Sydney Girls High School

2009

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

Extension 1

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2009 HSC Examination Paper in this subject.

Candidate Number

General Instructions

- Reading Time - 5 minutes
- Working time - 2 hours
- Attempt ALL questions
- ALL questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

Marks

Question 1 (12 marks)

(a) The remainder when the polynomial x^4 is divided by $x + a$ is 16. Find the value of a — 2

(b) Differentiate $\sin^{-1}\left(\frac{x}{2}\right)$ with respect to x — 1

(c) Evaluate $\int_0^3 \frac{1}{x^2+9}dx$ — 3

(d) The interval AB, where A is (2,1) and B is (3,2) is divided internally in the ratio 4:3 by the point P(x,y). Find the values of x and y. — 2

(e) Evaluate $\int_0^1 xe^{x^2}dx$ Leave your answer in exact form. — 2

(f) Let $f(x) = \sqrt{x^2 - 5x + 6}$ What is the domain of $f(x)$? — 2

Marks

Question 2 (12 marks)

(a) Use the substitution $u = 1 - \sin x$ to evaluate $\int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \cos x\sqrt{1-\sin x}\ dx$ 3

(b) A particle moves on the x-axis with velocity v.
Velocity is given by $v = x-3$
Using the fact that $a = \frac{d}{dx}(\frac{1}{2}v^2)$, find the acceleration of the particle at $x=4$ 3

(c) The polynomial $P(x) = x^2 + ax + b$ has a zero at $x = 2$. When $P(x)$ is divided by $x - 1$, the remainder is 2.
Find the value of a and b 3

(d) The function $f(x) = \sin x + \log x$ has a zero near $x = 0.5$
Use one application of Newton's method to obtain another approximation to this zero. Give your answer correct to three decimal places. 3

Marks

Question 3 (12 marks)

(a)
(i) Sketch the graph of $y = x(x-2)(x-3)$ showing the x and y intercepts. Do not use calculus. 2

(ii) Hence, or otherwise, solve $x^3 + 6x \geq 5x^2$ 2

(b) Use mathematical induction to prove that, for integers $n \geq 1$ 3

$$1\times 2 + 2\times 3 + 3\times 4 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

(c) The radius of a spherical balloon is expanding at a constant rate of $6cms^{-1}$

$(V = \frac{4}{3}\pi r^3, S = 4\pi r^2)$

(i) At what rate is the volume of the balloon expanding when its radius is 4cm? 2

(ii) At what rate is the surface area of the balloon expanding if the rate of change of volume is $15cm^3s^{-1}$ 3

Question 4 (12 marks) Marks

(a) A freshly baked chocolate cake is cooling in a room of constant temperature of 20^0C. At time t minutes its temperature T decreases according to the equation $\frac{dT}{dt} = -k(T-20)$ where k is a positive constant. The initial temperature of the cake is 150^0C and it cools to 100^0C after 15 minutes.

(i) Verify that $T = 20 + Ae^{-kt}$ is a solution of this equation, where A is a constant 1

(ii) Find the values of A and k, giving k correct to three decimal places. 2

(iii) How long will it take for the temperature of the cake to cool to 25^0C? Give your answer to the nearest minute. 2

(b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$

(i) Show that the gradient of PQ is $\frac{p+q}{2}$ 1

(ii) Show that if PQ passes through the focus then $pq = -1$ 2

(iii) Find the equation of the locus of the midpoint of PQ if PQ is a focal chord. 2

(c) Solve $\frac{5}{x-1} > 2$ 2

Question 5 (12 marks) Marks

(a) Let $f(x) = x^2 - 2x$ for $x \geq 1$. This function has an inverse, $f^{-1}(x)$ 3

(i) Sketch the graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same set of axes. (Use the same scale on both axes)

(ii) Find an expression for $f^{-1}(x)$ 2

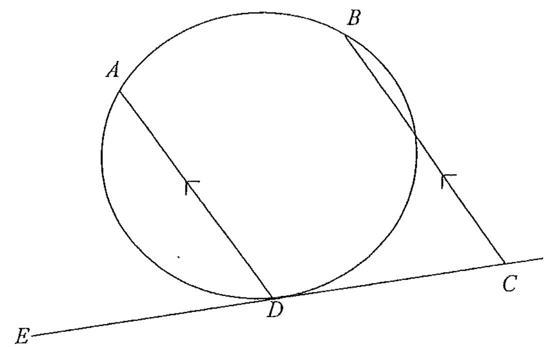
(iii) Evaluate $f^{-1}(2)$ 1

(b) A particle is moving in simple harmonic motion in a straight line. 3

Its amplitude is 3m and its period is $\frac{\pi}{2}$ seconds.

