Sydney Girls High School

2010
TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION

Mathematics

Extension 1

General Instructions

- Reading Time - 5 minutes
- Working time - 2 hours
- Attempt ALL questions
- ALL questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new page. Write on one side of the paper only.

This is a trial paper ONLY. It does not necessarily reflect the format or the content of the 2010 HSC Examination Paper in this subject.

Candidate Number

TABLE OF STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}; \quad n \neq -1$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

QUESTION 1: *(12 marks)*

a) Evaluate $\int_0^2 \frac{1}{x^2+4}\,dx$. **2**

b) Find the acute angle between the lines $2x-y-1=0$ and $3x-y-2=0$. Give your answer to the nearest minute if necessary. **2**

c) A is the point $(-1,1)$ and B is the point $(3,3)$. Find the point C which divides AB externally in the ratio 3 : 1. **2**

d) Evaluate $\int_{-1}^{0} x\sqrt{1+x}\,dx$ using the substitution $u=1+x$. **3**

e) Solve the inequality $\frac{2}{x-1}\le 1$. **3**

QUESTION 2 OVER PAGE

QUESTION 2: *(12 marks)*

a) If α,β and γ are the roots of the equation $x^3-2x+5=0$, find the values of:

(i) $\alpha+\beta+\gamma$ **1**

(ii) $\alpha\beta+\alpha\gamma+\beta\gamma$ **1**

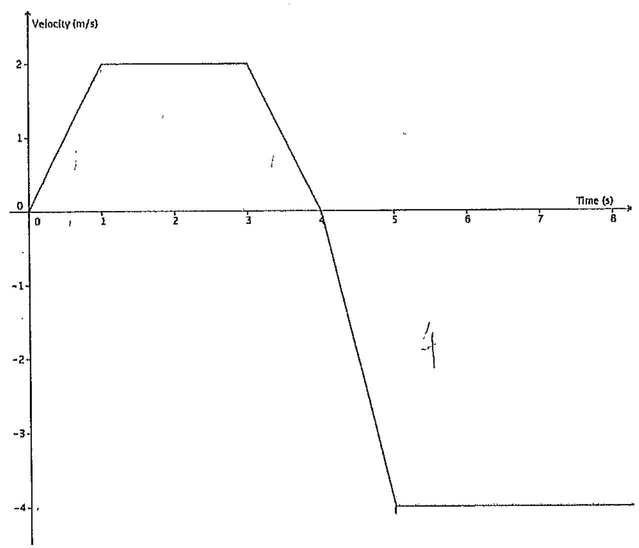
(iii) $\alpha\beta\gamma$ **1**

(iv) $\alpha^2+\beta^2+\gamma^2$ **2**

(v) $(\alpha-2)(\beta-2)(\gamma-2)$ **2**

b) Use mathematical induction to prove that n^3+2n is divisible by 3 for all $n\ge 1$. **3**

c) The graph below shows the velocity of a particle as it moves along a straight line. Initially the particle is at the origin. At what time does the particle return to the origin? **2**



QUESTION 3 OVER PAGE

QUESTION 3: *(12 marks)*

a) Points $P(2ap,ap^2)$ and $Q(2aq,aq^2)$ lie on the parabola $x^2=4ay$. PQ subtends a right angle at the vertex. The tangents at P and Q ~~meet at T.~~

(i) Show that $pq=-4$. 2

(ii) Find the equation of the locus of T. 3

b)

(i) Express $\frac{3}{2}\cos\theta+2\sin\theta$ in the form $A\cos(\theta-\alpha)$ where $A>0$. 2

(ii) Hence solve the equation $3\cos\theta+4\sin\theta=2$ for $0\le\theta\le360^0$. 2

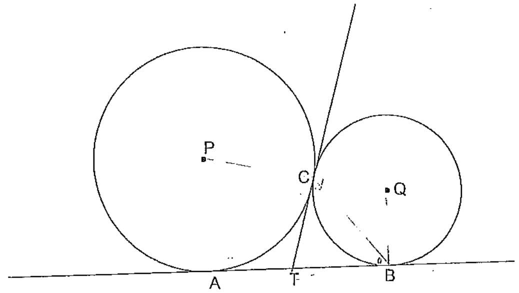
c) A polynomial is given by $P(x)=x^3+ax^2+bx-18$. Find values for a and b if $(x+2)$ is a factor of $P(x)$ and -24 is the remainder when $P(x)$ is divided by $(x-1)$. 3

QUESTION 4 OVER PAGE

QUESTION 4: *(12 marks)*

a) Use Newton's method to find a second approximation to the positive root of $x-2\sin x=0$. Take $x=1.7$ as the first approximation. 2

b) Two circles touch externally at C. The circles, which have centres P and Q are touched by a common tangent at A and B respectively. The common tangent through C meets the common tangent AB at T.



