SYDNEY GIRLS HIGH SCHOOL

LABOR OMNIA VINCIT

2008

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Diagrams are NOT drawn to scale.
- All necessary working should be shown in every question.
- Start each question on a new page.

Total marks – 84

- Attempt Questions 1 – 7.
- All questions are of equal value.

This is a trial paper ONLY. It does not necessarily reflect the format or contents of the 2008 HSC Examination in this subject.

Question 1:

a) Solve for x if $\frac{3x}{x-1} < 2$ *[3]*

b) Find the exact value of $\int_0^1 xe^{-x^2}.dx$ *[2]*

c) Find $\frac{d}{dx}\left(\sin^{-1}\sqrt{x}\right)$ *[2]*

d) If A is the point (3,-2) and B is (-3,1) find the point P that divides AB externally in the ratio 2:1. *[2]*

e) If the limiting sum of a geometric progression is 5 and the first term is 3, find the common ratio. *[3]*

Question 2:

a) Sketch the curve $y = 3\sin^{-1} 2x$ *[2]*

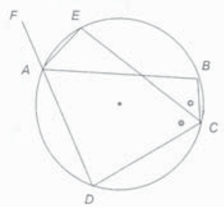
b) If $2\tan\theta = 3$ and θ is acute, find the exact value of $\frac{3\sin\theta + 2\cos\theta}{\sec\theta - \operatorname{cosec}\theta}$ *[3]*

c) The roots of $x^3 - 3x^2 + 6x - 5 = 0$ are α, β and γ. Find the value of

i) $(\alpha+1)+(\beta+1)+(\gamma+1)$ *[2]*

ii) $(\alpha+1)(\beta+1)(\gamma+1)$ *[2]*

e) In the diagram, DA is extended to F and EC bisects $\angle BCD$. Prove that AE bisects $\angle FAB$ *[3]*



Question 3:

a) Find the constant term in the expansion $\left(\frac{2}{x}-\frac{x^2}{3}\right)^{12}$ [3]

b) Find the value of k if $\int_0^k \sqrt{1+x}.dx = \frac{14}{3}$ [3]

c) Given that $x = 2$ is an approximation to $x^3 - 9 = 0$, use one application of Newton's Method to find a better approximation. [3]

d) Use induction to show that: $1 \times 3 \times 5 \times \ldots\ldots \times (2n-1) = \frac{(2n)!}{2^n.n!}$ [3]

Question 4:

a) Show that $7^n + 2$ is divisible by three for all positive integral n. [3]

b) Consider the polynomial $P(x) = x^3 - (k+1)x^2 + kx + 12$

i) Find the remainder when $P(x)$ is divided by $A(x) = x - 3$ [2]

ii) Find k if $P(x)$ is divisible by $A(x)$ [2]

iii) For this value of k, find the zeros of $P(x)$ [2]

c) Find $\int \frac{x.dx}{(25+x^2)^{\frac{3}{2}}}$ using the substitution $u = 25 + x^2$ [3]

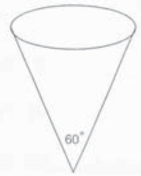
Question 5:

a) A projectile is fired at an angle of $\tan^{-1}\left(\frac{5}{12}\right)$ to the horizontal with an initial velocity of $130ms^{-1}$. Using $g = 10ms^{-2}$,

i) Derive the equations for the horizontal and vertical position of the projectile at time t. [2]

ii) Determine the horizontal range of the projectile. [3]

b) A right circular cone with vertical angle 60° is being filled with a liquid. The liquid is being poured into the cone at a rate of $3\pi cm^3.s^{-1}$.



i) Show that the volume of the cone is given by $V = \frac{\pi h^3}{9}$, where h is the height of the liquid [1]

ii) Find the rate the height of the liquid is increasing when that height is 9 cm. [4]

c) Find a general solution to $\cos 2x = \sin x$ [2]

Question 6:

a) A particle is moving in a straight line with an initial velocity of $48ms^{-1}$. At any time t, the acceleration in ms^{-2} is given by $\ddot{x} = -6t$. Find

i) The distance from its initial position when it stops

[2]

ii) The velocity as which it is travelling when it returns to its initial position.

