2^{k+1}x}{2^{k}\sin 2x} \times (\cos 2^{k+1}x)$$ (from induction step)

$$= \frac{\frac{1}{2}\sin 2.(2^{k+1}x)}{2^{k}\sin 2x}$$ (since $\sin x \cos x = \frac{1}{2}\sin 2x$)

$$= \frac{\sin 2^{k+2}x}{2^{k+1}\sin 2x}$$

$= RHS$.

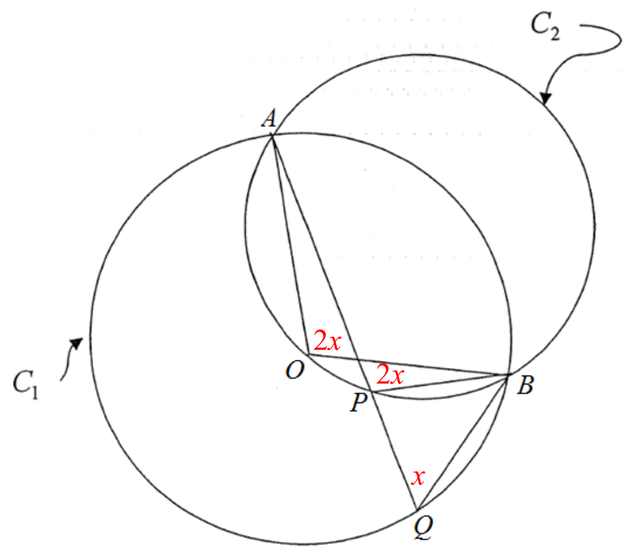
$\therefore$ true for $n = k+1$.

3

If students could do the case fotr n=1 then they could generally use the same identity for n=k+1. Generally this question was done well.

Question 14 Solutions

(a) O is the centre of the circle C_1.



(i) If $\angle AQB = x$, express $\angle AOB$ in the terms of x, giving reason(s) for your answer. **1**

In C_1: $\angle AOB = 2 \times \angle AQB$ (angles at centre and circumference)

$\therefore \angle AOB = 2x$

(ii)	Hence, or otherwise, show that $PB = PQ$.	**2**

In C_2: $\angle AOB = \angle APB$ (angles in the same segment)

$\therefore \angle APB = 2x$

$\angle APB = \angle AQB + \angle PBQ$ (exterior angle sum of ΔBPQ)

$\therefore \angle PBQ = x$

$\therefore PB = PQ$ (equal sides opposite equal angles)

Comment

Generally well done.
Penalties were given for reasons that did not fit or make sense.
If students are going to draw diagrams they need to be larger than postage stamps.

(b)	(i)	By considering the coefficient of x^r in the expansion of $(1+x)(1+x)^n$, Prove that for $1 \le r \le n$, $$\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$$	**2**

$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

$\binom{n}{r}$ is the coefficient of x^r in the expansion of $(1+x)^n$.

$\therefore \binom{n+1}{r}$ is the coefficient of x^r in the expansion of $(1+x)^{n+1}$.

Consider $(1+x)(1+x)^n = (1+x)^n + x(1+x)^n$:

Now $x(1+x)^n = x \times \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^{k+1}$

$\therefore$ the coefficient of x^r in the expansion of $x(1+x)^n$ is $\binom{n}{r-1}$ [letting $r = k+1$]

By collecting like terms in $(1+x)^n + x(1+x)^n$ the coefficient of x^r is $\binom{n}{r}+\binom{n}{r-1}$.

As $(1+x)(1+x)^n \equiv (1+x)^{n+1}$ then by equating coefficients of like terms:

$$\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}.$$

Comment

No marks were awarded for not following the instruction of the question.

This question was poorly set out by almost all students.

Penalties:

- Identities need to be stated properly, not just the right hand side of one.
- Trailing "=" signs caused some issues for students.

Many students are duplicating their answers at the end. If you are a fan of the ${}^nC_r$ notation, then do the whole question using this notation.

(b)	(ii)	The numbers $B_0, B_1, B_2, \ldots$ is defined by $$B_{2m} = \sum_{j=0}^{m} \binom{2m-j}{j} \text{ and } B_{2m+1} = \sum_{k=0}^{m} \binom{2m+1-k}{k}$$ Show that $B_{2n+3} - B_{2n+2} = B_{2n+1}$ for $n \ge 0$.	**3**

$$B_{2n+1} = \sum_{j=0}^{n} \binom{2n+1-j}{j},\ B_{2n+2} = \sum_{j=0}^{n+1} \binom{2n+2-j}{j} \text{ and } B_{2n+3} = \sum_{j=0}^{n+1} \binom{2n+3-j}{j}$$

