Sydney Girls High School

2003

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

Extension 1

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2003 HSC Examination Paper in this subject.

General Instructions

- Reading Time – 5 mins
- Working Time – 2 hours
- Attempt ALL questions
- ALL questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

Marks

Question 1 (12 marks)

a) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$ (1)

b) Evaluate i) $\int_0^{\pi} \sin^2 x\,dx$ (2)

ii) $\int_{-1}^{0} x\sqrt{1+x}\,dx$ (3)

using the substitution $u = 1 + x$

c) Find the point which divides the line joining A (1, 3) and B (-2, 6) externally in the ratio 2 : 1 (2)

(d) Write $3\cos\theta + 4\sin\theta$ in the form $R\cos(\theta - \alpha)$ and (4)

Hence solve $3\cos\theta + 4\sin\theta = 2$, $0 \le \theta \le 2\pi$

(Give you answer correct to 2 decimal places)

Marks

Question 2 (12 marks)

(a) Solve $\dfrac{2x}{x-1} < 1$ (3)

(b) Given that $x = 1.7$ is a first approximation to the positive root of $x = 2\sin x$, use Newton's method once to find a second approximation to this root. (Correct to 1 decimal place) (3)

(c) Find the size of the acute angle between the lines $4x - 3y + 1 = 0$ and $x + 4y + 1 = 0$. (Give answer to the nearest degree). (3)

(d) A polynomial is given by

$P(x) = x^3 + ax^2 + bx - 18$ (3)

Find the values of a and b if $(x+2)$ is a factor of $P(x)$ and if -24 is the Remainder when $P(x)$ is divided by $(x-1)$.

Marks

Question 3 (12 marks)

(a) For the function $y = 2\sin^{-1}x$, find the equation of the tangent to the curve at the point where $x = \frac{1}{\sqrt{2}}$ (3)

(b) The acceleration of a particle moving in a straight line is given by (5)

$$\frac{d^2x}{dt^2} = 2x - 3$$

where x is the position (in metres) from the origin 0 and t is the time in seconds. Initially the particle is at rest at $x = 4$ m.

i. If the velocity is v m/s show that $v^2 = 2x^2 - 6x - 8$

ii. Show that the particle does not pass through the origin.

iii. Find the position when $v = 10$ m/s

(c) The Volume (V) of a sphere of radius r cm is increasing at a constant rate of 200 cm^3 per second. (4)

i. Find $\frac{dr}{dt}$ in terms of r

ii. Hence find the rate of increase of the surface Area (A) when the radius of the sphere is 50 cm.

Question 4 (12 marks)

(a) Prove the identity $\frac{2\tan\Theta}{1+\tan^2\Theta} = \sin 2\Theta$ (1)

(b) For the function $f(x) = 2\cos^{-1}\frac{x}{3}$ (3)

i. Evaluate f (0)

ii. State the domain and range of y = f (x)

iii. Sketch the graph of y = f (x)

(c) A particle moves in simple harmonic motion about the origin 0. Its position x metres from 0 at time t seconds is given by (4)

$$x = 3\cos(2t + \frac{\Pi}{3})$$

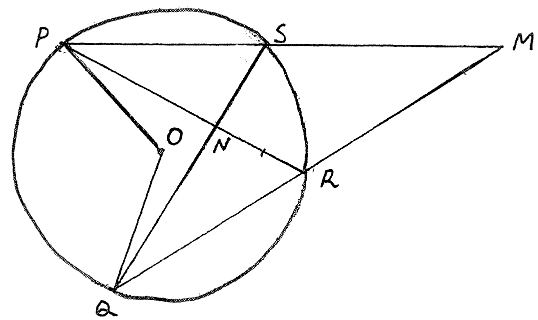
i. Find the acceleration in terms of position.

ii. Find its amplitude

iii. State the position x for maximum velocity.

iv. Find the maximum velocity.