[2]

b) A particle moving on a horizontal line has a velocity $v(ms^{-1})$, given by $v^2 = 64 + 24x - 4x^2$

i) Prove the particle moves in simple harmonic motion *[1]*

ii) Find the centre of motion *[1]*

iii) Write down the period and amplitude *[2]*

iv) If initially, the particle is at the centre of motion and is moving to the left, write down an expression for the displacement as a function of time.

[1]

c) i) Write the expression $\sqrt{3}\cos\theta + \sin\theta$ in the form $R\cos(\theta - \alpha)$

[1]

ii) Hence or otherwise solve $\sqrt{3}\cos\theta + \sin\theta = 1$, for $0^\circ \le \theta \le 360^\circ$.

[2]

Question 7:

a) The maximum charge of a battery that requires recharging is 12 volts. After 1 minute on the recharger, the charge is measured as 6 volts and after a further minute the charge has risen to 8 volts. Assuming the rate of increase of the charge is given by $\frac{dQ}{dt} = k(12 - Q)$,

i) Show this equivalent to $Q = 12 - Ae^{-kt}$

[2]

ii) Find the value of the charge on the battery at the time it is placed on the recharger.

[2]

iii) Find the exact value of the battery after it has been on the recharger for 3 minutes.

[3]

b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. If PQ passes through the point $(4a, 0)$

i) Show that $pq = 2(p+q)$

[2]

ii) Find the locus of M, the midpoint of PQ.

[3]

Question 1:

a) Solve for x if $\frac{3x}{x-1} < 2$ *[3]*

Solution:

$\frac{3x}{x-1} \cdot \frac{(x-1)^2}{1} < 2 \cdot \frac{(x-1)^2}{1}, \quad x \neq 1$

$\therefore 3x(x-1) < 2(x-1)^2$

or $2(x-1)^2 > 3x(x-1)$

$\therefore 2(x-1)^2 - 3x(x-1) > 0$

$\therefore (x-1)(2x-2-3x) > 0$

$\therefore (x-1)(-x-2) > 0$

or $(x-1)(x+2) < 0$

$\therefore -2 < x < 1$

b) Find the exact value of $\int_0^1 xe^{-x^2}.dx$ *[2]*

Solution:

$$\int_0^1 xe^{-x^2}.dx = -\frac{1}{2}\int_0^1 -2xe^{-x^2}.dx$$
$$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$$
$$= \frac{1}{2}\left[e^{-x^2}\right]_1^0$$
$$= \frac{1}{2}\left[1-e^{-1}\right] = \frac{1}{2}\left[1-\frac{1}{e}\right]$$

c) Find $\frac{d}{dx}\left(\sin^{-1}\sqrt{x}\right)$ *[2]*

Solution:

$$\frac{d}{dx}\left(\sin^{-1}\sqrt{x}\right) = \frac{1}{2\sqrt{x}}\frac{1}{\sqrt{1-x}}$$
$$= \frac{1}{2\sqrt{x(1-x)}}$$

d) If A is the point (3,-2) and B is (-3,1) find the point P that divides AB externally in the ratio 2:1. *[2]*

Solution:

$$P(x,y) = \left(\frac{k_1x_2+k_2x_1}{k_1+k_2}, \frac{k_1y_2+k_2y_1}{k_1+k_2}\right)$$
$$= \left(\frac{2(-3)-1(3)}{2-1}, \frac{2(1)-1(-2)}{2-1}\right)$$
$$= \left(\frac{-6-3}{1}, \frac{2+2}{1}\right)$$
$$= (-9, 4)$$

e) If the limiting sum of a geometric progression is 5 and the first term is 3, find the common ratio. *[3]*