Consider $B_{2n+1} + B_{2n+2}$:

$$B_{2n+1} + B_{2n+2} = \sum_{j=0}^{n} \binom{2n+1-j}{j} + \sum_{j=0}^{n+1} \binom{2n+2-j}{j}$$

$$= \binom{2n+1}{0} + \binom{2n}{1} + \binom{2n-1}{2} + \ldots + \binom{n+2}{n-1} + \binom{n+1}{n} +$$

$$\binom{2n+2}{0} + \binom{2n+1}{1} + \binom{2n}{2} + \binom{2n-1}{3} + \ldots + \binom{n+2}{n} + \binom{n+1}{n+1}$$

Note $\binom{2n+3}{0} = \binom{2n+2}{0} = 1$

Now using part (b) (i):

$$B_{2n+1} + B_{2n+2} = \binom{2n+3}{0} + \binom{2n+2}{1} + \binom{2n+1}{2} + \ldots + \binom{n+3}{n} + \binom{n+2}{n+1}$$

$$= \binom{2n+3-0}{0} + \binom{2n+3-1}{1} + \binom{2n+3-2}{2} + \ldots + \binom{2n+3-n}{n} + \binom{2n+3-(n+1)}{n+1}$$

$$= \sum_{j=0}^{n+1} \binom{2n+3-j}{j}$$

$$= B_{2n+3}$$

Alternative solution on next page

Alternative solution to (b) (ii)

$$B_{2n-3} - B_{2n+2} = \sum_{k=0}^{n+1}\binom{2n+3-k}{k} - \sum_{k=0}^{n+1}\binom{2n+2-k}{k}$$

$$= \binom{2n+3}{0} - \binom{2n+2}{0} + \sum_{k=1}^{n+1}\left[\binom{2n+3-k}{k} - \binom{2n+2-k}{k}\right]$$

$$= 1 - 1 + \sum_{k=1}^{n+1}\left[\binom{2n+2-k}{k-1}\right] \qquad \left[\text{Using part (i)}\right]$$

$$= \sum_{k=1}^{n+1}\left[\binom{2n+2-k}{k-1}\right]$$

Now let $u = k - 1$

$$B_{2n-3} - B_{2n+2} = \sum_{u=0}^{n}\binom{2n+2-(u+1)}{u}$$

$$= \sum_{u=0}^{n}\binom{2n+1-u}{u}$$

$$= B_{2n+1}$$

Comment

This question would have been better left blank by a lot of students rather than wasting their ink on it. A handful were successful in using factorials to answer this question.

Penalties:

- No care was taken with the definitions, so that most people just could not score many marks, if any at all.
- Many students didn't elaborate how they made use of part (i).
- Some students didn't think through their calculations and ended up with terms like $\binom{n+1}{-1}$ or $\binom{n-1}{n+1}$ and then tried to just cross them out or hope that the marker wouldn't see them.

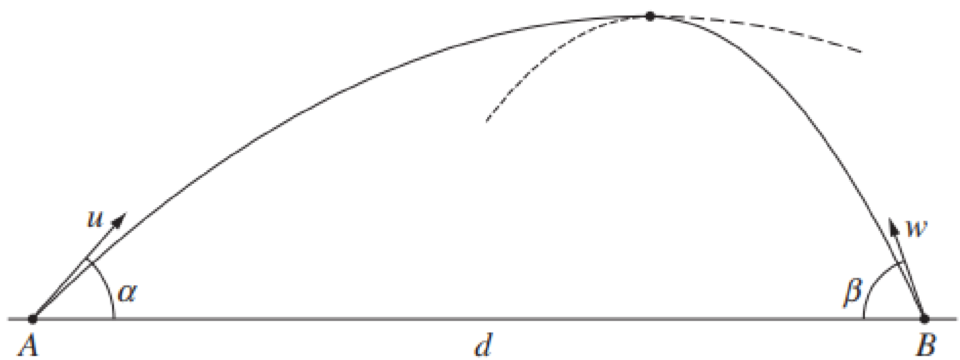
(c) When an object is projected from the origin with an initial velocity of V at an angle θ to the horizontal, the equations of motion of the object are:

$$x(t) = Vt\cos\theta \text{ and } y(t) = Vt\sin\theta - \frac{1}{2}gt^2 . \text{ (Do NOT prove these)}$$

Two particles, A and B, are projected simultaneously towards each other from two points which are d distance apart in a horizontal plane.
Particle A has mass m and is projected at speed u at angle α above the horizontal. Particle B has a mass M and is projected at speed w at angle β above the horizontal.

The trajectories of the two particles lie in the same vertical plane. The particles collide directly when each is at its point of greatest height above the plane.