(i) Show that $AT = TB$. 2

(ii) Show that $\angle ACB$ is a right angle. 2

c)

(i) Differentiate $y=\tan^{-1}x+\tan^{-1}\frac{1}{x}$ 3

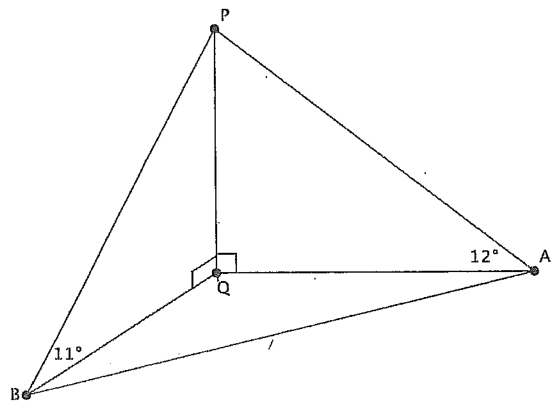
(ii) Hence or otherwise sketch the graph of $y=\tan^{-1}x+\tan^{-1}\frac{1}{x}$. 3

QUESTION 5 OVER PAGE

QUESTION 5: *(12 marks)*

a) Evaluate $\lim_{x\to 0}\frac{\sin 3x}{5x}$. 2

b) The angle of elevation of a tower PQ of height h metres from a point A, due east of it is 12^0. From another point B, due south of the tower, the angle of elevation is 11^0. The points A and B are 100 m apart and on the same level as Q, the base of the tower. Calculate h to the nearest metre. 4



c) Prove, without the use of a calculator: $\sin^{-1}\frac{1}{3}+\cos^{-1}\frac{2}{3}=\sin^{-1}\frac{2\left(1+\sqrt{10}\right)}{9}$. 3

d) Prove that $\frac{\sin 3\theta}{\sin\theta}-\frac{\cos 3\theta}{\cos\theta}=2$ $\left(\text{for } \sin\theta\neq 0, \cos\theta\neq 0\right)$. 3

QUESTION 6 OVER PAGE

QUESTION 6: *(12 marks)*

a) Consider the equation $x^3+6x^2-x-30=0$. One of the roots of the equation is equal to the sum of the other two roots. Find the value of the three roots. 4

b) Find $2\int\cos^2 4x\,dx$. 2

c) Callum has baked a chocolate cake. At 2 p.m, he takes it out of a 180^0C hot oven and places it on a cooling rack in the kitchen, where the temperature is 20^0C. According to Newton's Law of Cooling the temperature T, of Callum's cake t minutes after it comes out of the oven satisfies the equation:

$$\frac{dT}{dt}=-k\left(T-20\right) \text{ where } k \text{ is a constant.}$$

(i) Show that $T=20+160e^{-kt}$ is a solution of the equation. 2

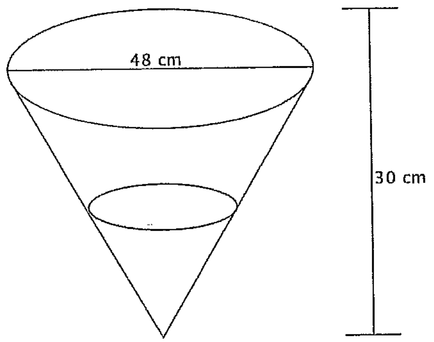
(ii) At 2:15 p.m, the cake's temperature is 100^0C. Find the value of k, correct to 3 significant figures. 2

(iii) The cake must cool to 35^0C before Callum can ice it. What is the earliest time that the cake can be iced? 2

QUESTION 7 OVER PAGE

QUESTION 7: *(12 marks)*

a) Water is pouring into a conical vessel of height 30 cm and diameter 48 cm.