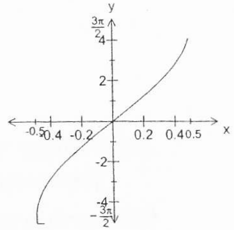
Solution:

$$S_\infty = \frac{a}{1-r}$$
$$5 = \frac{3}{1-r}$$
$$5-5r = 3$$
$$2 = 5r$$
$$\therefore r = \frac{2}{5}$$

Question 2:

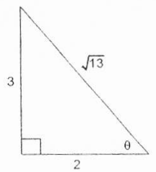
a) Sketch the curve $y = 3\sin^{-1}2x$ *[2]*

Solution:



b) If $2\tan\theta = 3$ and θ is acute, find the exact value of $\dfrac{3\sin\theta + 2\cos\theta}{\sec\theta - \operatorname{cosec}\theta}$ *[3]*

Solution:



$$\frac{3\sin\theta+2\cos\theta}{\sec\theta-\operatorname{cosec}\theta} = \frac{3\left(\frac{3}{\sqrt{13}}\right)+2\left(\frac{2}{\sqrt{13}}\right)}{\frac{\sqrt{13}}{2}-\frac{\sqrt{13}}{3}}$$

$$= \frac{\frac{9}{\sqrt{13}}+\frac{4}{\sqrt{13}}}{\frac{3\sqrt{13}}{6}-\frac{2\sqrt{13}}{6}}$$

$$= \frac{13}{\sqrt{13}} \div \frac{\sqrt{13}}{6} = 6$$

c) The roots of $x^3 - 3x^2 + 6x - 5 = 0$ are α, β and γ. Find the value of

i) $(\alpha+1)+(\beta+1)+(\gamma+1)$ *[2]*

Solution: $\alpha+\beta+\gamma = -\dfrac{b}{a} = 3$

$\therefore (\alpha+1)+(\beta+1)+(\gamma+1) = 3+3 = 6$

ii) $(\alpha+1)(\beta+1)(\gamma+1)$ *[2]*

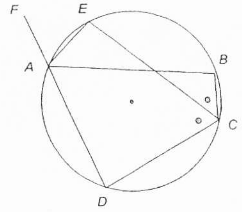
Solution:

$\alpha\beta+\beta\gamma+\gamma\alpha = \dfrac{c}{a} = 6$

$\alpha\beta\gamma = -\dfrac{d}{a} = 5$

$$(\alpha+1)(\beta+1)(\gamma+1) = \alpha\beta\lambda + \alpha\beta + \beta\gamma + \gamma\alpha + \alpha + \beta + \gamma + 1$$
$$= 5+6+3+1 = 15$$

e) In the diagram, DA is extended to F and EC bisects $\angle BCD$. Prove that AE bisects $\angle FAB$ *[3]*



Solution:

$\angle ECD = \angle EAF$ (exterior angle of cyclic quad)

$\angle ECB = \angle EAB$ (angles on the same arc)

Since $\angle ECB = \angle ECD$ (given), $\angle EAB = \angle EAF$

$\therefore$ AE bisects $\angle FAB$

Question 3:

a) Find the constant term in the expansion $\left(\dfrac{2}{x} - \dfrac{x^2}{3}\right)^{12}$ *[3]*

Solution:

$U_{r+1} = {}^nC_r a^{n-r} b^r = {}^{12}C_r \left(\dfrac{2}{x}\right)^{12-r} \left(-\dfrac{x^2}{3}\right)^r$

The x coefficient is $x^{r-12}x^{2r} = x^{3r-12}$, so $x = 4$

$U_5 = {}^{12}C_4 \left(\dfrac{2}{x}\right)^8 \left(-\dfrac{x^2}{3}\right)^4 = \dfrac{495 \text{ x } 256}{81} = \dfrac{14080}{9}$

b) Find the value of k if $\displaystyle\int_0^k \sqrt{1+x}.dx = \frac{14}{3}$ *[3]*