(d) (4)



0 is the centre of a circle and $\angle POQ = \Theta°$ Lines PS and QR produced intersect at M and lines PR and QS intersect at N

i. Copy this diagram into your exam booklet

ii. Prove that $\angle PRM = (180 - \frac{1}{2}\Theta)°$

iii. Prove that $\angle PNQ + \angle PMQ = \Theta$

Marks

Question 5 (12 marks)

(a) If α, β, γ are the roots of $x^3 - 3x + 1 = 0$ (3)

Find i $\alpha + \beta + \gamma$

ii $\alpha \beta \gamma$

iii $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(b) i. Factorise the polynomial $f(x) = 3x^3 - 7x^2 + 4$ (3)

ii Hence solve $f(x) = 0$

(c) The points P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$ (3)

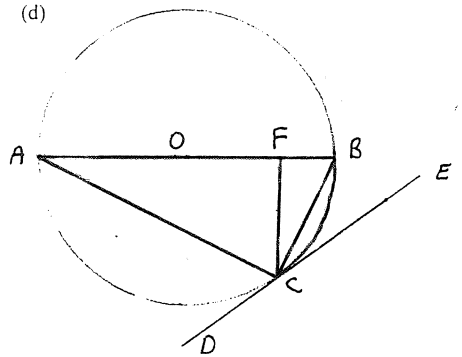
i. Derive the equation of the tangent at P

ii. Find the coordinates of the point of intersection T of the tangents to the parabola at P and Q

iii. If these tangents intersect at 45^0 show that

$p = 1 + q + pq$, if $p > q$

(d)



CF is perpendicular to AB (3)

O is the centre of the circle

DCE is a tangent at C

i Copy this diagram into your workbook

ii Prove that BC bisects angle FCE

Marks

Question 6 (12 marks)

(a) In a population study, the population N is given by the equation (3)

$$N = 200 + Ae^{kt}$$

Initially N = 300 and when t = 3 seconds, N = 500

i Find the values of A and k (correct to 4 decimal places)

ii Find the population after 5 seconds

(b) i If $x = 4\sin\Theta$ show that $\cos\Theta = \frac{\sqrt{16 - x^2}}{4}$ (5)

ii By using the substitution $x = 4\sin\Theta$

show that $\int \frac{x^2 dx}{\sqrt{16 - x^2}} = 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16 - x^2}}{2} + c$

(c) The chord of contact of the tangents to the parabola $x^2 = 4ay$ from an external point P (x_1, y_1) cuts the directrix at Q. Prove that PQ subtends a right angle at the focus of the parabola. (4)

Marks

Question 7 (12 marks)

(a) Prove, using the Principle of Mathematical Induction, that (3)

$1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$

for all positive integers $n \geq 1$

(b) An projectile at the highest point of its trajectory has a velocity (5)

8 metres per second and its position is 8 metres above the ground.

Find i. the angle of projection (to nearest degree)

ii the initial velocity (correct to 1 decimal place)

(take $g = 9.8 \text{ ms}^{-2}$)

(c) (4)

i. Restrict the domain of $y = x^2 - 4x$ so that it will have an inverse function of the largest possible domain, and will include the point $x = 3$.

ii. Determine the equation of the inverse, writing y as the subject.

iii. Write down the co-ordinates of any points shared by the original curve and its inverse.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