Find the maximum speed and maximum acceleration of the motion.

(c)

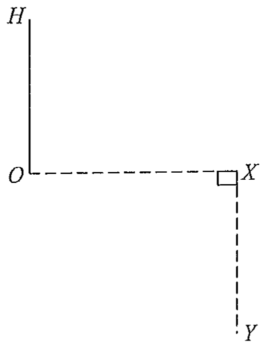


In the diagram above, $AD \parallel BC$ and the line EC is a tangent to the circle at D. 3
Copy or trace the diagram
Prove that $BD^2 = AD.BC$

Marks

Question 6 (12 marks)

(a) From a point X due east of a tower, the angle of elevation of the top of the tower H is 31^0. From another point Y due south of X, the angle of elevation of H is 25^0. The distance XY is 150m.



(i) Copy or trace the diagram, adding the given information to your diagram 1

(ii) Hence find the height of the tower 3

(b) Find all values of θ in the range $0 \le \theta \le 2\pi$ for which $\sqrt{3}\cos\theta - \sin\theta = 1$ 4

(c)

(i) Find the equation of the normal to the curve $y = \dfrac{8}{x^2+4}$ at the point $(2,1)$ 2

(ii) Show that the normal in (i) does not touch the curve again. 2

Marks

Question 7 (12 marks)

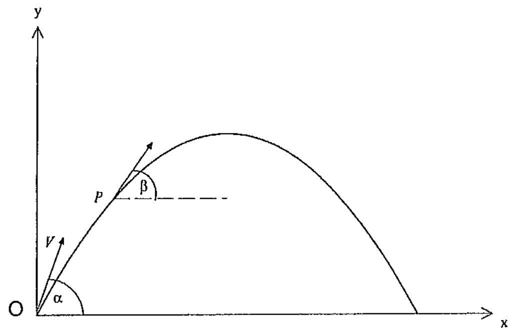
(a) $y = f(x)$ is an odd function 3

$f(x) \ge 0$ if $x \ge 0$

$\int_0^a f(x)dx = a^2 \times 2^a$ if $a > 0$

Find the area bounded by $y = f(x+3)$, $x = -4$, $x = 4$ and the x axis

(b) A particle is projected from a point O on horizontal ground, with speed $V ms^{-1}$ at an angle of elevation to the horizontal of α



Its equations of motion are $\ddot{x} = 0$, $\ddot{y} = -g$

(i) Show that $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$ 2

(ii) Show that the time of flight of the particle is $\dfrac{2V\sin\alpha}{g}$ 2

(iii) The particle reaches a point P, as shown, where the direction of the flight makes an angle β with the horizontal. 2

Show that the time taken to travel from O to P is $\dfrac{V\sin(\alpha-\beta)}{g\cos\beta}$ seconds

(iv) Consider the case where $\beta = \frac{\alpha}{2}$ If the time taken to travel from O to P is then one-third of the total time of flight, find the value of α 3

- End of Exam -

Ext (1)

Question 1.

a) $f(x) = x^4$

divided by $x+a$

$R = 16$

$f(-a) = (-a)^4$

$= a^4$

$a^4 = 16$

$\therefore a = \pm 2$ (2)

b) $\frac{d}{dx}\left(\sin^{-1}\frac{x}{2}\right) = \frac{1}{\sqrt{4-x^2}}$ (1)

$\left(\frac{1}{2\sqrt{1-\frac{1}{4}x^2}}\right)$

c) $\int_0^3 \frac{1}{x^2+9}\,dx = \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$

$= \frac{1}{3}\tan^{-1}1 - \frac{1}{3}\tan^{-1}0$

$= \frac{1}{3}\cdot\frac{\pi}{4} - 0$

$= \frac{\pi}{12}$ (3)

d) $x = \frac{nx_1 + mx_2}{m+n}$ $\quad y = \frac{ny_1 + my_2}{m+n}$ $\quad \overset{m}{4} : \overset{n}{3}$

$x = \left(\frac{3(2) + 4(3)}{7}\right)$ $\quad y = \left(\frac{3(1) + 4(2)}{7}\right)$