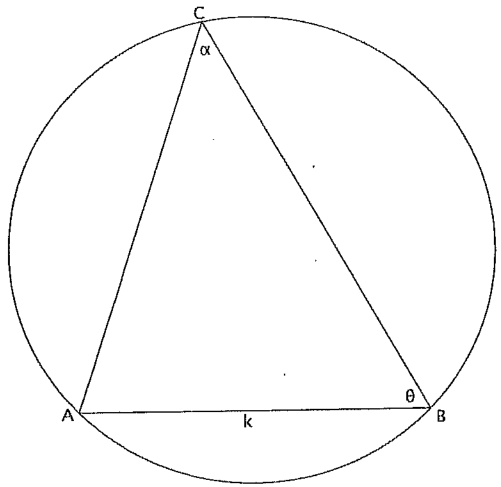


(i) Show that when there is water in the vessel to a depth of h cm, the volume of water $(V\ cm^3)$ is given by $V=\frac{16\pi h^3}{75}$. **2**

(ii) If the height of water is increasing at 0.5 cm/min, find the rate of increase of the volume when the depth of water is 10 cm from the top. Give your answer in exact form. **3**

QUESTION 7 CONTINUES OVER PAGE

b) Points A, B and C lie on a circle. The length of the chord AB is a constant k. The radian measures of $\angle ABC$ and $\angle BCA$ are θ and α respectively.



(i) Let l equal the sum of the lengths of chords CA and CB. Show that **3**

$$l=\frac{k}{\sin\alpha}\left[\sin\theta+\sin(\theta+\alpha)\right].$$

(ii) Why is α a constant? **1**

(iii) Evaluate $\frac{dl}{d\theta}$ when $\theta=\frac{\pi}{2}-\frac{\alpha}{2}$. **3**

END OF TEST ☺

TRIAL HSC EXTENSION 1 MATHEMATICS 2010 - SOLUTIONS

Question 1:

a)

$$\int_0^2 \frac{1}{x^2+4}dx = \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \frac{1}{2}\left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

b)

$$2x - y - 1 = 0 \qquad 3x - y - 2 = 0$$

$$y = 2x - 1 \qquad y = 3x - 2$$

$$m_1 = 2 \qquad m_2 = 3$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{2-3}{1+2\times3}\right|$$

$$= \left|\frac{-1}{7}\right|$$

$$= \frac{1}{7}$$

$$\theta = \tan^{-1}\frac{1}{7}$$

$$= 8.130102\ldots$$

$$= 8^0 8' \text{ (nearest minute)}$$

c)

$$m:n = -3:1$$

$$x = \frac{mx_2 + nx_1}{m+n} \qquad y = \frac{my_2 + ny_1}{m+n}$$

$$= \frac{(-3\times3)+(1\times-1)}{-3+1} \qquad = \frac{(-3\times3)+(1\times1)}{-3+1}$$

$$= 5 \qquad = 4$$

$$\therefore D = (5,4)$$

d)

$$u = 1 + x \qquad \text{When } x = -1: \; u = 1 - 1 = 0 \qquad \text{When } x = 0: \; u = 1 + 0 = 1$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$\int_{-1}^{0} x\sqrt{1+x}\,dx = \int_0^1 (u-1)\sqrt{u}\,du$$

$$= \int_0^1 (u-1)u^{\frac{1}{2}}du$$

$$= \int_0^1 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

$$= \left[\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3}\right]_0^1$$

$$= \left[\frac{2\sqrt{u^5}}{5} - \frac{2\sqrt{u^3}}{3}\right]_0^1$$

$$= \frac{2}{5} - \frac{2}{3}$$

$$= -\frac{4}{15}$$

e)

$$\frac{2}{x-1} \le 1$$

$$2(x-1) \le (x-1)^2$$

$$(x-1)^2 - 2(x-1) \ge 0$$

$$(x-1)[(x-1)-2] \ge 0$$

$$(x-1)(x-3) \ge 0$$

$$\therefore x < 1 \text{ or } x \ge 3$$

Question 2:

a)

i. $\alpha+\beta+\gamma=0$

ii. $\alpha\beta+\alpha\gamma+\beta\gamma=-2$

iii. $\alpha\beta\gamma=-5$

iv.
$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$=0^2-2(-2)$$
$$=4$$

v.
$$(\alpha-2)(\beta-2)(\gamma-2)=(\alpha\beta-2\alpha-2\beta+4)(\gamma-2)$$
$$=\alpha\beta\gamma-2\alpha\gamma-2\beta\gamma+4\gamma-2\alpha\beta+4\alpha+4\beta-8$$
$$=\alpha\beta\gamma-2(\alpha\gamma+\beta\gamma+\alpha\beta)+4(\alpha+\beta+\gamma)-8$$
$$=(-5)-2(-2)+4(0)-8$$
$$=-5+4-8$$
$$=-9$$

b)

Step 1: Prove true for $n=1$:
$1^3+2(1)=3$
which is divisible by 3.
$\therefore$ true for $n=1$

Step 2: Assume true for $n=k$:
Let $k^3+2k=3M$ where M is a positive integer.