Note $\quad \ln x = \log_e x, \quad x > 0$

(a) $\lim_{x \to 0} \frac{\sin 2x}{x}$

$= 2 \lim_{x \to 0} \frac{\sin 2x}{2x}$

$= 2$

(b) i $\int_0^{\pi} \sin^2 x \, dx$

$= \frac{1}{2} \int_0^{\pi} 1 - \cos 2x \, dx$

$= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi}$

$= \frac{1}{2} \left[\left(\pi - \frac{1}{2} \sin 2\pi \right) - (0) \right]$

$= \frac{\pi}{2}$

ii $\int_{-1}^{0} x \sqrt{1+x} \, dx$

Let $u = 1 + x$ | $x = -1, u = 0$; $x = 0, u = 1$

$\frac{du}{dx} = 1$

$\therefore du = 1 dx$

$\therefore \int_{-1}^{0} x \sqrt{1+x} \, dx$

$= \int_0^1 (u-1) u^{\frac{1}{2}} du$

$= \int_0^1 u^{3/2} - u^{\frac{1}{2}} du$

$= \left[\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3} \right]_0^1$

$= \frac{2}{5} - \frac{2}{3}$

$= -\frac{4}{15}$

(c) $A = 1, 3 = x_1 y_1$

$B = -2, 6 = x_2 y_2$

Divide in ratio $= -2 : 1$

$= k_1 : k_2$

$x = \frac{x_1 k_2 + x_2 k_1}{k_1 + k_2}$

$= \frac{(1 \times 1) + (-2 \times -2)}{-2 + 1}$

$= \frac{1 + 4}{-1}$

$= -5$

$y = \frac{y_1 k_2 + y_2 k_1}{k_1 + k_2}$

$= \frac{(3 \times 1) + (6 \times -2)}{-2 + 1}$

$= \frac{3 - 12}{-1}$

$= 9$

$\therefore (x, y) = (-5, 9)$

11 (d) $a \cos\theta + b \sin\theta = A \cos(\theta - \alpha)$

$= A \cos\theta \cos\alpha + A \sin\theta \sin\alpha$

$a = A \cos\alpha$, $b = A \sin\alpha$

$a^2 + b^2 = A^2 (\cos^2\alpha + \sin^2\alpha)$

$\therefore A = \sqrt{a^2 + b^2}$

$\frac{A \sin\alpha}{A \cos\alpha} = \frac{b}{a}$

$\therefore \tan\alpha = \frac{b}{a}$

Now solve $3 \cos\theta + 4 \sin\theta = 2$

$A = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\tan\alpha = \frac{4}{3}$ $\therefore \alpha = 0.9273$

$A \cos(\theta - \alpha) = 2$

$5 \cos(\theta - 0.9273) = 2$

$\cos(\theta - 0.9273) = 2/5$

$\theta - 0.9273 = 1.1593$ or 5.1239

$\therefore \theta = 2.09$ or 6.05 ans.

62. a $\frac{2x}{x-1} < 1$

Put $x - 1 = 0$

$\therefore x = 1$ is 1st c.v.

Put $\frac{2x}{x-1} = 1$

$2x = x - 1$

$\therefore x = -1$ is 2nd c.v.

-1 $\quad$ 1

Test $x = -2$, $\frac{-4}{-3} = 1\frac{1}{3} \not< 1$

Test $x = 2$, $\frac{4}{2-1} = 4 \not< 1$

Test $x = 0$ $\frac{0}{-1} = 0 < 1$

$\therefore -1 < x < 1$ ans.

b $x = 2 \sin x$

$x - 2 \sin x = 0$

Let $f(x) = x - 2 \sin x$

$f'(x) = 1 - 2 \cos x$

Let $a = 1.7$ be the first approx

$a_1 = a - \frac{f(a)}{f'(a)}$

$= 1.7 - \frac{[1.7 - 2 \sin 1.7]}{[1 - 2 \cos 1.7]}$

$= 1.9$

$\therefore$ Second approximation to the root is $x = 1.9$

62 (c) $4x - 3y + 1 = 0$

$4x + 1 = 3y$

$\therefore y = \frac{4}{3}x + \frac{1}{3}$ $\quad (m_1 = \frac{4}{3})$

$x + 4y + 1 = 0$

$4y = -x - 1$

$\therefore y = -\frac{1}{4}x - \frac{1}{4}$ $\quad (m_2 = -\frac{1}{4})$

$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

$= \left| \frac{4/3 + 1/4}{1 - \frac{4}{3} \times \frac{1}{4}} \right|$

$= \left| 2\,3/8 \right|$

$\theta = 67^{\circ}$ ans

(d) $P(x) = x^3 + ax^2 + bx - 18$

$(x+2)$ is a factor

$\therefore P(-2) = -8 + 4a - 2b - 18 = 0$

$4a - 2b = 26$

$2a - b = 13$ ①

Remainder $= -24$ when $P(x)$ is divided by $(x-1)$

$\therefore P(1) = 1 + a + b - 18 = -24$

$a + b = -7$ ②

add ① + ② $3a = 6$

$\therefore a = 2$ } ans.