$x = \left(\frac{6+12}{7}\right)$ $\quad y = \left(\frac{3+8}{7}\right)$

$x = \frac{18}{7}$ $\quad y = \frac{11}{7}$

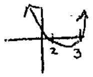
$P\left(\frac{18}{7}, \frac{11}{7}\right)$ (2)

e) $\int_0^1 xe^{x^2}\,dx = \frac{1}{2}e^{x^2}\Big]_0^1$

$= \frac{1}{2}e - \frac{1}{2}e^0$ (2)

$= \frac{1}{2}e - \frac{1}{2} = \frac{1}{2}(e-1)$

f) Domain:



$x^2 - 5x + 6 \geq 0$

$(x-3)(x-2) \geq 0$

$x \leq 2$ and

$x \geq 3$ (2)

3U Q2

(a) $u = 1 - \sin x$

$\frac{du}{dx} = -\cos x$

$-du = \cos x\,dx$

when $x = \frac{\pi}{6}$

$u = 1 - \sin\frac{\pi}{6}$

$= 1 - \frac{1}{2} = \frac{1}{2}$

when $x = \frac{\pi}{2}$

$u = 1 - \sin\frac{\pi}{2}$

$= 1 - 1$

$= 0$

$\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \cos x\sqrt{1-\sin x}\,dx$

$= \int_0^{\frac{1}{2}} -\sqrt{u}\,.du$

$= -\left[\frac{2u^{\frac{3}{2}}}{3}\right]_0^{\frac{1}{2}}$

$= -\left(\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} - 0\right)$

$\left(\sqrt{\frac{1}{2}}\right)^3 = \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{2\sqrt{2}}$

$= -\frac{2}{3}\left(\frac{1}{2\sqrt{2}}\right)$

$= -\frac{1}{3\sqrt{2}}$ /3

(b) $V = x - 3$

$V^2 = (x-3)^2$

$\frac{1}{2}V^2 = \frac{1}{2}(x^2 - 6x + 9)$

$= \frac{1}{2}x^2 - 3x + \frac{9}{2}$

$a = \frac{d}{dx}\left(\frac{1}{2}V^2\right)$

$= x - 3$

when $x = 4$

$a = 4 - 3$

$= 1$ /3

(c) $P(x) = x^2 + ax + b$

Factor $(x-2)$

$P(2) = 4 + 2a + b$

$0 = 4 + 2a + b$

$\therefore b = -2a - 4$ .....(1)

$P(1) = 1 + a + b$

$2 = 1 + a + b$

$b = -a + 1$ ....(2)

(1) = (2):

$-2a - 4 = -a + 1$

$-5 = a$

Sub a in (2):

$b = -(-5) + 1$

$= 6$

$\therefore a = -5, \quad b = 6$ /3

(d) $P(x) = \sin x + \log x$

$P(0.5) = \sin 0.5 + \log 0.5$

$P'(x) = \cos x + \frac{1}{x}$

$P'(0.5) = \cos 0.5 + \frac{1}{0.5}$

$= \cos 0.5 + 2$

$x = a - \frac{P(a)}{P'(a)}$

$= 0.5 - \frac{\sin 0.5 + \log 0.5}{\cos 0.5 + 2}$

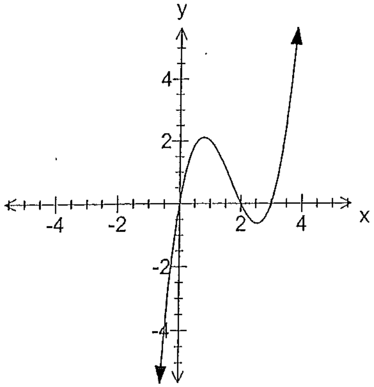
$\doteqdot 0.574$ /3

Extension 1 – Trial 2009

Question 3 Solutions

a.

i.



ii.

From part i. $0 \le x \le 2$, or $x \ge 3$

b.