Step 3: Prove true for $n=k+1$:
$$(k+1)^3+2(k+1)=k^3+3k^2+3k+1+2k+2$$
$$=3M+3k^2+3k+3 \text{ (using assumption)}$$
$$=3(M+k^2+k+1)$$
which is divisible by 3.
Proven true for $n=k+1$

Step 4: If true for $n=k$, then proven true for $n=k+1$. Proven true for $n=1$, hence proven true for $n=2$. If true for $n=2$, then proven true for $n=3$ and so on. Hence, by mathematical induction, statement is true for all integers $n\geq 1$.

c) $t=6$ seconds.

Question 3:

a)

i.
$$m_{PO}=\frac{ap^2-0}{2ap-0}=\frac{p}{2}$$
$$m_{QO}=\frac{aq^2-0}{2aq-0}=\frac{q}{2}$$

As $OP\perp OQ$, $m_{OP}\times m_{OQ}=-1$:
$$\frac{p}{2}\times\frac{q}{2}=-1$$
$$pq=-4$$

ii. At $P: m_T=p$
$$y-ap^2=p(x-2ap)$$
$$y=px-ap^2$$

Eqn tangent at P: $y=px-ap^2$

Eqn tangent at Q: $y=qx-aq^2$

At T:
$$px-ap^2=qx-aq^2$$
$$px-qx=ap^2-aq^2$$
$$x(p-q)=a(p-q)(p+q)$$
$$x=a(p+q)$$

When $x=a(p+q)$:
$$y=p(a[p+q])-ap^2$$
$$=apq$$
$$=-4a$$

Locus of T: $y=-4a$

b)

i.
$$A=\sqrt{\left(\frac{3}{2}\right)^2+2^2}=\sqrt{\frac{9}{4}+4}=\frac{5}{2}$$

$$\tan\theta=\frac{2}{\left(\frac{3}{2}\right)}=\frac{4}{3}$$
$$\theta=\tan^{-1}\frac{4}{3}$$
$$=53^\circ 8' \text{ (nearest minute)}$$

$$\frac{3}{2}\cos\theta+2\sin\theta=\frac{5}{2}\cos(\theta-53^\circ 8')$$

ii. $3\cos\theta + 4\sin\theta = 2$

$$\frac{3}{2}\cos\theta + 2\sin\theta = 1$$

$$\frac{5}{2}\cos(\theta - 53^0 8') = 1$$

$$\cos(\theta - 53^0 8') = \frac{2}{5}$$

$\theta - 53^0 8' = 66^0 25'$ $\theta - 53^0 8' = 293^0 35'$

$\theta = 119^0 33'$ $\theta = 346^0 42'$

c) $P(x) = x^3 + ax^2 + bx - 18$

$P(-2) = (-2)^3 + a(-2)^2 + b(-2) - 18$

$0 = -8 + 4a - 2b - 18$

$4a - 2b = 26$

$2a - b = 13$

$P(1) = (1)^3 + a(1)^2 + b(1) - 18$

$-24 = 1 + a + b - 18$

$a + b = -7$

$2a - b = 13$

$a + b = -7$

$3a = 6$

$a = 2$

$b = -9$

Question 4:

a) $f(x) = x - 2\sin x$

$f(1.7) = 1.7 - 2\sin 1.7$

$= -0.2833296...$

$f'(x) = 1 - 2\cos x$

$f'(1.7) = 1 - 2\cos(1.7)$

$= 1.257688989...$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.7 - \frac{f(1.7)}{f'(1.7)}$$

$\cong 1.9$ (to 1 dec. pl)

b) i. $AT = TC$ (equal tangents from an external point)

$TB = TC$ (equal tangents from an external point)

$\therefore AT = TB$

ii. As $AT = TC = TB$:

T is the centre of a circle with diameter AB.