As a result of the collision $m\cot\alpha = M\cot\beta$. (Do NOT prove this result)



(i) Show that $u\sin\alpha = w\sin\beta$. **2**

A particle reaches its maximum height when $\dot{y} = 0$.

$$\dot{y} = V\sin\theta - gt \Rightarrow t = \frac{V\sin\theta}{g}$$

Particles A and B collide at the same time $\Rightarrow t = \dfrac{u\sin\alpha}{g} = \dfrac{w\sin\beta}{g}$

$$\therefore u\sin\alpha = w\sin\beta$$

Comment

The aim of this question is to show why the initial velocities are the same. To state that they have to be, as many physics students insisted, begs the question.

(c)	(ii)	Show that the collision occurs at a point which is a horizontal distance, b metres, from the point of projection of A where $b=\dfrac{Md}{m+M}$.	3

How far has a particle travelled horizontally when it reaches maximum height?

$$\begin{aligned} x &= Vt\cos\theta \\ &= V\times\frac{V\sin\theta}{g}\times\cos\theta \\ &= \frac{V^2\sin\theta\cos\theta}{g} \end{aligned}$$

$$\therefore b=\frac{u^2\sin\alpha\cos\alpha}{g} \text{ and } d-b=\frac{w^2\sin\beta\cos\beta}{g}$$

$$\therefore g=\frac{u^2\sin\alpha\cos\alpha}{b}=\frac{w^2\sin\beta\cos\beta}{d-b}$$

$$\therefore \frac{(u\sin\alpha)^2\cos\alpha\sin\beta}{b}=\frac{(w\sin\beta)^2\cos\beta\sin\alpha}{d-b}$$

Applying $u\sin\alpha=w\sin\beta$ from (c) (i) and re-arranging $\Rightarrow \dfrac{(d-b)\cos\alpha}{\sin\alpha}=\dfrac{b\cos\beta}{\sin\beta}$

$$\therefore (d-b)\cot\alpha=b\cot\beta$$

Now $m\cot\alpha=M\cot\beta \Rightarrow \dfrac{\cot\alpha}{\cot\beta}=\dfrac{M}{m}$

$$\therefore (d-b)\frac{\cot\alpha}{\cot\beta}=b$$

$$\therefore (d-b)\frac{M}{m}=b \Rightarrow M(d-b)=bm$$

$$\therefore b(m+M)=Md$$

$$\therefore b=\frac{Md}{m+M}$$

Comment

Some very impressive alternative answers provided by some students. That said, most students didn't handle this question well.

(c) (iii) Show that the height, h, above the horizontal plane at which the collision occurs is given by **2**

$$h = \frac{b \tan\alpha}{2}.$$

What is the maximum height a particle reaches?

$$y_{\max} = V \times \frac{V\sin\theta}{g} \times \sin\theta - \frac{g}{2}\left(\frac{V\sin\theta}{g}\right)^2$$

$$= \frac{V^2\sin^2\theta}{g} - \frac{V^2\sin^2\theta}{2g}$$

$$= \frac{V^2\sin^2\theta}{2g}$$

$$h = \frac{u^2\sin^2\alpha}{2g}$$

$$= \frac{u^2\sin\alpha.\sin\alpha}{2g}$$

$$= \frac{u^2\sin\alpha\cos\alpha.\sin\alpha}{2g\cos\alpha}$$

$$= \frac{1}{2} \times \frac{u^2\sin\alpha\cos\alpha}{g} \times \frac{\sin\alpha}{\cos\alpha}$$

$$= \frac{1}{2} \times b \times \tan\alpha \qquad \left[\text{From (c) (ii)}\right]$$

$$= \frac{b\tan\alpha}{2}$$

Alternative solution on next page

$$b = ut\cos\alpha \Rightarrow t = \frac{b}{u\cos\alpha}$$

The time to get to (b, h) is $t = \dfrac{b}{u\cos\alpha}$

$$\begin{aligned}
h &= ut\sin\alpha - \tfrac{1}{2}gt^2 \\
&= u\left(\frac{b}{u\cos\alpha}\right)\sin\alpha - \tfrac{1}{2}g\left(\frac{b}{u\cos\alpha}\right)^2 \\
&= b\tan\alpha - \frac{b^2}{2gu^2\cos^2\alpha} \\
&= b\tan\alpha - \frac{b}{2}\times\frac{b}{gu^2\cos^2\alpha}\times\frac{\sin\alpha}{\sin\alpha} \\
&= b\tan\alpha - \frac{b\tan\alpha}{2}\times\frac{b}{gu^2\cos\alpha\sin\alpha} \\
&= b\tan\alpha - \frac{b\tan\alpha}{2} \qquad \left[\text{Proved in (ii) } b = \frac{u^2\sin\alpha\cos\alpha}{g}\right] \\
&= \frac{b\tan\alpha}{2}
\end{aligned}$$

Comment

Some interesting alternative answers provided by some students. That said, most students didn't handle this question well.

End of solutions