Step 1 – Prove true for n=1

$$LHS = 1\times 2$$
$$= 2$$
$$RHS = \frac{1(1+1)(1+2)}{3}$$
$$= \frac{6}{3}$$
$$= 2$$
$$= LHS$$
$$\therefore \text{ true for } n = 1$$

Step 2 – Assume true for n=k

$$1\times 2 + 2\times 3 + ... + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

Step 3 – Prove true for n=k+1

Required to prove that:

$$1\times 2 + 2\times 3 + ... + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

$$LHS = 1\times 2 + 2\times 3 + ... + k(k+1) + (k+1)(k+2)$$
$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$
$$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}$$
$$= \frac{(k+3)(k+1)(k+2)}{3}$$
$$= RHS$$
$$\therefore \text{ true for } n = k+1$$

Therefore the statement is true for all positive integral values of n by induction.

c.

$$\frac{dr}{dt} = 6$$
$$\frac{dV}{dr} = 4\pi r^2$$
$$\frac{dS}{dr} = 8\pi r$$

i.

$$\frac{dV}{dt} = \frac{dr}{dt}\times\frac{dV}{dr}$$
$$= 6\times 4\pi r^2$$
$$= 24\pi r^2$$

when $r = 4$

$$\frac{dV}{dt} = 24\times\pi\times 4^2$$
$$= 384\pi cm^3 s^{-1}$$

ii.

$$\frac{dV}{dt} = 15$$
$$15 = 24\pi r^2$$
$$r^2 = \frac{15}{24\pi}$$
$$r = \sqrt{\frac{15}{24\pi}}$$
$$\frac{dS}{dt} = \frac{dS}{dr}\times\frac{dr}{dt}$$
$$= 8\pi r\times 6$$
$$= 48\pi r$$
$$= 48\pi\times\sqrt{\frac{15}{24\pi}}$$
$$= 67.26 cm^2 s^{-1}$$

Question 4

(a). (i). $T = 20 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$
$$= -k(20 + Ae^{-kt} - 20)$$
$$= -k(T - 20)$$

(ii).

$$T = 20 + Ae^{-kt}$$
$$150 = 20 + A$$
$$\therefore A = 130$$
$$T = 20 + 130e^{-kt}$$
$$100 = 20 + 130e^{-15k}$$
$$e^{-15k} = \frac{8}{13}$$
$$k = -\frac{1}{15}\ln\left(\frac{8}{13}\right)$$
$$\therefore k \approx 0.032$$

(iii).

$T = 25$ when $t = ?$

$$T = 20 + 130e^{-0.032t}$$
$$25 = 20 + 130e^{-0.032t}$$
$$e^{-0.032t} = \frac{5}{130}$$
$$t = \ln\left(\frac{5}{130}\right)\div(-0.032)$$

$\therefore t \approx 101.8$ (i.e. 102 minutes)

(b). (i). $m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$

$$= \frac{a(p+q)(p-q)}{2a(p-q)}$$
$$= \frac{p+q}{2}$$

(ii).

Equation of line PQ:

$$y - ap^2 = \frac{p+q}{2}(x - 2ap)$$

If PQ passes through the focus $(0, a)$ then :

$$a - ap^2 = \frac{p+q}{2}(-2ap)$$
$$2a - 2ap^2 = -2ap^2 - 2apq$$
$$2a = -2apq$$
$$pq = -1$$

(iii).

Midpoint of PQ is :

$$M = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$$
$$= \left(a(p+q), \frac{a}{2}\left(p^2 + q^2\right)\right)$$

$\therefore x = a(p+q)$ i.e. $p + q = \frac{x}{a}$

$$y = \frac{a}{2}\left(p^2 + q^2\right)$$
$$= \frac{a}{2}\left((p+q)^2 - 2pq\right)$$

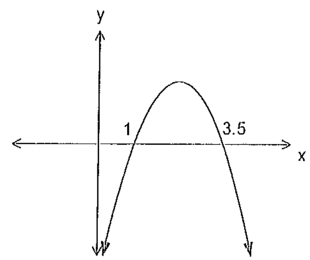
$\therefore y = \frac{a}{2}\left(\frac{x^2}{a^2} + 2\right)$ or $y = \frac{x^2}{2a} + a$

since $pq = -1$ (a focal chord)

(c).

$$\frac{5}{x-1}\times(x-1)^2 > 2(x-1)^2$$
$$5(x-1) - 2(x-1)^2 > 0$$
$$(x-1)(5 - 2(x-1)) > 0$$
$$(x-1)(7 - 2x) > 0$$

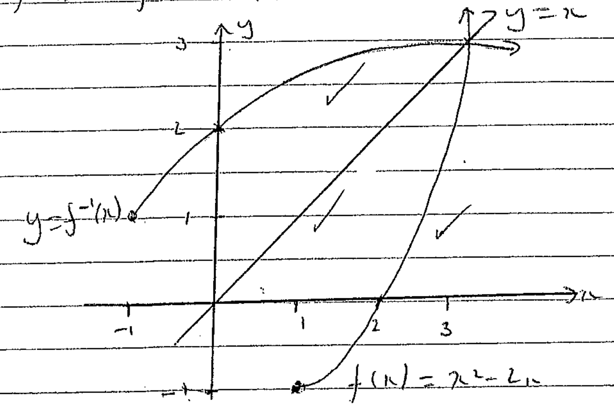
$$1 < x < 3.5$$



Extn 1 2009

Question 5.

a) i. $f(x) = x^2 - 2x$ for $x \geq 1$



−1 if curves do not meet on $y = x$

−1 if restriction not applic.