$\therefore \angle ACB = 90^0$ (angle in a semi-circle)

c) i. $y = \tan^{-1} x + \tan^{-1}\frac{1}{x}$

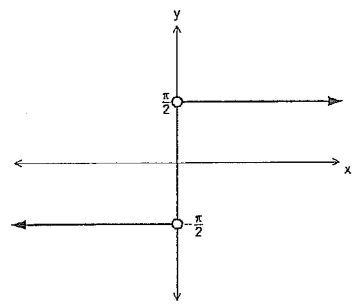
$= \tan^{-1} x + \tan^{-1} x^{-1}$

$$= \frac{1}{1+x^2} - \frac{1}{x^2}\left(\frac{1}{1+\frac{1}{x^2}}\right)$$

$$= \frac{1}{1+x^2} - \frac{1}{x^2+1}$$

$= 0$

ii.



Question 5:

a)
$$\lim_{x\to 0}\frac{\sin 3x}{5x}=\frac{3}{5}\lim_{x\to 0}\frac{\sin 3x}{3x}$$
$$=\frac{3}{5}\left(\text{as }\lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

b) In ΔBPQ:

$$\tan 11^0=\frac{h}{BQ}$$
$$BQ=h\cot 11^0$$

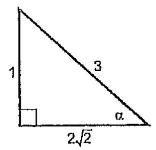
In ΔAPQ:

$$\tan 12^0=\frac{h}{AQ}$$
$$AQ=h\cot 12^0$$

$$AB^2=BQ^2+AQ^2$$
$$100^2=\left(h\cot 11^0\right)^2+\left(h\cot 12^0\right)^2$$
$$=h^2\left(\cot^2 11+\cot^2 12\right)$$
$$h^2=\frac{100^2}{\cot^2 11+\cot^2 12}$$
$$h=14.344...\text{ m}$$
$$\cong 14\text{ m (to the nearest metre)}$$

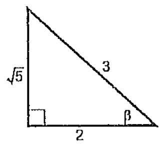
c) Let $\alpha=\sin^{-1}\frac{1}{3}$

$$\sin\alpha=\frac{1}{3}$$



Let $\beta=\cos^{-1}\frac{2}{3}$

$$\cos\beta=\frac{2}{3}$$



$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$=\frac{1}{3}\times\frac{2}{3}+\frac{2\sqrt{2}}{3}\times\frac{\sqrt{5}}{3}$$
$$\alpha+\beta=\sin^{-1}\left(\frac{2}{9}+\frac{2\sqrt{10}}{9}\right)$$
$$\sin^{-1}\frac{1}{3}+\cos^{-1}\frac{2}{3}=\sin^{-1}\frac{2\left(1+\sqrt{10}\right)}{9}$$

d)
$$LHS=\frac{\sin 3\theta}{\sin\theta}-\frac{\cos 3\theta}{\cos\theta}$$
$$=\frac{\sin 3\theta\cos\theta-\cos 3\theta\sin\theta}{\sin\theta\cos\theta}$$
$$=\frac{\sin(3\theta-\theta)}{\sin\theta\cos\theta}$$
$$=\frac{\sin 2\theta}{\sin\theta\cos\theta}$$
$$=\frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$$
$$=2$$
$$LHS=RHS$$

Question 6:

a) $x^3+6x^2-x-30=0$

Let roots be $\alpha,\beta,\alpha+\beta$

$$2\alpha+2\beta=-6$$
$$\alpha+\beta=-3 \quad ---(1)---$$
$$\alpha\beta(\alpha+\beta)=30$$
$$\alpha\beta(-3)=30$$
$$\alpha\beta=-10$$
$$\beta=\frac{-10}{\alpha} \quad ---(2)---$$

Sub (2) into (1):

$$\alpha-\frac{10}{\alpha}=-3$$
$$\alpha^2+3\alpha-10=0$$
$$(\alpha+5)(\alpha-2)=0$$
$$\alpha=-5 \text{ or } \alpha=2$$

When $\alpha=-5$: $\beta=2$

When $\alpha=2$: $\beta=-5$

$\therefore$ The roots are -5, 2 and -3.