(a) $y = 2\sin^{-1}x$

$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

Put $x = \frac{1}{\sqrt{2}}$

$y = 2\sin^{-1}\frac{1}{\sqrt{2}} = 2\frac{\pi}{4} = \frac{\pi}{2}$

Let $x = \frac{1}{\sqrt{2}}$, $\frac{dy}{dx} = \frac{2}{\sqrt{1-\frac{1}{2}}} = \frac{2}{\sqrt{\frac{1}{2}}} = 2\sqrt{2}$

Eqn of tangent is

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{2} = 2\sqrt{2}\,(x - \frac{1}{\sqrt{2}})$

$y - \frac{\pi}{2} = 2\sqrt{2}x - 2$

$\therefore y = 2\sqrt{2}x + \frac{\pi}{2} - 2$

(b) i $v^2 = 2\int a\,dx$

$= 2\int 2x - 3\,dx$

$= 2\left[\frac{2x^2}{2} - 3x\right] + C$

Data: $t=0$, $v=0$, $x=4$

$= 2x^2 - 6x + C$

Put $v=0$, $x=4$

$0 = 32 - 24 + C$

$\therefore C = -8$

Hence $v^2 = 2x^2 - 6x - 8$

ii Let $x = 0$

$\therefore v^2 = -8$

a squared velocity can't be negative

$\therefore x \neq 0$

iii Let $v = 10$

$100 = 2x^2 - 6x - 8$

$50 = x^2 - 3x - 4$

$\therefore x^2 - 3x - 54 = 0$

$\therefore (x-9)(x\ \ (\ )$

$x = 9$ or -6 It can't get to $x=-6$ if it does not pass through $x=0$

$\therefore x = 9$ m Ans

Q3 (a) i $\frac{dV}{dt} = 200$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$

$200 = 4\pi r^2\cdot\frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{200}{4\pi r^2} = \frac{50}{\pi r^2}$

ii $S = 4\pi r^2$

$\frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = \frac{dS}{dr}\cdot\frac{dr}{dt}$

$= 8\pi r\cdot\frac{50}{\pi r^2}$

$= \frac{400}{r}$ put $r = 50$

$\therefore \frac{dS}{dt} = \frac{400}{50} = 8$ cm²/s

Q4

(a) i $x = 3\cos(2t + \frac{\pi}{3})$

$\dot{x} = -6\sin(2t + \frac{\pi}{3})$

$\ddot{x} = -12\cos(2t + \frac{\pi}{3})$

$\therefore \ddot{x} = -4x$

ii $x = a\cos(nt + \alpha)$ where a = amplitude

$\therefore$ amplitude $= 3$ m

iii maximum vel occurs at $x = 0$ (since $\ddot{x} = 0$)

iv $\dot{x} = -6\sin(2t + \frac{\pi}{3})$

Since $x = 0$ $\therefore 0 = 3\cos(2t + \frac{\pi}{3})$

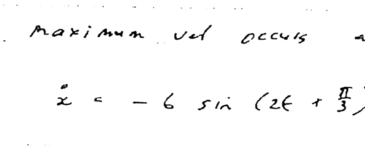
Thus $2t + \pi/3 = \frac{\pi}{2}$

$\therefore \dot{x} = -6\sin(\frac{\pi}{2}) = -6$

This means the particle is travelling to the left. Hence when it travels to the right its max vel = 6 m/s.

Q4 (b)

i



ii Aim: Prove $\angle PRM = (180 - \frac{1}{2}\theta)$

Proof: $\angle PRQ = \frac{1}{2}\theta$ ($\angle$ at centre is $2\times\angle$ at circum)

$\therefore \angle PRM = 180 - \frac{1}{2}\theta$ (adj supp $\angle$'s)

Aim Prove $\angle PNQ + \angle PMQ = \theta$

Proof $\angle PSQ = \frac{1}{2}\theta$ ($\angle$ at centre is $2\times\angle$ at circum)

$\therefore \angle QSM = 180 - \frac{1}{2}\theta$ (adj supp $\angle$'s)

$\angle SNR + \angle SMR = 360 - (180 - \frac{1}{2}\theta) - (180 - \frac{1}{2}\theta)$ ($\angle$ sum of quad)

$= \theta$

$\angle PNQ = \angle SNR$ (vert opp $\angle$s)

$\angle SMR = \angle PMQ$ (PSM and QRM are st. lines)

$\therefore \angle PNQ + \angle PMQ = \theta$

Q4 (a) Prove $\frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$

L.H.S $= \frac{2\tan\theta}{\sec^2\theta}$

$= 2\frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta$

$= 2\sin\theta\cos\theta$

$= \sin 2\theta$

$=$ R.H.S.