Solution:

$$\left[\frac{2(1+x)^{3/2}}{3}\right]_0^k = \frac{14}{3}$$

$$\therefore \left[(1+x)^{3/2}\right]_0^k = 7$$

$$\therefore (1+k)^{3/2} - 1 = 7$$

$$(1+k)^{3/2} = 8$$

$$1+k = 4 \rightarrow k = 3$$

c) Given that $x = 2$ is an approximation to $x^3 - 9 = 0$, use one application of Newton's Method to find a better approximation. *[3]*

Solution:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f(x) = x^3 - 9 \quad \rightarrow \quad f(2) = 8 - 9 = -1$$

$$f'(x) = 3x^2 \quad \rightarrow \quad f'(2) = 12$$

$$x_1 = 2 - \frac{f(3)}{f'(3)} = 2 - \frac{-1}{12} = 2\frac{1}{12}$$

d) Use induction to show that: $1 \text{ x } 3 \text{ x } 5 \text{ x } \ldots\ldots\text{x } (2n-1) = \dfrac{(2n)!}{2^n.n!}$ *[3]*

Solution:

at $n = 1$, $LHS = 1$

$RHS = \dfrac{2!}{2^1} = 1 = LHS \quad \rightarrow \quad \therefore$ true for $n = 1$

assume true for $n = k \quad \rightarrow \quad 1 \text{ x } 3 \text{ x } 5 \text{ x } \ldots\ldots\text{x } (2k-1) = \dfrac{(2k)!}{2^k.k!}$

prove for $n = k+1$, *i.e.* prove $1 \text{ x } 3 \text{ x } 5 \text{ x } \ldots\ldots\text{x } (2k-1)(2k+1) = \dfrac{(2k+2)!}{2^{k+1}.(k+1)!}$

$LHS = 1 \text{ x } 3 \text{ x } 5 \text{ x } \ldots\ldots\text{x } (2k-1)(2k+1)$

$$= \frac{(2k)!}{2^k.k!}(2k+1) \qquad \text{[using assumption]}$$

$$= \frac{(2k)!}{2^k.k!}\frac{(2k+1)}{1}\left(\frac{2k+2}{2k+2}\right)$$

$$= \frac{(2k)!}{2^k.k!}\frac{(2k+1)(2k+2)}{2(k+1)}$$

$$= \frac{(2k+2)!}{2^{k+1}.(k+1)!}$$

$= RHS$

$\therefore$ if true for $n = k$, it is true for $n = k+1$.

Since true for $n = 1$, it is true for $n = 2$, then $n = 3$ and so on for all positive integral values of n.

Question 4:

a) Show that $7^n + 2$ is divisible by three for all positive integral n. *[3]*

Solution:

at $n = 1$, $7^1 + 2 = 9 = 3 \text{ x } 3$, so is true for $n = 1$

assume true for $n = k \quad \rightarrow \quad 7^k + 2 = 3M$ (M an integer)

prove for $n = k+1$

$7^{k+1} + 2 = 7 \text{ x } 7^k + 2$

$= 7 \text{ x } (3M - 2) + 2$

$= 21M - 14 + 2$

$= 21M - 12$

$= 3(7M - 4)$

$\therefore$ if true for $n = k$, it is true for $n = k+1$.