(3)

ii) $x = y^2 - 2y$ ✓

$= y^2 - 2y + 1 - 1$

$= (y-1)^2 - 1$

$(y-1)^2 = x + 1$

$y = 1 + \sqrt{x+1}$ ✓

(2)

iii) $f^{-1}(2) = 1 + \sqrt{3}$ ✓ (1)

No ECFA

No carry through from trivial answer in ii)

b) $\frac{\pi}{2} = \frac{2\pi}{n}$, $a = 3$ $v^2 = n^2(a^2 - x^2)$

$n = \frac{2\pi}{1} \times \frac{2}{\pi}$ $v^2 = 16(9 - x^2)$

$= 4$ ✓ max velocity $x = 0$

$v = \sqrt{16(9)}$

$= 12\,ms^{-1}$ 1 ✓

(3)

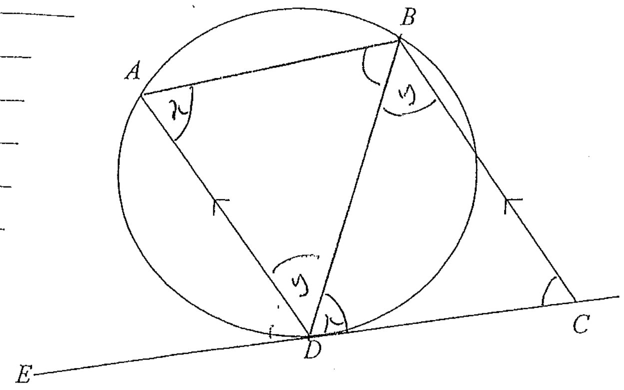
$a = -n^2 x$

$= -16x$

max acceleration when $x = -3$

$a = -16(-3)$

$= 48\,ms^{-2}$ 1 ✓



In Δ's DBC, ABD

∠BDC = ∠BAD x (∠ in alt segment) ✓

∠CBD = ∠BDA y (alt ∠'s AD || BC) ✓

∴ Δ ABC ||| Δ DCB (equiangular)

$\frac{BD}{AD} = \frac{BC}{BD}$ (corres sides sim Δ's) ✓

$BD^2 = AD \cdot BC$

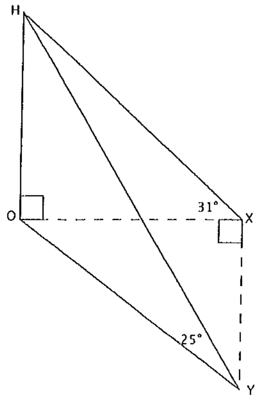
(3)

1/ 2/ 4/ 5/ 6/ 10/ 11/ 9/ 7/ 8/ 13/

Check ... ECFA

Question 6:

a)



In ΔOHX:

$$\tan 31^0 = \frac{h}{OX}$$

$$OX = \frac{h}{\tan 31^0}$$

In ΔOHY:

$$\tan 25^0 = \frac{h}{OY}$$

$$OY = \frac{h}{\tan 25^0}$$

In ΔOXY:

$$OY^2 = OX^2 + XY^2$$

$$\frac{h^2}{\tan^2 25^0} = \frac{h^2}{\tan^2 31^0} + 150^2$$

$$\frac{h^2}{\tan^2 25^0} - \frac{h^2}{\tan^2 31^0} = 150^2$$

$$\frac{h^2\tan^2 31 - h^2\tan^2 25}{\tan^2 25\tan^2 31} = 150^2$$

$$h^2\left(\tan^2 31 - \tan^2 25\right) = 150^2\tan^2 25\tan^2 31$$

$$h^2 = \frac{150^2\tan^2 25\tan^2 31}{\tan^2 31 - \tan^2 25}$$

$$h = 110.91096\ldots$$

$$h \cong 110.9\text{ m } \left(\text{to 1 dec. pl}\right)$$

b) $\sqrt{3}\cos\theta - \sin\theta = R\cos\left(\theta+\alpha\right)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$R\cos\alpha = \sqrt{3}$