(b) $f(x) = 2\cos^{-1}\frac{x}{3}$

i $f(0) = 2\cos^{-1}0 = 2\cdot\frac{\pi}{2} = \pi$

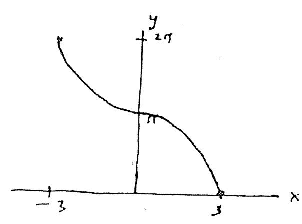
ii $-1 \le \frac{x}{3} \le 1$

$\therefore -3 \le x \le 3$ is domain of $f(x)$

$0 \le \cos^{-1}\frac{x}{3} \le \pi$

$\therefore 0 \le 2\cos^{-1}\frac{x}{3} \le 2\pi$ is Range of $f(x)$

iii



(a) $x^3 + 0x^2 - 3x + 1 = 0$

$a = 1 \quad b = 0 \quad c = -3 \quad d = 1$

i $\alpha + \beta + \gamma = \frac{-b}{a} = 0$

ii $\alpha\beta\gamma = -\frac{d}{a} = \frac{-1}{1} = -1$

iii $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{c/a}{-1} = \frac{-3/1}{-1} = 3$

(b) i $f(x) = 3x^3 - 7x^2 + 4$

$f(1) = 3 - 7 + 4 = 0 \quad \therefore (x-1)$ is a factor

$$\begin{array}{r l}
 & 3x^2 - 4x - 4 \\
x - 1 \,\big)\, & 3x^3 - 7x^2 + 4 \\
 & 3x^3 - 3x^2 \quad \text{(Subtract)} \\
 & -4x^2 + 4 \\
 & -4x^2 + 4x \\
 & -4x + 4 \quad \text{(Subtract)} \\
 & -4x + 4 \\
 & 0 \quad \text{(Subtract)}
\end{array}$$

$\therefore f(x) = (x-1)(3x^2 - 4x - 4)$

$= (x-1)(3x+2)(x-2)$

Now Solve $(x-1)(3x+2)(x-2) = 0$

$\therefore x = 1, -\frac{2}{3}, 2$ ans.

Q5 (c) i $x = 2ap \qquad y = ap^2$

$\frac{dx}{dp} = 2a \qquad \frac{dy}{dp} = 2ap$

$\therefore \frac{dy}{dx} = \frac{2ap}{2a} = p$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$\therefore y = px - ap^2$ is tangent at P

ii $y = px - ap^2$

$y = qx - aq^2$

$0 = (p-q)x - ap^2 + aq^2$ Subtract

$a(p^2 - q^2) = (p-q)x$

$\therefore x = a(p+q)$

$y = ap(p+q) - ap^2$

$= ap^2 + apq - ap^2$

$= apq$

$\therefore$ Pt of intersection T $= (a(p+q), apq)$

Q5 (c) iii If Tangents intersect at 45° $(= \theta)$

$m_1 = p, \quad m_2 = q$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

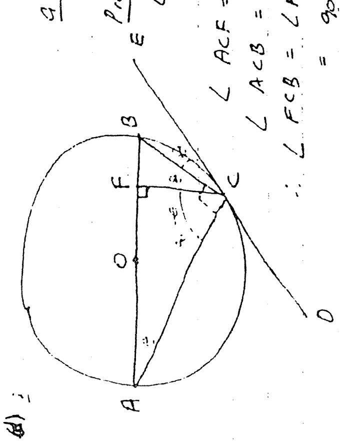
$\therefore \tan 45 = \left|\frac{p-q}{1+pq}\right|$

Hence $\frac{p-q}{1+pq} = 1$ since $p > q$

$p - q = 1 + pq$

$\therefore p = 1 + q + pq$

Q5 (d) i



Aim: Prove BC bisects $\angle FCE$

Proof: Let $\angle BCE = \theta$

$\angle BCE = \angle CAB$ $(= \theta)$ ($\angle$ in alt. seg.)