Since true for $n = 1$, it is true for $n = 2$, then $n = 3$ and so on for all positive integral values of n.

b) Consider the polynomial $P(x) = x^3 - (k+1)x^2 + kx + 12$

i) Find the remainder when $P(x)$ is divided by $A(x) = x - 3$ *[2]*

Solution:

$P(3) = 27 - 9k - 9 + 3k + 12$

$= 30 - 6k$

ii) Find k if $P(x)$ is divisible by $A(x)$ *[2]*

Solution:

If $P(3) = 0, 30 - 6k = 0$

and $k = 5$

iii) For this value of k, find the zeros of $P(x)$ *[2]*

Solution:

$P(x) = x^3 - 6x^2 + 5x + 12$

$= (x-3)(ax^2 + bx + c)$ as $(x-3)$ is a factor

$= (x-3)(x^2 - 3x - 4)$

$= (x-3)(x-4)(x+1)$

$\therefore x = -1, 3, 4$

c) Find $\int \frac{x.dx}{\left(25+x^2\right)^{3/2}}$ using the substitution $u = 25 + x^2$

[3]

Solution:

$$\text{let } u = 25 + x^2,$$
$$\therefore du = 2x.dx$$
$$I = \frac{1}{2}\int \frac{du}{u^{3/2}} = \frac{1}{2}\int u^{-3/2}.du$$
$$= \frac{1}{2}.2u^{-1/2} + c = \frac{1}{\sqrt{u}} + c$$
$$= \frac{1}{\sqrt{25+x^2}} + c$$

Question 5:

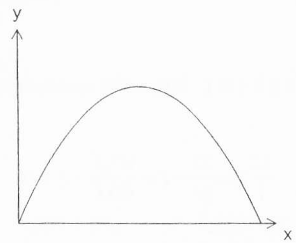
a) A projectile is fired at an angle of $\tan^{-1}\left(\frac{5}{12}\right)$ to the horizontal with an initial velocity of $130ms^{-1}$. Using $g = 10ms^{-2}$,

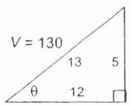
i) Derive the equations for the horizontal and vertical position of the projectile at time t.

[2]

ii) Determine the horizontal range of the projectile.

[3]





Solution:

$\ddot{x} = 0, \quad \ddot{y} = -10$

$\dot{x} = c_1, \quad \dot{y} = -10t + c_2$

at $t = 0, \dot{x} = V\cos\theta, \quad \dot{y} = V\sin\theta$

$$= 130\left(\frac{12}{13}\right), \quad = 130\left(\frac{5}{13}\right)$$
$$= 120 = c_1, \quad = 50 = c_2$$
$$\therefore \dot{x} = 120, \quad \dot{y} = -10t + 50$$
$$\therefore x = 120t + c_3, \quad y = -5t^2 + 50t + c_4$$

at $t = 0, x = 0 = c_3, y = 0 = c_4$

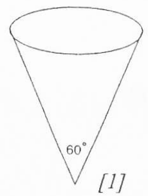
$$\therefore x = 120t, \quad y = -5t^2 + 50t$$

ii) at $y = 0, \ t = 0, 10$

range at $t = 10$

sub into $x = 120t$ and range is 600 metres

b) A right circular cone with vertical angle 60^o is being filled with a liquid. The liquid is being poured into the cone at a rate of $3\pi cm^3.s^{-1}$.



i) Show that the volume of the cone is given by $V = \frac{\pi h^3}{9}$, where h is the height of the liquid

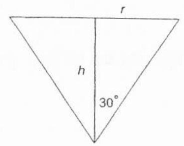
[1]

ii) Find the rate the height of the liquid is increasing when that height is 9 cm.

[4]

Solution:

i) $V = \frac{1}{3}\pi r^2 h \qquad \tan 30^o = \frac{r}{h} \rightarrow \frac{1}{\sqrt{3}} = \frac{r}{h}$ and $r = \frac{h}{\sqrt{3}}$



$$\therefore V = \frac{\pi}{3}\left(\frac{h}{\sqrt{3}}\right)^2 (h) = \frac{\pi h^3}{9}$$

ii) $\frac{dV}{dt} = \frac{dV}{dh}.\frac{dh}{dt}$ and $\frac{dV}{dh} = \frac{\pi h^2}{3}$

$$\therefore \frac{dV}{dt} = \frac{\pi h^2}{3}.\frac{dh}{dt}$$

$$\therefore 3\pi = \frac{\pi(81)}{3}.\frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{1}{9}cm.\sec^{-1}$$

c) Find a general solution to $\cos 2x = \sin x$

[2]