b)

$$\cos 2x=2\cos^2 x-1$$
$$\cos 8x=2\cos^2 4x-1$$
$$2\cos^2 4x=1+\cos 8x$$
$$2\int\cos^2 4x\,dx=\int(1+\cos 8x)\,dx$$
$$=x+\frac{1}{8}\sin 8x+C$$

c) i.

$$T=20+160e^{-kt}$$
$$\frac{dT}{dt}=-160ke^{-kt}$$

$$\frac{dT}{dt}=-k(T-20)$$
$$=-k\left(\left[20+160e^{-kt}\right]-20\right)$$
$$=-k\left(160e^{-kt}\right)$$
$$=-160ke^{-kt}$$

$\therefore T=20+160e^{-kt}$ is a solution of the equation.

ii. At 2:15p.m ($t=15$):

$$100=20+160e^{-15k}$$
$$80=160e^{-15k}$$
$$e^{-15k}=\frac{1}{2}$$
$$-15k=\ln\left(\frac{1}{2}\right)$$
$$k=-\frac{1}{15}\ln\left(\frac{1}{2}\right)$$
$$=0.0462098...$$
$$\cong 0.0462 \text{ (3 sig. figs)}$$

iii. When $T=35$:

$$35=20+160e^{-0.046t}$$
$$15=160e^{-0.046t}$$
$$e^{-0.046t}=\frac{3}{32}$$
$$-0.046t=\ln\left(\frac{3}{32}\right)$$
$$t=-\frac{1}{0.046}\ln\left(\frac{3}{32}\right)$$
$$=51.459209...\text{min}$$
$$\cong 51 \text{ min}$$

The earliest time Callum can ice the cake is 2:51p.m.

Question 7:

a) i. Let radius of water surface = r

$$\frac{h}{30}=\frac{r}{24}$$
$$30r=24h$$
$$r=\frac{4h}{5}$$

$$V=\frac{1}{3}\pi r^2 h$$
$$=\frac{1}{3}\pi\left(\frac{4h}{5}\right)^2 h$$
$$=\frac{1}{3}\pi\times\frac{16h^2}{25}\times h$$
$$=\frac{16\pi h^3}{75}$$

ii. $\frac{dh}{dt}=0.5$ cm/min

$$\frac{dV}{dh}=\frac{48\pi h^2}{75}$$
$$=\frac{16\pi h^2}{25}$$

$$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$$
$$=\frac{16\pi h^2}{25}\times 0.5$$
$$=\frac{8\pi h^2}{25}$$

When h = 20 (i.e. 10 cm from the top):

$$\frac{dV}{dt}=\frac{8\pi\times(20)^2}{25}$$
$$=128\pi \text{ cm}^3/\text{min}$$

b) i. $\frac{CA}{\sin\theta}=\frac{k}{\sin\alpha}$

$$CA=\frac{k}{\sin\alpha}\sin\theta$$

$$\frac{CB}{\sin(180-[\alpha+\theta])}=\frac{k}{\sin\alpha}$$
$$\frac{CB}{\sin(\alpha+\theta)}=\frac{k}{\sin\alpha}$$
$$CB=\frac{k}{\sin\alpha}\sin(\alpha+\theta)$$

$$l=CA+CB$$
$$=\frac{k}{\sin\alpha}\sin\theta+\frac{k}{\sin\alpha}\sin(\alpha+\theta)$$
$$=\frac{k}{\sin\alpha}\left[\sin\theta+\sin(\alpha+\theta)\right]$$

ii. α is the angle at the circumference standing on chord AB and the length of AB, k is a constant.

iii. $\frac{dl}{d\theta}=\frac{k}{\sin\alpha}\left[\cos\theta+\cos(\theta+\alpha)\right]$

When $\theta=\frac{\pi}{2}-\frac{\alpha}{2}$:

$$\frac{dl}{d\theta}=\frac{k}{\sin\alpha}\left[\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}+\alpha\right)\right]$$
$$=\frac{k}{\sin\alpha}\left[\sin\left(\frac{\alpha}{2}\right)+\cos\left(\frac{\pi}{2}-\left\{-\frac{\alpha}{2}\right\}\right)\right]$$
$$=\frac{k}{\sin\alpha}\left[\sin\left(\frac{\alpha}{2}\right)+\sin\left(-\frac{\alpha}{2}\right)\right]$$
$$=\frac{k}{\sin\alpha}\left[\sin\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right]$$
$$=\frac{k}{\sin\alpha}\times 0$$
$$=0$$