$\angle ACF = 90 - \theta$ ($\angle$ sum of Δ)

$\angle ACB = 90^\circ$ ($\angle$ in semicircle)

$\therefore \angle FCB = \angle ACB - \angle ACF$

$= 90^\circ - (90 - \theta)$

$= \theta$

$\therefore$ Since $\angle BCE = \angle FCB = \theta$

Then BC bisects $\angle FCE$

Q6 (a) i $N = 200 + Ae^{kt}$

Let $N = 300$ and $t = 0$

$300 = 200 + Ae^0$

$\therefore A = 100$

Let $N = 500$ and $t = 3$

$500 = 200 + 100e^{3k}$

$\frac{300}{100} = e^{3k}$

$3 = e^{3k}$

$\log_e 3 = \log_e e^{3k}$

$\log_e 3 = 3k \log_e e$

$\therefore k = \frac{1}{3} \times \log_e 3 = 0.3662$

Q6 (a) ii Put $t = 5$

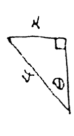
$N = 200 + 100e^{0.3662 \times 5}$

$= 824$ = population.

Q6 (b) i $x = 4\sin\theta$

$\sin\theta = \frac{x}{4}$

$\therefore \cos\theta = \frac{\sqrt{16 - x^2}}{4}$



$y^2 + x^2 = 4^2$

$y^2 = 16 - x^2$

$y = \sqrt{16 - x^2}$

Q6 b) i

$x = 4\sin\theta \qquad \therefore x^2 = 16\sin^2\theta$

$\dfrac{dx}{d\theta} = 4\cos\theta$

$\therefore dx = 4\cos\theta \, d\theta$

$$\int \frac{x^2\,dx}{\sqrt{16-x^2}} = \int \frac{16\sin^2\theta \cdot 4\cos\theta\, d\theta}{\sqrt{16-16\sin^2\theta}}$$

$$= \int \frac{16\sin^2\theta \cdot 4\cos\theta \, d\theta}{4\cos\theta}$$

$$= 16 \int \sin^2\theta \, d\theta$$

$$= 16 \cdot \tfrac{1}{2} \int 1 - \cos 2\theta \, d\theta$$

$$= 8\left[\theta - \tfrac{1}{2}\sin 2\theta\right] + C$$

$$= 8\left[\theta - \tfrac{1}{2} \cdot 2\sin\theta\cos\theta\right] + C$$

$$8\theta - 8\sin\theta\cos\theta + C$$

$$8\sin^{-1}\frac{x}{4} - 8\,\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4} + C$$

$$8\sin^{-1}\frac{x}{4} - \tfrac{1}{2}x\sqrt{16-x^2} + C$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$

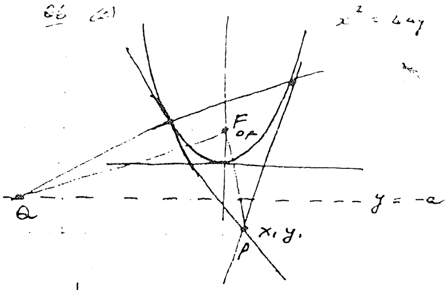
$\sin\theta = \dfrac{x}{4}$

$\theta = \sin^{-1}\left(\dfrac{x}{4}\right)$

$\cos\theta = \dfrac{\sqrt{16-x^2}}{4}$

From part (i)

Q6 (c)



Chord of contact from $P(x_1 y_1)$ is

$x x_1 = 2a(y + y_1)$

This cuts directrix $(y = -a)$ at Q

$x x_1 = 2a(y_1 - a)$

$x = \dfrac{2a(y_1 - a)}{x_1}$

$\therefore$ Point Q $= \left[\dfrac{2ay_1 - 2a^2}{x_1}, -a\right]$

Grad PF $= \dfrac{(y_1 - a)}{x_1}$ $(= m_1)$

Grad QF $= \dfrac{-a-a}{\left(\dfrac{2ay_1 - 2a^2}{x_1}\right)} = \dfrac{-2a x_1}{2ay_1 - 2a^2}$ $(= m_2)$

$m_1 \times m_2 = \dfrac{(y_1 - a)}{x_1} \cdot \dfrac{-2a x_1}{2a(y_1 - a)} = -1$

$\therefore PF \perp QF$

Thus PQ subtends a right angle at the focus

Q7 a Prove $1 + 2 + 4 + \dots + 2^{n-1} = 2^n - 1$ for $n \geq 1$

Step 1 Prove true for $n = 1$

L.H.S $= 1 \qquad$ R.H.S $= 2^1 - 1 = 1$

$\therefore$ True for $n = 1$

Step 2 Assume true for $n = k$

$\therefore 1 + 2 + 4 + \dots + 2^{k-1} = 2^k - 1$

Step 3 Prove true for $n = k+1$ (i.e. Prove $1+2+4+\dots+2^{k-1}+2^k = 2^{k+1} - 1$)