$R\sin\alpha = 1$

$$R = \sqrt{\left(\sqrt{3}\right)^2 + 1^2}$$

$$= 2$$

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$$2\cos\left(\theta + \frac{\pi}{6}\right) = 1$$

$$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$$

$$\theta + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\theta = \frac{\pi}{6}, \frac{3\pi}{2}$$

c) (i)

$$y = \frac{8}{x^2+4}$$

$$= 8\left(x^2+4\right)^{-1}$$

$$y' = \frac{-16x}{\left(x^2+4\right)^2}$$

At $x = 2$:

$$m_T = \frac{-16\times 2}{\left(2^2+4\right)^2}$$

$$= -\frac{32}{64}$$

$$= -\frac{1}{2}$$

For normal: $m_N = -\dfrac{1}{m_T}$

$= 2$

At $x = 2$:

$$y = \frac{8}{2^2+4}$$

$$= 1$$

Equation of normal at $\left(2,1\right)$:

$$y - y_1 = m\left(x - x_1\right)$$

$$y - 1 = 2\left(x-2\right)$$

$$y = 2x - 3$$

(ii) Points of intersection:

$$2x - 3 = \frac{8}{x^2+4}$$

$$\left(2x-3\right)\left(x^2+4\right) = 8$$

$$2x^3 + 8x - 3x^2 - 20 = 0$$

Let $P\left(x\right) = 2x^3 + 8x - 3x^2 - 20$

$P\left(1\right) = -13$

$P\left(-1\right) = -33$

$P\left(2\right) = 0$

$\therefore \left(x-2\right)$ is a factor of $P\left(x\right)$

Using division of polynomials:

$$P\left(x\right) = \left(x-2\right)\left(2x^2 + x + 10\right)$$

$$\Delta = 1^2 - 4\cdot 2\cdot 10$$

$$= -79$$

As $\Delta < 0$, $2x^2 + x + 10 = 0$ has no real roots. (i.e. no more points of intersection).

So the normal $y = 2x - 3$ does not touch the curve again.

7 (a) $f(x+3)$ is $f(x)$ translated 3 units to the left

$$\int_{-4}^{4} f(x+3)\,dx = \int_{-1}^{7} f(x)\,dx$$ ✓

$$= \int_{-1}^{0} f(x)\,dx + \int_{0}^{7} f(x)\,dx$$

$$A = \int_0^1 f(x)\,dx + \int_0^7 f(x)\,dx$$ ✓

$$= 1^2 \times 2^1 + 7^2 \times 2^7$$

$$= 6274$$ ✓

(b)(i) $\ddot{x} = 0$ $\qquad \ddot{y} = -g$

$\dot{x} = V\cos\alpha$ ✓ $\qquad \dot{y} = V\sin\alpha - gt$ ✓

$x = Vt\cos\alpha$ $\qquad y = Vt\sin\alpha - \frac{1}{2}gt^2$

(ii) Let $y = 0$

$t(V\sin\alpha - \frac{1}{2}gt) = 0$ ✓

$V\sin\alpha = \frac{1}{2}gt$

$\frac{2V\sin\alpha}{g} = t$ ✓

(iii) $\tan\beta = \frac{dy}{dx}$ ✓

$= \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{V\sin\alpha - gt}{V\cos\alpha}$

$V\cos\alpha\tan\beta = V\sin\alpha - gt$

$gt = V\sin\alpha - V\cos\alpha\tan\beta$

$= \frac{V\sin\alpha\cos\beta - V\cos\alpha\sin\beta}{\cos\beta}$

$t = \frac{V\sin(\alpha-\beta)}{g\cos\beta}$ ✓

(iv) $\frac{2V\sin\alpha}{3g} = \frac{V\sin(\alpha - \frac{\alpha}{2})}{g\cos\frac{\alpha}{2}}$ ✓

$\frac{2\sin\alpha}{3g} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$

$\frac{4\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{3} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$

$\frac{4\cos^2\frac{\alpha}{2}}{3} = 1$ since $\sin\alpha \neq 0$

$\cos^2\frac{\alpha}{2} = \frac{3}{4}$ ✓

$\cos\frac{\alpha}{2} = \frac{\sqrt{3}}{2}$

$\frac{\alpha}{2} = 30°$

$\alpha = 60°$ ✓