Solution:

$\therefore 1 - 2\sin^2 x = \sin x$

$\therefore 2\sin^2 x + \sin x - 1 = 0$

$\therefore (2\sin x - 1)(\sin x + 1) = 0$

$\therefore \sin x = \frac{1}{2}$ or $\sin x = -1$

$\therefore x = n\pi + (-1)^n\left(\frac{\pi}{6}\right)$ or $\frac{\pi}{2}(4n+3)$

Note: This may also be answered in degrees .

Question 6:

a) A particle is moving in a straight line with an initial velocity of $48ms^{-1}$. At any time t, the acceleration in ms^{-2} is given by $\ddot{x} = -6t$. Find

i) The distance from its initial position when it stops

[2]

ii) The velocity as which it is travelling when it returns to its initial position.

[2]

Solution:

$\ddot{x} = -6t$

$\dot{x} = -3t^2 + c$

at $t = 0, \dot{x} = 48 = c$

$\therefore \dot{x} = -3t^2 + 48$

$x = -t^3 + 48t + c_2$

at $t = 0, x = 0 = c_2$

$\therefore x = -t^3 + 48t$

at $\dot{x} = 0, 3(-t^2 + 16) = 0 \rightarrow t = 4$

at $t = 4, x = -4^3 + 48 \text{ x } 4 = 128$ Distance is $128m$

at $x = 0, -t^3 + 48t = 0 \rightarrow t = \sqrt{48} = 4\sqrt{3}$

at $t = 4\sqrt{3}, v = -3\left(4\sqrt{3}\right)^2 + 48 = -96$ Velocity is $-96ms^{-1}$

b) A particle moving on a horizontal line has a velocity $v(ms^{-1})$, given by $v^2 = 64 + 24x - 4x^2$

i) Prove the particle moves in simple harmonic motion

[1]

ii) Find the centre of motion

[1]

iii) Write down the period and amplitude

[2]

iv) If initially, the particle is at the centre of motion and is moving to the left, write down an expression for the displacement as a function of time.

[1]

Solution:

i) $\frac{1}{2}v^2 = 32 + 12x - 2x^2$

$\ddot{x} = \frac{d}{dx}\left(\frac{v^2}{2}\right) = 12 - 4x = -4(x-3)$ which is the form of SHM

ii) At $\ddot{x} = 0, x = 3$

iii) $n^2 = 4 \rightarrow n = 2, \therefore T = \frac{2\pi}{2} = \pi$

At $v = 0, 4x^2 - 24x - 64 = 0 \rightarrow 4(x-8)(x+2) = 0$

Particle moves from -2 to 8, so amplitude is 5

iv) $x = 3 - 5\sin 2t$

c) i) Write the expression $\sqrt{3}\cos\theta + \sin\theta$ in the form $R\cos(\theta - \alpha)$

[1]

ii) Hence or otherwise solve $\sqrt{3}\cos\theta + \sin\theta = 1$, for $0^\circ \le \theta \le 360^\circ$.

[2]

Solution:

i) $R\cos(\theta - \alpha) = R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$\sqrt{3}\cos\theta + \sin\theta = 2\left(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right), \quad \therefore R = 2, \ \theta = \frac{\pi}{6}$

$\therefore R\cos(\theta - \alpha) = 2\cos\left(\theta - \frac{\pi}{6}\right)$

ii) $2\cos\left(\theta - \frac{\pi}{6}\right) = 1 \rightarrow \cos\left(\theta - \frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore \theta - \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3} \quad \rightarrow \quad \theta = \frac{\pi}{2}, \frac{11\pi}{6}$