L.H.S $= 1 + 2 + 4 + \dots + 2^{k-1} + 2^k$

$= 2^k - 1 + 2^k$

$= 2(2^k) - 1$

$= 2^{k+1} - 1$

R.H.S $= 2^{k+1} - 1$

$\therefore$ True for $n = k+1$

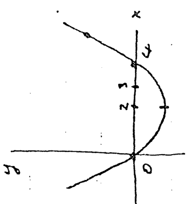
Step 4 If true for $n = 1$, then true for $n = 2$ etc.

Hence by Math. Induction

$1 + 2 + 4 + \dots + 2^{n-1} = 2^n - 1$ for all $n \geq 1$

Q7 (c) $y = x^2 - 4x$

$= x(x-4)$



i Inverse Domain = $x \geq 2$

ii $x = y^2 - 4y$

Complete the square

$4 + x = y^2 - 4y + 4$

$\therefore (y-2)^2 = x + 4$

$y - 2 = \pm\sqrt{x+4}$

$\therefore y = 2 \pm \sqrt{x+4}$

Test $x = 5$, $y = 2 + \sqrt{9} = 5$

$\therefore$ Inverse function is $y = 2 + \sqrt{x+4}$

iii To find the common point.

Solve $y = x$

$y = x^2 - 4x$

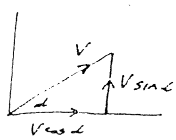
$x^2 - 4x = x$

$\therefore x^2 - 5x = 0$

$x(x-5) = 0$

$\therefore x = 5, y = 5 \qquad$ Ans $(5, 5)$

7 (b) i



Vertical motion. Take up ↑ positive

Data: $t = 0$, $y = 0$, $\dot{y} = V \sin \alpha$

$\ddot{y} = -g$

$\dot{y} = -gt + c$

$V \sin \alpha = 0 + c$

$\therefore \dot{y} = -gt + V \sin \alpha$ (*)

$y = -\frac{gt^2}{2} + Vt \sin \alpha + c$

$0 = 0 + 0 + c$

$\therefore y = -\frac{gt^2}{2} + Vt \sin \alpha$ (*)

Horizontal motion Data: $t = 0$, $x = 0$, $\dot{x} = V \cos \alpha$

$\ddot{x} = 0$

$\dot{x} = 0 + c$

$V \cos \alpha = c$

$\therefore \dot{x} = V \cos \alpha$ (*)

$x = Vt \cos \alpha + c$

$0 = 0 + c$

$\therefore x = Vt \cos \alpha$ (*)

7 b i At highest point $\dot{y} = 0$

$0 = -gt + V \sin \alpha$

$gt = V \sin \alpha$

$t = \frac{V \sin \alpha}{g}$

At highest point $y = 8$

$8 = -g\frac{t^2}{2} + Vt \sin \alpha$

$= -\frac{g}{2} \frac{V^2 \sin^2 \alpha}{g^2} + V \sin \alpha \frac{V \sin \alpha}{g}$

$= -\frac{V^2 \sin^2 \alpha}{2g} + \frac{V^2 \sin^2 \alpha}{g}$

$\therefore 8 = \frac{V^2 \sin^2 \alpha}{2g}$ (1)

Also at highest point $\dot{x} = 8$

$8 = V \cos \alpha$

$\therefore V = \frac{8}{\cos \alpha}$ sub into (1) at

$8 = \frac{64}{\cos^2 \alpha} \cdot \frac{\sin^2 \alpha}{2g}$

$\frac{16 \times 9.8}{64} = \frac{\sin^2 \alpha}{\cos^2 \alpha} = \tan^2 \alpha$

$2.45 = \tan^2 \alpha$

$\therefore \tan \alpha = \sqrt{2.45} = 1.5652$

$\alpha = 57^\circ 26' \doteqdot 57^\circ$ = angle of projection

So $V = \frac{8}{\cos \alpha} = 14.[illegible]$ m/s

= initial velocity