Question 7:

a) The maximum charge of a battery that requires recharging is 12 volts. After 1 minute on the recharger, the charge is measured as 6 volts and after a further minute the charge has risen to 8 volts. Assuming the rate of increase of the charge is given by $\frac{dQ}{dt} = k(12 - Q)$,

i) Show this equivalent to $Q = 12 - Ae^{-kt}$

[2]

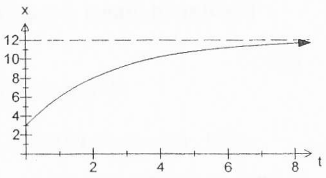
ii) Find the value of the charge on the battery at the time it is placed on the recharger.

[2]

iii) Find the exact value of the battery after it has been on the recharger for 3 minutes.

[3]

Solution:



i) $Q = 12 - Ae^{-kt}$

$\frac{dQ}{dt} = k\left(Ae^{-kt}\right)$

and since $Ae^{-kt} = 12 - Q$, $\frac{dQ}{dt} = k(12 - Q)$

ii) $Q = 12 - Ae^{-kt}$

at $t = 1, Q = 6 \rightarrow \therefore 6 = 12 - Ae^{-k}$ and $Ae^{-k} = 6$ or $A = 6e^{k}$

at $t = 2, Q = 8 \rightarrow \therefore 8 = 12 - Ae^{-2k}$ and $Ae^{-2k} = 4$ or $A = 4e^{2k}$

Equating for A, $6e^{k} = 4e^{2k} \rightarrow 3/2 = 3^{k}$

Since $A = 6e^{k}$, $A = 6(3/2) = 9$ and $Q = 12 - 9e^{-kt}$

At $t = 0, Q = 12 - 9e^{0} = 3$

iii) At $t = 3, Q = 12 - 9e^{-3k}$

$= 12 - 9\left(e^{-k}\right)^3$

Since $e^{k} = 3/2, e^{-k} = 2/3$

and $Q = 12 - 9(2/3)^3 = 9\,1/3$

b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. If PQ passes through the point $(4a, 0)$

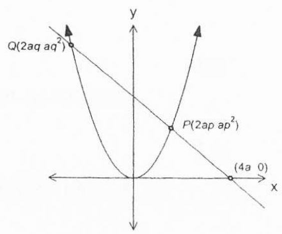
i) Show that $pq = 2(p+q)$

[2]

ii) Find the locus of M, the midpoint of PQ.

[3]

Solution:



i) $m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq} = \frac{a(p-q)(p+q)}{2a(p-q)} = \frac{p+q}{2}$

$\therefore$ Eqn PQ is $y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$y - ap^2 = \left(\frac{p+q}{2}\right)x - ap^2 - apq \rightarrow y = \left(\frac{p+q}{2}\right)x - apq$

at $(4a, 0)$, $0 = \left(\frac{p+q}{2}\right)4a - apq \rightarrow pq = 2(p+q)$

ii) Midpoint PQ is $a(p+q), \ \frac{a}{2}(p^2 + q^2)$

Let $x = a(p+q), \ y = \frac{a}{2}(p^2 + q^2)$

$\frac{x}{a} = p + q \rightarrow \frac{2x}{a} = 2(p+q), \ \therefore \frac{2x}{a} = pq$

Write y as $y = \frac{a}{2}\left\{(p+q)^2 - 2pq\right\}$

$\therefore y = \frac{a}{2}\left\{\left(\frac{x}{a}\right)^2 - 2\left(\frac{2x}{a}\right)\right\}$

$= \frac{a}{2}\left\{\frac{x^2}{a^2} - \frac{4x}{a}\right\}$

$= \frac{x^2}{2a} - 2x$

$\therefore 2ay = x^2 - 4ax$

or $2ay + 4a^2 = x^2 - 4ax + 4a^2 \rightarrow (x - 2a)^2 = 2a(y + 2a)$